Phonon hot spot in pure substances

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This paper analyzes in detail the evolution of the "hot spot" which is produced by intense optical or thermal excitation in semiconductors or insulators of high purity when ordinary heat-conduction conditions prevail in the heated region. The spatial asymptotic behavior of the temperature distribution is singled out for special study. Two characteristic regions can be distinguished in this asymptotic behavior. Directly adjacent to the high-temperature part of the spot is a region with "diffusion" asymptotic behavior, within which the motion of a front with a given temperature is described by an $x \propto t^{1/2}$ law. For the real phonon relaxation mechanisms, however, this asymptotic behavior can occur only in one-dimensional geometry, i.e., only at distances from the surface shorter than the spot radius. The second characteristic region—that of the "runaway" asymptotic behavior—makes it possible to satisfy the physical limitations at infinity which stem from the limitation on ballistic heat transfer. The space-time dependence of the temperature is found for three-dimensional experimental geometry and also two-dimensional geometry (a film) for various heat-conduction laws. The occupation numbers of ballistic phonons emitted by the spot are found. Their spectrum is shown to be non-Planckian, characterized in particular by depletion of the low-frequency region. The time evolution is determined by both the details of the phonon kinetics and the experimental geometry. In particular, the corresponding dependence may be nonmonotonic and may even have a local minimum. This analysis yields an interpretation of several recent experimental results.

INTRODUCTION

Both the intense thermal excitation of the surface of a semiconductor or insulator which can be achieved using a metallic heater ("heat generator") and intense optical excitation of the surface of a semiconductor are known to be accompanied by the formation of a phonon nonequilibrium region or "hot spot" in the surface layer (Refs. 1–3, for example). This spot exists for a comparatively long time, which is significantly longer than the time scale for the removal of energy by ballistic phonons; it also depends on the pump. There are two mechanisms which might be responsible for this behavior, which lead to a "cutoff" of the phonons in the surface layer and which prevent their free escape into the "cold" crystal. The first of these mechanisms consists of phonon-phonon umklapp processes which do not conserve the total momentum of the phonon system; the second mechanism is the scattering of phonons by defects (including isotopic defects). The efficiency of each of these mechanisms obviously increases with increasing characteristic energy of the phonons and thus with the extent to which the surface is heated. For the first mechanism, this dependence, which is accompanied by dependence on the filling numbers, is sharper, and it is usually important at fairly high temperatures. With regard to the second mechanism, we note that it is strongly affected by normal phonon-phonon processes which redistribute the energy of the phonon system over the spectrum.

Kazakovtsev and Levinson[4] have derived a theory for hot spots for the case of strong scattering by defects, in which the condition $l_i(\omega) \ll l_N(\omega)$ holds for all phonon frequencies (here l_i and l_N are the phonon mean free paths with respect to scattering by defects and with respect to normal phonon-phonon processes, respectively). In this case there is a nonlocal heat conduction regime.[5,6] The conditions under which a spot forms and the subsequent evolution of the spot were studied. In particular, it was predicted that a spot would decay rapidly as soon as its thickness became comparable to its radius r_0 (i.e., after the transition from one-dimensional to three-dimensional geometry).

Since under the condition $\hbar\omega \ll \Theta$ (Θ is the Debye temperature) we have $l_i(\hbar\omega \sim T) \propto T^{-4}$, while we have $l_N(\hbar\omega \sim T) \propto T^{-5}$ (Ref. 7), the relation between l_i and l_N is evidently determined by both the purity of the sample and the intensity of the excitation. If experiments are carried out on samples of pure substances, and the excitation level is sufficiently high (in several actual experiments, the spot temperature exceeds Θ; Ref. 8), we would expect the inequality $l_N \ll l_i$ to hold. In this case we would be dealing with ordinary heat conduction. Processes accompanied by scattering might also play a substantial role in the vicinity of a spot. This limiting case was studied by Guseĭnov[9] for germanium, where the relation $\varkappa \propto T^{-1}$ ($\varkappa$ is the thermal conductivity) holds over a broad temperature range. Guseĭnov carried out a semiquantitative analysis of the nonlinear heat-conduction problem on the basis of energy conservation.[1)] However, in Ref. 9 Guseĭnov studied only the evolution of the region in which most of the spot energy is concentrated (this comment also applies to Ref. 4). At the same time, it is obvious that there cannot be a temperature drop at the boundary of a spot. The argument is that if there were such a temperature drop the high-energy phonons would be able to escape freely into the cold volume, where they should undergo umklapp events and so give rise to a nonzero temperature, but this result would contradict the concept of a sharp boundary. There thus exists a wide region of a comparatively slow temperature decay—the "atmosphere" of a spot—which controls the escape of energy from the spot and thus its lifetime. An analysis of this region is of further importance in that if the frequency of the nonequilibrium

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phonons observed experimentally is sufficiently high then the only place they could be produced is in this atmosphere, since phonons of the corresponding frequency generated in the high-temperature region would ultimately attach to thermalized phonons, forming a spot. In such a case, it would be the evolution of the atmosphere which determined the time evolution of the observable filling numbers.

In this paper we carry out a detailed study of the hot spot in the case in which the condition $l_N \ll l_i$ is satisfied in the spot region. When this condition holds, we can use the ordinary heat-conduction equation, for an arbitrary temperature dependence $\varkappa(T)$. We will be particularly interested in the spatial asymptotic behavior of the temperature distribution and also the time evolution of the filling numbers of the ballistic phonons emitted by the spot and detected by a narrow-band phonon detector.

The spot itself and the hottest part of the atmosphere are analyzed through an exact solution of the nonlinear heat-conduction problem with appropriate boundary and initial conditions.[10] Such a solution exists if $\varkappa(T)$ is described by a power law: $\varkappa(T) \propto T^{3-n}$, where n is not too large. In this regime, which we will call a "diffusion" regime, the behavior $T(x, t)$ is determined by the relation $D(T)t \sim x^2$, where $D(T)$ is the diffusion coefficient of phonons with an energy $\hbar\omega \sim T$. In other words, the law describing the shift of a point on the temperature relief with a temperature T is characteristic of a diffusive motion and is determined exclusively by the value of $D(T)$. An important property of such a temperature distribution is that most of the energy is concentrated at small values of x; i.e., a given state of the spot is reproduced in the course of its evolution. The maximum temperature T_m can be estimated from energy conservation and the circumstance that most of the energy is concentrated in a region with a typical thickness $x_s \sim (D(T_m)t)^{1/2}$.

As we have already mentioned, however, such "diffusive" spreading of a spot can occur if $D(T)$ does not increase too rapidly with decreasing T. With $D \propto T^n$, in the one-dimensional situation, this situation corresponds to a limitation $n < 8$; in three dimensions it corresponds to $n < 8/3$; and in two dimensions (a film) we have $n < 4$. [In the one-dimensional case, the corresponding condition is obviously violated only if the thermal conductivity is limited by phonon-phonon umklapp processes; the exponential factor in the functional dependence $D(T)$ plays a fundamental role. In the three-dimensional situation, on the other hand, a diffusion solution is impossible for essentially any scattering mechanism.] In the opposite case, the nonlinear heat-conduction equation does not have a solution which satisfies homogeneous boundary conditions at infinity (the natural conditions for a hot spot propagating in a cold volume). This means that in a heat-conduction description the energy transfer into an infinitely remote region would occur in infinitely short times.[10] Since a conclusion of this sort is unphysical, it is necessary to impose a restriction on the energy flux (which corresponds to the solution of the heat-conduction equation): This flux cannot be greater than permitted by the ballistic regime for the propagation of phonons. In particular, it is obvious that for $x > wt$ (w is the sound velocity) we have $I = 0$. At the same time, if we focus on some particular coordinate $x_1 \ll wt$ then we would expect, under the conditions of this heat-transfer regime, that the energy of the phonon system corresponding to the region $x > x_1$ would be considerably greater than that concentrated at $x < x_1$. We will accordingly call the corresponding asymptotic behavior of the temperature relief, which supports a rapid energy transfer, a "runaway" asymptotic behavior. Since the time scale of the heat transfer in this situation is short in comparison with the time scale for the heat-conduction equation $[\sim (D(T)/x^2)^{1/2}]$, in analyzing the latter it is natural to assume that the energy flux has already reached a steady state and is independent of the time: $I(x) = \text{const}$. To estimate I we can use the value corresponding to a ballistic regime with $x \sim wt$.

Under the conditions corresponding to this runaway solution, a spot obviously could not form (so that, in particular, the initial stage of the evolution of a spot is necessarily one-dimensional). However, a runaway asymptotic behavior may be realized in a certain intermediate region of the atmosphere—under conditions such that most of the energy of the spot is concentrated in the region in which the diffusion solution applies. At any rate, a transition from one-dimensional to three-dimensional geometry necessarily involves a transition to a runaway asymptotic behavior. An asymptotic behavior of this sort may also be manifested in a purely one-dimensional problem—if the temperature dependence $\varkappa(T)$ becomes exponential at low temperatures. An important question is how the transition from diffusive to runaway behavior occurs. It would appear at first glance that this transition would occur at specifically those coordinates at which the dependence $\varkappa(T)$ changes or at which there is a transition to a three-dimensional (or two-dimensional) geometry (i.e., at $x \sim r_0$). This is not always true, however. The reason is that the energy flux satisfies $I \propto \varkappa\nabla T \propto T^{4-n}$, and the temperature growth under the conditions corresponding to the diffusive solution at $r \sim r_0$, for $n > 4$, would imply a decrease in I, while in the case of a runaway solution, controlled by the ballistic removal of energy, the temperature growth should have lead to an increase in I. Accordingly, the merging of the two asymptotic expressions at $r \sim r_0$ is possible only if $n < 4$ [i.e., only if the dependence $\varkappa(T)$ is of a nature which is not really typical from the experimental standpoint]. For $n > 4$, on the other hand, since the spatial variation of the heat flux must be monotonic in the asymptotic region, we see that the diffusive solution becomes inapplicable quite early, at some $x_r(t) \ll r_0$: It is near x_r that the transition to a quasisteady runaway asymptotic behavior occurs. For this reason, the decay of the spot occurs earlier than we would expect on the basis of the estimate $x_s(t) \sim r_0$ [specifically, at $x_s(t) \sim x_r(t) \ll r_0$].

The three-dimensional geometry has an important distinctive feature, analogous to a spreading resistance. As we move away from the center of a spot, the effective "potential" $U = \int \varkappa dT$ falls off rapidly. Its gradient determines the heat flux density: $j = \nabla U$. At $n > 4$ ($U \propto T^{4-n}$), this circumstance would obviously be incompatible with the requirement $T|_{r\to\infty} \to 0$. This result means that even at $r \sim r_0$ the concept of a temperature and the heat-conduction description become meaningless, and the subsequent propagation of the phonons is in a regime of ballistics, quasiballistics, or quasidiffusion.[11] The lowest temperature which is still meaningful (it can be assigned to the "edge of the spot"), $T^* = T(r \sim r_0)$ is determined by the estimate $T^* \sim T_b$, where $l(T_b) \sim r_0$ (it does not change during the existence of a spot).

If, on the other hand, we have $n = 4$, then we have $j \propto d(\ln T)/dx$, so the temperature decay in the region of the quasisteady asymptotic behavior is very sharp. The quantity x_r turns out to be on the order of r_0, and in this region (which essentially coincides with the region of the transition from a one-dimensional to a three-dimensional geometry) the temperature decays from values $T_r = T(x_r) \propto t^{1/4}$, determined by the diffusion asymptotic behavior, to values $\sim T_b$.

The case of two-dimensional geometry, in which the sample is a plate or film, is quite distinctive. In this case there is always a phonon scattering mechanism at the boundaries for which we have $D = \text{const}$. At the same time, heat removal through the film surface can play an important role. It is because of this latter circumstance that (as we will see) there can be a sharp temperature decay near $r \sim r_0$ in the case $n = 4$. The temperature of the "edge of the spot," T^*, increases in time in proportion to $t^{1/4}$.

In the final section we will show how we can use a known temperature distribution in a spot and its atmosphere to find the time evolution of the filling numbers of the ballistic phonons which could be detected by a detector. In particular, we show that these dependences may be rather complex and nonmonotonic (dependences of this type were seen in the experiments of Ref. 8). We will also demonstrate that at low frequencies there is a depletion of the spectrum of occupation numbers of nonequilibrium phonons (again, this result has been seen experimentally[12,13]).

1. HIGH-TEMPERATURE REGION OF A SPOT; DIFFUSIVE ASYMPTOTIC BEHAVIOR

We consider the half-space $x > 0$, filled with a semiconductor or insulator, in which an energy ε is introduced at the time $t = 0$, in a process which takes a negligibly short time t_0. We assume that the characteristic radius of the excited region, r_0, is considerably greater than the thickness of the excitation layer, x_0, which we will also assume to be negligibly small (Fig. 1a). We examine the case of high excitation levels, in which we have $l_N \ll l_i$ in the spot region for the typical phonon energies, and the typical thickness of the layer in which most of the spot energy is localized satisfies $x_s(t) \gg l_N$. When these conditions are satisfied, we can use the heat-conduction equation

$$C\frac{dT}{dt} = \operatorname{div}(\varkappa \nabla T). \tag{1}$$

The boundary condition on Eq. (1) at the boundary $x = 0$ corresponds to the requirement that the heat flux vanish there: $\nabla T|_{x=0} = 0$ (it turns out that heat removal can be ignored in this situation). At infinity, on the other hand, we

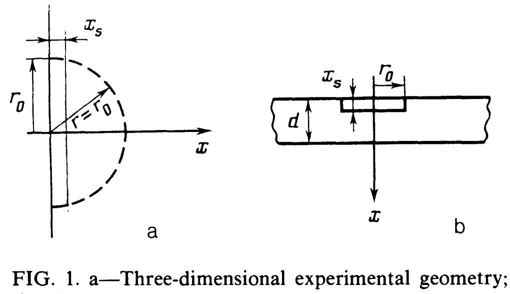

FIG. 1. a—Three-dimensional experimental geometry; b—two-dimensional.

require $T|_{x\to\infty} \to 0$ (the condition of a "cold" volume). With regard to the temperature dependence of $\varkappa$, we assume that it is determined by both the temperature dependence of the heat capacity, $C(T)$, and that of the phonon diffusion coefficient D [i.e., actually that of the phonon mean free path l with respect to momentum-loss processes: $\varkappa \sim C(T)D(T)$, $D \sim wl/3$]. For simplicity we will be ignoring the distinction between the various branches of the phonon spectrum.

The nature of the temperature dependence will obviously differ in the regions $T > \Theta$ and $T < \Theta$. Following Ref. 9, we introduce a temperature T_0, at which we will join the high-temperature and low-temperature asymptotic expressions. For the region $T > T_0$ we have $C = C_0 = \text{const}$, $l \sim l_N \sim l_0 T_0/T$, and $\varkappa \propto T^{-1}$ (Refs. 7 and 14). For the low-temperature region we have $C \sim C_0(T/T_0)^3$. The behavior $l(T)$ requires a more detailed analysis. We know that the phonon-phonon component of l^{-1} is described at $T \ll \Theta$ by[14]

$$l^{-1} \sim l_0^{-1}(T/T_0)^m \exp(-T_0/\alpha T), \tag{2}$$

where m and α depend on the particular details of the phonon spectrum. It is important that the relation $\alpha \gg 1$ usually holds, since the exponential dependence begins to be seen only at extremely low temperatures. In turn, it is at these extremely low temperatures that impurity scattering is usually predominant (for this type of scattering, under the condition $l_N \lesssim l$, we have $l \propto T^{-4}$). It thus turns out that over a broad temperature range (up to the temperatures at which the concept of a local thermal conductivity becomes meaningless) the dependent $l(T)$ can be described by a power law $l \propto T^{-n}$. This behavior corresponds to the actual experimental situation.[14]

We will first analyze one-dimensional spot evolution. An important point is that in the case of a power function $\varkappa(T) \sim C_0 T_0 (T/T_0)^{\nu - n - 1}$ [where $C(T) \equiv C_0(T/T_0)^{\nu - 1}$] the nonlinear equation (1) has a simple solution which satisfies the boundary and initial conditions of interest here[10]:

$$Q \equiv \int_0^T C\,dT = \left(\frac{4E}{at}\right)^{\nu/(2\nu-n)} \left[\frac{n}{2(2\nu-n)}\left(\xi_0^2 + \frac{x^2}{[at/(2E)^{n/\nu}]^{2\nu/(2\nu-n)}}\right)\right]^{-\nu/n}, \tag{3}$$

where

$$a \equiv \varkappa Q^{n/\nu}/C|_{T=T_0} \sim D_0 (C_0 T_0)^{n/\nu}/\nu^{n/\nu},$$

$$\xi_0^{2-n/\nu} \equiv 2\nu(2-n/\nu)\pi^{n/2\nu} n^{-1}\Gamma^{n/\nu}(\nu/n - 1/2)\Gamma^{-n/\nu}(\nu/n) \sim 1,$$

and $E \sim \mathscr{E}/\pi r_0^2$ is the surface density of the injected energy. The asymptotic form of this solution as $x \to \infty$ is quite simple:

$$\left(\frac{T}{T_0}\right) \approx \left(\frac{\nu}{C_0 T_0}\right)^{1/\nu} \left(\frac{\gamma a t}{x^2}\right)^{1/n} \sim \left(\frac{\gamma D_0 t}{x^2}\right)^{1/n}, \quad \gamma = \frac{2(2\nu - n)}{n}. \tag{4}$$

It does not depend on E, and it has a clear physical meaning: A front corresponding to some given temperature T is propagating into the interior of the sample in accordance with a diffusion law $D(T(x,t))t \sim x^2$. The trajectory $x(t)$ is determined exclusively by the value of T and does not depend on the temperature distribution at smaller values of x. The spot region proper, where we find most of the energy of the spot, corresponds to small values of x, for which the term ξ_0^2 on the right side of (3) is important. Since $\xi_0 \sim 1$, the boundary of this region, x_s, is found from the condition

$$x_s = \left[\frac{at}{(2E)^{n/\nu}}\right]^{\nu/(2\nu-n)} \sim (D_0 t)^{\nu/(2\nu-n)} \left(\frac{C_0 T_0}{E}\right)^{n/(2\nu-n)}, \quad (5)$$

while the maximum temperature in the spot is, in order of magnitude,

$$T_m \approx T_0 \left[\frac{4E^2}{D_0 t (C_0 T_0)^2}\right]^{1/(2\nu-n)} \left[\xi_0^2 \frac{n}{2(2\nu-n)}\right]^{1/\nu}$$
$$\sim \left(\frac{\nu E}{C_0 T_0 x_s}\right)^{1/\nu}. \quad (6)$$

If $T_m > T_0$, then in a certain region of the atmosphere at $x \sim x_0(t)$ there should be a transition from a regime with $T > T_0$ to one with $T < T_0$. Correspondingly, there will be changes in the indices n and ν; according to (4) we have $x_0 \sim (D_0 t)^{1/2}$. Solution (3) of course presupposes n, $\nu = \text{const}$, and strictly speaking we cannot use it for $T_m > T_0$. However, the fact that the law of motion of a $T = \text{const}$ front is essentially independent of the T distribution in the hotter region, according to (3) and (4), being determined only by the value of $D(T)$, makes it possible to join the high-temperature and low-temperature asymptotic expressions self-consistently at $T \sim T_0$ [since the numerical factors in (4) for $T > T_0$ and $T < T_0$ are different, the joining occurs in a certain transition region, but the presence of this region can be ignored if we are content with order-of-magnitude estimates].

2. RUNAWAY ASYMPTOTIC BEHAVIOR

Solution (3) evidently holds only if $(2\nu - n) > 0$. Since this condition clearly does hold at $T > T_0$, we will restrict the present section of this paper to the case $T < T_0$, for which the condition becomes $8 - n > 0$. To see the physical content of this limitation, we note that in addition to the natural normalization condition

$$\int_0^\infty dx Q(x) = E, \quad (7)$$

from which we find

$$\lim_{x\to\infty} [x T^4(x)] = 0, \quad (8)$$

the function $T(x, t)$ must satisfy the requirement that the energy flux vanish at infinity.

$$\lim_{x\to\infty} (\varkappa \nabla T) = 0. \quad (9)$$

We will discuss the meaning of this condition in more detail in just a moment. It is not difficult to see that conditions (8) and (9) are compatible only if $(8 - n) > 0$. If, on the other hand, we have $n \geqslant n_c = 8$, and conditions (7) and (8) hold, then we necessarily have $\varkappa \nabla T|_{x\to\infty} \to \infty$.

It is not difficult to generalize these considerations to the d-dimensional situation ($d = 2$ or 3), writing conditions (8) and (9) in the form

$$\lim_{r\to\infty} r^d T^4(r) = 0, \quad (8a)$$

$$\lim_{r\to\infty} I(r) = 0, \quad I(r) = 2\pi r^{d-1} \varkappa \nabla T \quad (9a)$$

(here we have taken account of the circumstance that in the typical experimental situation with $d = 3$ the heat is propagating into a half-space, while for $d = 2$ the heat is propagating in all directions). The critical value of n, which corresponds to a disruption of the compatibility of (8a) and (9a), is $n_c = 4$ ($d = 2$) or $n_c = 8/3$ ($d = 3$). What are the consequences of the incompatibility of (8a) and (9a)? Let us first refine the meaning of (9) and (9a). In the limit $r \to \infty$, $t > 0$, the condition $dT/dt > 0$ must hold [since we have $T(t = 0) = 0$, and the heat transfer rate is finite]. It follows that in the limit $r \to \infty$ the function $I(r)$ falls off monotonically. Integrating (1) over a region bounded by the radius r, we find

$$dE(r)/dt = -I(r), \quad (10)$$

so that, on the one hand, condition (9a) means an infinite time for the heat transfer to an infinitely remote point, while on the other hand the result $I(r) \to \infty$ [which follows from (8a) in the case $n \geqslant n_c$] means an instantaneous heat transfer over infinitely long distances.[10]

In principle, we could assume that the nonlinear equation (1) has a solution which vanishes at finite distances $r = r'$. In this case, however, the boundedness of I as $r \to r' - 0$ would require that the condition $\nabla T|_{r\to r'-0} \to 0$ be satisfied [since we have $\varkappa(T)|_{T\to 0} \to \infty$]. The simultaneous vanishing of T and ∇T at the point r', however, would not let us have a nonzero solution at $r < r'$. In this regard, the situation is quite different from nonlinear heat conduction under explosive conditions (cf. Ref. 15), for which we would have $n < 0$ and $C = \text{const}$, i.e., $\varkappa|_{T\to 0} \to 0$. At the same time, for $n \geqslant n_c$ it is not possible to realize an "explosive" self-similar solution (as proposed in Ref. 4), according to which heat transfer over an infinite distance would occur in a finite time.

The picture of instantaneous heat transfer obviously does not correspond to the physical meaning. For $n \geqslant n_c$ we should thus take account of the relevant physical limitations. In the first place, Eq. (1) can be used only if $r < l(T)$, i.e., $r < r_b$, where r_b corresponds to the transition to the ballistic regime and is determined by the condition $r_b \sim l(T)_b$, $T_b \equiv T(r_b)$. Second, for $r \sim r_b$ the energy flux corresponding to (1) should join with that determined by the ballistic transport of phonons:

$$I = I_b = 2\pi r_b^{d-1} j_b, \quad j_b \sim C(T_b) T_b w$$

(an estimate of I_b will be derived below for specific situations). Since the energy flux coming out of the spot is limited only by the ballistic removal, the corresponding asymptotic behavior can be called the "runaway" asymptotic behavior. Clearly, under such conditions ($n \geqslant n_c$), with rapid removal of energy, a spot cannot form. We can thus discuss specifically this asymptotic region in a case in which the condition $n < n_c$ holds in the hotter part of a spot.

As we mentioned earlier, the condition $dI/dr < 0$ holds in the atmosphere of a spot. In the limit $r \to \infty$ with $n \geqslant n_c$, that condition is incompatible with (8a), but since we are talking about finite distances $r < wt$ this incompatibility is unimportant. We can thus write the following expression in the region of the runaway asymptotic behavior:

$$E(r \sim wt)/E(r_1) \gg 1, \quad \text{if} \quad r_1 \ll wt.$$

Since

$$I(r_1) \gtrsim I(r \sim wt), \quad E(r \sim wt)/I(r \sim wt) \sim t,$$

we have $E(r_1)/I(r_1) \ll t$. Since the quantity $E(r)/I(r)$ is an estimate of the time scale of the evolution of the heat distribution, it follows that for $r \ll wt$ the solution is quasisteady. In other words, it satisfies the equation

$$2\pi r^{(d-1)}\varkappa(T)\nabla T = I(r) \approx \text{const} \approx I_b. \qquad (11)$$

Let us examine the question of joining the diffusion asymptotic behavior (4) with the runaway asymptotic behavior (11). It might appear that this joining should be done at those values $r = r_c$ (or $x = x_c$) for which the transition from the $n < n_c$ regime to the $n \geqslant n_c$ regime occurs. Such a transition evidently could occur when we go from the one-dimensional geometry to $d = 2$ or $d = 3$; in this case we would have $x_c \sim r_0$ (Fig. 1a). In principle, this transition could also occur within the confines of a one-dimensional geometry, if at some $T = T_1$ there were a corresponding change in the heat-conduction law. However, matching at $r = r_c$ can be realized only for $n < 4$. The reason is that the expression for the energy flux density,

$$j = \varkappa\nabla T \propto T^{(4-n)}/r, \quad n \neq 4, \quad \text{or} \quad j \propto d(\ln T)/dr, \quad n = 4,$$

shows us that a power-law increase in the temperature over time as described by solution (4) would not be accompanied by a corresponding growth of j for $n \geqslant 4$ (for $n > 4$, j would decrease with increasing T). In the case of the runaway asymptotic behavior, on the other hand, a temperature increase would evidently lead to an increase in the ballistic flux and thus in j. As we will see from some estimates which follow, in the case $n \geqslant 4$ we have $I_{\text{diff}}(r = r_c - 0) \ll I(r = r_c + 0) \sim I_b$ (the equality $I_{\text{diff}} \sim I_b$ holds only at the time $t \sim r_c/w$). It follows that for $n \geqslant 4$ we have intermediate asymptotic behavior, which falls in the region of a one-dimensional geometry, for which the relation $I(x) > I_{\text{diff}}(x)$ holds. It is easy to see that in this region we have $T(x) \lesssim T_{\text{diff}}(x)$. We thus have the estimate

$$C\frac{dT}{dt} \lessdot \frac{CT}{t} < \frac{CT_{\text{diff}}}{t} \sim \frac{I_{\text{diff}}}{x} \ll \frac{I}{x};$$

i.e., for the intermediate asymptotic behavior we have $CdT/dt \ll I/x$, or $dI/dx \ll I/x$. Since the x in this relation is the length scale of the variation along the coordinate, we have

$$I(x) = \text{const}$$

in this region. The conclusion which we draw is that the quasisteady asymptotic behavior described by (11) can be continued into the region with $n < n_c$. Matching with solution (4) is done at $x = x_r$, where

$$I_{\text{diff}}|_{x \sim x_r} \sim I_b. \qquad (12)$$

From (12) and (4) we find the following results for x_r and for the corresponding temperature T_r:

$$x_r \approx (C_0 T_0 D_0 / 2nI_b)^{n/(8-n)} (\gamma D_0 t)^{(4-n)/(8-n)}, \qquad (13)$$

$$T_r \approx T_0 (\gamma D_0 t / x_r^2)^{1/n}. \qquad (13a)$$

We see that we have $x_r = x_r(t)$. In particular, with $I_b = \text{const}$ we have $x_r \propto t^{(4-n)/(8-n)}$.

Figure 2 is a sketch of the temperature distribution in a spot and its atmosphere.

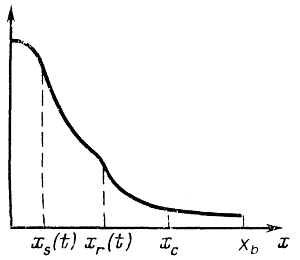

FIG. 2. Temperature distribution in a spot. $(0, x_r)$—Region of diffusion solution (3); $x > x_r$—region of quasisteady solution (11); x_c—boundary between the $n < n_c$ and $n > n_c$ regimes (for $d = 2$ and 3 we have $x_c = r_0$); r_b—boundary of the transition to the ballistic regime.

3. RUNAWAY ASYMPTOTIC BEHAVIOR IN ONE-DIMENSIONAL GEOMETRY

Analogous behavior is possible in substances which are so pure (including isotopically pure) that $\varkappa$ has an exponential dependence at low temperatures. Although the preceding analysis was carried out for a power law $D(T)$, this assumption is not important for the quasisteady runaway asymptotic behavior (11). It is sufficient that the growth in $D(T)$ with decreasing T occur more rapidly than T^{-n_c} (this condition is met in the present case). We consider the $l(T)$ dependence [see (2)] at temperatures $\exp(T_\alpha/T) > (T_\alpha/T)^{n-3}$ ($T_\alpha \equiv T_0/\alpha$). The general solution of (11) with $d = 1$ is

$$\varkappa(T_\alpha)T_\alpha \int_{T_\alpha/T_c}^{T_\alpha/T} du\, u^{n-5} e^u \sim \varkappa(T_\alpha)T_\alpha \exp\left(\frac{T_\alpha}{T}\right)\left(\frac{T_\alpha}{T}\right)^{n-5}$$

$$= I_6(x - x_c), \qquad (14)$$

so we have

$$T = T_\alpha\left\{\ln\left[\frac{I_6(x-x_1)}{\varkappa(T_\alpha)T_\alpha}\right] + (n-5)\ln\left(\frac{T_\alpha}{T}\right)\right\}^{-1}. \qquad (15)$$

We find I_b on the basis of the following considerations. For $x \sim wt$, there is evidently a transition from diffusive propagation of phonons to ballistic propagation. At $x \lesssim wt$, the concept of a temperature is meaningful (since we are assuming $l_N \ll l_i$). For $x \lesssim wt$, this is correct in order of magnitude. We see that the value $T(x \sim wt) \equiv T_b$ is determined by the condition $l(T_b) \sim wt$. On the one hand, we have $l(T_b) \gtrsim wt$, while in the opposite case phonons with $\hbar\omega \sim T_b$ could not be at point x at time t. At the same time, if the condition $l(T_b) \gg wt$ held, then phonons with energies greater than T_b could be at point x, having arrived from the heated regions of the spot. Using (2), we then find

$$T_b \approx T_\alpha[n\ln(T_\alpha/T_b) + \ln(wt/l(T_\alpha))]^{-1}. \qquad (16)$$

Knowing the characteristic temperature which we can assign to the ballistic front, we find the following estimate of the energy flux:

$$I_b \approx \frac{C_0 T_0}{4}\left(\frac{T_b}{T_0}\right)^4 w. \qquad (17)$$

The quantity I_b depends on the time, but this dependence is weak, and it does not disrupt the quasisteady nature of solution (11).

4. THREE-DIMENSIONAL GEOMETRY (FIG. 1a)

Since we have $n_c = 8/3$ in this case, for realistic values of n we are in the region of runaway asymptotic behavior. Let us estimate the temperature T^* corresponding to $r \sim r_0$. We first assume that this temperature is so high that the condition $l_N(T^*) \lesssim l(T^*) \ll r_0$ holds. We can then use Eq. (11) for $r > r_0$; the solution of this equation is

$$(T^*/T)^{n-4} = 1 + (n-4) I_b [2\pi\varkappa(T^*) T^*]^{-1} (1/r_0 - 1/r), \quad n>4, \tag{18a}$$

$$\ln(T^*/T) = I_b [2\pi\varkappa(T^*) T^*]^{-1} (1/r_0 - 1/r), \quad n=4. \tag{18b}$$

For $n > 4$, the dependence $T(r)$ is a power law, and since r_0 is the only length scale characterizing the spatial distribution of the temperature we have $\nabla T \lesssim T^*/r_0$. Accordingly, the maximum value of I for $r \sim r_0$ is $2\pi r_0^2 \varkappa(T^*) T^*/r_0$. It is easy to see that in this case expressions (18) do not satisfy the requirement $T|_{r\to\infty} \to 0$. The assumption $l(T^*) \ll r_0$ thus leads to a contradiction. On the other hand, by analogy with the arguments in Sec. 3, we can conclude that we have $l(T^*) \lesssim r_0$. As a result we find the estimate $l(T^*) \sim r_0$, which we presented in the Introduction. This result means that even for $r \sim r_0$ there is a ballistic regime (or quasiballistic regime[11]). The typical temperature of the periphery of the spot, T^*, does not vary with time and is given by the estimate

$$T^* \sim T_0 [l(T_0)/r_0]^{1/n}. \tag{19}$$

We also have

$$I_b \sim {}^1\!/_2 \pi r_0^2 C(T^*) T^* w. \tag{20}$$

Knowing I_b, we find the lifetime of the spot:

$$t \sim 2E/\pi C(T^*) T^* w. \tag{21}$$

This value turns out to be lower than that which would be found from the estimate $x_s(t) \sim r_0$.

There is a slightly more complicated situation at $n = 4$, since in this case we have $I \propto \nabla(\ln T)$. In the region of the diffusion asymptotic behavior (4), we have $I(t) = \text{const}$ but $T(t) \neq \text{const}$. Using the characteristic value for the transition to the ballistic regime, r_b, from the condition $l(T_b) \sim r_b$, we find the following estimate for I_b: $I_b \sim 2\pi\varkappa(T) T r_b$. "Cutting off" the solution (18b) at $r \sim r_b$, joining it with the corresponding solution for the one-dimensional quasisteady asymptotic behavior, (11), which takes the form

$$\ln(T/T^*) \approx (r_0 - x) r_b / r_0^2, \quad T^* = T(x \sim r \sim r_0), \tag{22}$$

in the case $n = 4$, and, finally, joining (22) with the diffusion solution (4) for $x \sim r_r$, we find an equation for x_r:

$$8(\xi - 1) \sim \ln(t/t^*) + 3 \ln \xi, \quad \xi = r_0/x_r = r_b/r_0. \tag{23}$$

Here t^* is determined by the condition $T(t^*, r_0) = T_0^*$, $l(T_0^*) = r_0$. In turn we have

$$T_b \sim T_0^* \xi^{-1/4}, \quad I_b \sim 2\pi\varkappa T r_0 \xi. \tag{24}$$

We see that we have $\xi \sim 1$; i.e., for $x \sim r_0$, in the region of the transition from the one-dimensional regime to the three-dimensional regime, there is a sharp decay of the temperature, from values $T_r \sim T_0^* (t/t^*)^{1/4} \gg T_0^*$ to a value $T_b \sim T_0^*$.

For $n < 4$ we have $x_r \sim r_0$ (i.e., $T^* \propto t^{1/n}$) and $r_b \sim r_0 (T^*/T_0^*)^{n(4-n)/(2n-4)}$. This is not a really typical regime from the standpoint of the experimental curves of $\varkappa(T)$, but it does have an analogy in the case of a nonlocal heat conduction, as we will see.

5. TWO-DIMENSIONAL GEOMETRY (FIG. 1b)

In this case we have $n_c = 4$, and for $r > r_0$ runaway asymptotic behavior occurs for the typical scattering mechanisms. The physical picture, however, is noticeably different from that shown above. One of the important factors responsible for the differences is the diffuse scattering of phonons by the boundaries of the film. Because of this scattering, the mean free path is limited by the condition $l \lesssim d$. We introduce a temperature T_d such that we have $l(T_d) \sim d$. Clearly, for $T < T_d$ the propagation of the phonons is determined by surface scattering, and the solution is different in the regions $T > T_d$ (we denote its boundary by r_d) and $T < T_d$ (i.e., for $r > r_d$). Another important factor is heat removal across the surface. It is a complicated matter to take this factor into account, particularly because of the nonlinear behavior as a function of T (for example, if liquid He is the heat reservoir, heating leads to a change in its properties near the surface and to a sharp degradation of the heat removal[16]). The heat removal is more important for the parts of the film which have been heated relatively little (both because of the large values of the coefficient for the escape of phonons across the surface, k, and because of the larger area corresponding to these parts). In view of the discussion above, it appears possible to describe at least qualitatively the basic details of the physical picture on the basis of the following simple model. We assume that for $r < r_d$ the coefficient k does not depend on the phonon frequency and that for $r > r_d$ (i.e, for $T > T_d$) we have $k = 0$.

We first consider the region $r > r_d$. Here Eq. (1) reduces to the linear equation

$$\frac{dQ}{dt} = \frac{D_d}{r} \frac{d}{dr} r \frac{dQ}{dr} - \beta Q, \tag{25}$$

where $\beta \sim kw/d$. Equation (25) also holds if the concept of a temperature becomes meaningless, but in such a case Q should be understood as the total energy density of the phonon system. We also assume $t \gg \beta^{-1}$. In this case the situation is in a steady state, and the solution is

$$Q = K((\beta/D_d)^{1/2}/r), \tag{26}$$

where $K(z)$ is the modified Bessel function, with the properties

$$K(z)|_{z\to\infty} \sim (\pi/2z)^{1/2} e^{-z}, \quad K(z\to 0) \sim \ln(z/2).$$

For the total energy flux at $r \sim r_d$ we then have the interpolation estimate

$$I_d \sim 2\pi r_d \varkappa(T_d) \nabla T|_{r\sim r_d} \sim 2\pi\varkappa(T_d) T_d B^{-1},$$
$$B = \ln\left[e^{2/3} + \left(\frac{D_d}{\beta}\right)^{1/2} \frac{1}{r_d}\right]. \tag{27}$$

We now consider the region $r_d > r > r_0$, $T > T_d$. We assume $r_d \gg r_0$ (this assumption is, at any rate, valid for $r_0 \gg d$). From (11) and (27) we find

$$1 - (T_d/T)^{n-4} = (n-4) \ln(r_d/r) B^{-1}, \quad n>4,$$
$$\ln(T/T_d) \approx \ln(r_d/r) B^{-1}, \quad n=4. \tag{28}$$

Setting $r = r_0$ and $T = T^*[T^* = T(r \sim r_0)]$ in (28), we find a relationship between r_d and T^*:

$$\frac{r_d}{r_0} \sim \left[\exp\left(\frac{2(n-3)}{3(n-4)}\right) + \left(\frac{D_d}{\beta}\right)^{1/2} \frac{1}{r_0}\right]^{1/(n-3)},$$

$$B = \ln\left[e^{2/3} + \left(\frac{D_d}{\beta r_0^2}\right)^{(n-4)/2(n-3)}\right], \quad n>4,$$

$$\frac{r_d}{r_0} \sim \left(\frac{T^*}{T_d}\right)^{2/3} + \left(\frac{D_d}{\beta r_0^2}\right)^{(1-\zeta)/2}, \quad B = \ln\left[e^{2/3} + \left(\frac{D_d}{\beta r_0^2}\right)^{\zeta/2}\right], \tag{29}$$

$$\zeta = \frac{1}{\ln(T^*/T_d)}, \quad n=4$$

(these expressions are again interpolations). In turn, we find the value of T^* from the condition for joining with the asymptotic behavior corresponding to the one-dimensional situation at $x \sim r_0$ (for simplicity we set $r_0 \sim d$). For $n > 4$ we have

$$\frac{T^*}{T_0} = \left[\frac{r_0(I_d/r_0)}{\varkappa(T_0)T_0(n-4)} + 1\right]^{1/(n-4)} \sim \frac{T_d}{T_0} B^{1/(n-4)}. \tag{30}$$

At $n = 4$ we find, using $\varkappa T = \text{const}$,

$$I_d \sim 2\pi T^* \varkappa(T^*) B^{-1}. \tag{31}$$

This quantity, to within a logarithmic factor, is on the same order of magnitude as that corresponding to the diffusion solution at $x \sim r_0$. Consequently, it is at $x \sim r_0$ that we join with the diffusion solution; at this level of accuracy, we have

$$T^* \sim T_0 (D_0 t / r_0^2)^{1/4}. \tag{32}$$

It can be concluded from this analysis that for $n > 4$ the temperature decay for $r_0 < r < r_d$ is slow:

$$T \sim T^*[1-(n-3)\ln(r/r_0)B^{-1}]^{1/(n-4)}. \tag{33}$$

The temperature of the edge of the spot, $T^* = T(r_0)$, is comparatively low and does not vary with the time. At $n = 4$, on the other hand, the temperature decay for $r > r_0$ is quite sharp:

$$T \sim T^*(r_0/r)^{1/B}; \tag{34}$$

as for $d = 3$, the temperature of the edge of the spot, T^*, increases with the time, in proportion to $t^{1/4}$.

6. PROPERTIES OF THE ASYMPTOTIC BEHAVIOR IN THE CASE OF NONLOCAL HEAT CONDUCTION

Since we have $l_i(\hbar\omega \sim T) \propto T^4$, $l_N(\hbar\omega \sim T) \propto T^5$ even when ordinary heat conduction takes place in the hottest part of the spot, the condition $l_N > l_i$ may hold in the cold part of the "atmosphere" of the spot, and there may be a transition to nonlocal heat conduction.[4–6] The heat transfer is implemented by subthermal phonons ($\hbar\omega < T$) which arrive at a given point from hotter regions. It can be shown (see the Appendix), however, that if the temperature falls off sufficiently rapidly with distance the heat flux at point x, with temperature T, is determined primarily by the value of T and depends only weakly on the temperature distribution at $x' \ll x$ (we will discuss the case $d = 1$ first). We can thus use considerations analogous to those of Refs. 4–6 in order to evaluate the temperature distribution. We know[5,17] that in the case $l_N > l_i$ three regimes are possible: 1) $x/w\tau_N(T) \ll \delta^{1/3}$, $\delta \equiv \tau_N/\tau_i$, corresponding to definitely nonlocal heat conduction; 2) $\delta^{1/3} < x/w\tau_N < \delta$; 3) $x/w\tau_N > \delta > 1$. We restrict the present discussion to an analysis of situation 1). In this case the typical frequencies ω_{ph} of the phonons which implement the heat transfer at the point x are determined by the condition

$$x^2 \sim D(\omega_{ph})\tau_N(\omega_{ph}, T) \tag{35}$$

[where $\tau_N^{-1}(\omega, T) \propto \omega T^4$], and the time scale of the heat transfer is determined by the time scale for spectral pumping of energy from frequencies $\omega \sim \omega_{ph}$ to $\hbar\omega \sim T$:

$$t \sim \tau_N(\hbar\omega \sim T)(T/\hbar\omega_{ph})^4. \tag{36}$$

Solving Eqs. (35) and (36) for T, we find

$$T = T_3 \left(\frac{D(T_3)t}{x^2}\right)^{4/11} \left(\frac{1}{\tau_N(T_3)}\right)^{1/11}, \tag{37}$$

where T_3 is some normalization temperature.

We turn now to the question of the transition to three-dimensional geometry. For $r > r_0$ the heat transfer is implemented by phonons with frequencies ϖ such that $D(\varpi)\tau_N(\varpi, T) \approx r^2$, and the energy flux is

$$I \sim 2\pi r^2 C(\hbar\varpi)\hbar\varpi D(\varpi)\nabla T/\hbar\varpi \sim r^{12/5} T^{4/5} \nabla T.$$

Analysis of conditions of the form (8a) and (9a) shows that we are dealing with runaway asymptotic behavior. By analogy with Sec. 2, we find $I \approx \text{const}$. Hence

$$T \approx T_D (r_0/r)^{7/9} (I/I_D)^{5/9}, \tag{38}$$

where $D(T_D)\tau_N(T_D) \sim r_0^2$, $I_D = \varkappa(T)T \cdot 2\pi r_0$. On the other hand, for solution (37) we have $I(x \sim r_0) \propto T^{9/5}(r_0)$; i.e., in the case of nonlocal heat conduction the energy flux increases with increasing T, as in the case of local heat conduction with $n < 4$. This result means that solutions (38) and (37) join at $r \sim r_0$. We thus find

$$T^* \equiv T(r_0) \propto t^{5/11}, \quad I = I_D(T^*/T_D)^{9/5}.$$

It follows from estimates in Ref. 17 that in regime 2) we have $I \propto T$, and this energy flux is qualitatively the same as in the case above, while in regime 3) we have $I \propto T$, and the energy flux is similar to that in the case $n > 4$ (Secs. 2 and 4). It is easy to see that as the temperature at point x rises there is a sequence of transitions from regime 1) to regimes 2) and 3) (and then to local heat conduction with $n = 4$). If we instead look at the edge of the spot, at $x \sim r_0$, we conclude by analogy with the discussion above (Sec. 4) that $T(r_0)$ initially increases [regimes 1) and 2)] and then, at the transition to regime 3) becomes approximately constant: $T(r_0) \approx \text{const}$.

7. SPECTRUM OF OCCUPATION NUMBERS OF BALLISTIC PHONONS EMITTED BY A SPOT

Let us find the occupation numbers N_ω of low-frequency phonons which are emitted by a spot and which then propagate ballistically. Here it is sufficient to consider three-phonon interaction processes involving thermal phonons. The collision operator can be described in the relaxation-time approximation. We thus write the kinetic equation in the form

$$w\nabla N + (N - N_0(T))/\tau_N(\omega, T) = 0, \tag{39}$$

where $N_0 \sim T/\omega$ is the equilibrium distribution function. For simplicity we restrict the analysis to the case $T_m < T_0$. The expression for $\tau_N(\omega, T)$ is[7,14]

$$\tau_N^{-1}(\omega, T) = \tau_0^{-1}(\hbar\omega/T_0)^\alpha (T/T_0)^{5-\alpha}. \quad (40)$$

In this section of the paper, where we are assuming a given temperature distribution, we will not restrict the discussion to the single-branch model. We allow α to deviate from unity (in particular, for longitudinal phonons in crystals of cubic symmetry we would have $\alpha = 2$, Refs. 7 and 14). We assume that the time required for the phonons to propagate to the observation point is much shorter than the spot evolution time. We can thus regard the temperature distribution in (39) as quasisteady. The solution of (39) is expressed in terms of an integral along a trajectory:

$$N_\omega(x) = \int_0^x \frac{dx'}{w\tau_N(\omega, T)} N_0(\omega, T) \exp\left(-\int_{x'}^x \frac{dx''}{w\tau_N(\omega, T)}\right). \quad (41)$$

Here $T = T(x')$, and for simplicity we have written (41) for the case $d = 1$. In the $d = 2$ cases, we should understand x' as a trajectory variable along the corresponding trajectory directed along the wave vector **q**, and we should understand x as the coordinates of the observation point. We then need to sum solution (41) over **q**, i.e., over the trajectories. If the region $r \lesssim r_0$ is dominant, however, and if the condition $x \gg r_0$ holds, this summation reduces to multiplication of (41) by a coefficient $V(\Omega)$ determined by the solid angle Ω which the spot subtends at the point x. In the case $x \gg r_0$ and $d = 3$, we obviously have $V(\Omega) \propto x^{-2}$; in the case $d = 2$ we have $V(\Omega) \propto x^{-1}$.

Clearly, expression (41) depends strongly on the integral in the argument of the exponential function. The latter cuts off the range of the integation over x', which corresponds to

$$\int_{x_1}^x \frac{dx''}{w\tau_N(\omega, T)} \sim 1. \quad (42)$$

We will first discuss the situation in which (42) cannot be satisfied regardless of the value of x_1, i.e., the situation in which we have

$$\left(\frac{\hbar\omega}{T_0}\right)^\alpha \int_0^x \frac{dx'}{w\tau_0}\left(\frac{T}{T_0}\right)^{5-\alpha} < 1. \quad (43)$$

This condition obviously holds for phonons with frequencies below a certain $\omega_{\lim}$, at which the left side of (43) is equal to unity. Clearly, in a given situation, for realistic values of α and n, the integral in (41) is determined by the hottest part of the spot, where we have $T \sim T_m$; i.e., here we have

$$N_\omega \sim V(\Omega)\left(\frac{\hbar\omega}{T_0}\right)^{\alpha-1}\left(\frac{T_m}{T_0}\right)^{6-\alpha}$$

$$\times\left(\frac{x_s}{w\tau_0}\right) \propto T_m^{2-\alpha} \propto t^{-(2-\alpha)/(8-n)}. \quad (44)$$

According to (44), in the case $\omega < \omega_{\lim}$ we have $N(\omega)/N_0(\omega)|_{\omega\to 0} \to 0$; i.e., the low-frequency region is depleted in comparison with a Planckian distribution determined by the typical energies of nonequilibrium phonons at point x. Such depletion of the low-frequency part of the phonon spectrum has been observed in several experiments.[12,13]

At $\omega > \omega_{\lim}$, three situations are possible: a) $x_1 < x_s$; b) $x_1 > x_s$, $T(x_1) \gg T(x)$; c) $x_1 > x_s$, $T(x_1) \sim T(x)$. [Since we have $t_N(\omega, T) > \tau_N(\hbar\omega \sim T)$, condition c) can hold only if the concept of a temperature at point x is meaningful.]

We will evaluate the integral in (41) in order of magnitude, restricting the integration over x' to a lower value of x_1 and replacing the exponential function by unity. In case a), as before, the integral is determined by the region $x' < x_s$, and we have

$$N_\omega \sim V(\Omega) N_0(\omega, T = T_m). \quad (45)$$

In case b), because of the strong dependence of τ_N^{-1} on x for realistic values of α and n, the integral in (41) is determined primarily by the neighborhood of the point x_1. In the $d = 1$ case, this is true regardless of whether x_1 falls in the region of a diffusion or runaway asymptotic behavior (the case analyzed in Sec. 3 is exceptional). We thus write

$$N_\omega \sim V(\Omega) N_0(\omega, T = T(x_1)), \quad T[w\tau_N(T)\nabla T]^{-1}|_{x=x_1} \sim 1. \quad (46)$$

The evolution of the observed signal is thus determined by the evolution of the value $t(x_1)$; the latter increases as the decay of $T(x)$ becomes sharper. If the point x_1 corresponds to the region $d = 1$ $(x_1 \ll r_0)$ and to the region of diffusive asymptotic behavior, then we have, as can be seen easily from (4) and (42),

$$T(x_1) \propto t^{-1/(10-2\alpha-n)}. \quad (47)$$

For realistic values of α, and n, the value of N_ω falls off with increasing t up to the time at which the point x_1 falls in the region of quasisteady asymptotic behavior.

With $d = 3$ and $n > 4$ we have $T^* = \text{const}$; at $x_1 > x_r$ there is no change in N_ω before the spot disappears. In the $n = 4$ case, the region of a quasisteady asymptotic behavior is comparatively narrow and the temperature decay in it is sharp. If in this case we have

$$\int_{x_r}^x \frac{dx'}{w\tau_N} < 1, \quad \text{but} \quad [w\tau_N(T(x_r))]^{-1} x_r > 1, \quad (48)$$

then we have $x_1 \sim x_r$ and

$$N_\omega \sim N_0(T(x_r)). \quad (49)$$

Since $T(x_r)$ increases with the time, growth $N_\omega(t) \propto t^{1/4}$ occurs under the conditions (48). According to (23) and (24), with $d = 3$ and $n = 4$ in the region $x > x_r$ we have

$$d(\ln T)/dr \propto I_b \propto \xi \sim 1 + (1/8)\ln(t/t^*),$$

so, according to (46), at $x_r < x_1 < r_0$ we have

$$N_\omega \propto \xi^{1/(5-\alpha)}.$$

If, on the other hand, we have $x_1 > r_0$, i.e., if we are dealing with a spherical surface, then the growth in the area of this surface with increasing t comes into play (gives rise to an increase in Ω), and we have

$$N_\omega \propto \xi^2.$$

Correspondingly, in the case of nonlocal heat conduction we find, using (37) and (38),

$$N_\omega \propto t^{-5/[8(5-\alpha)-11]} \quad (x_1<r_0), \quad N_\omega \propto t^{45(10-2\alpha)/11(45-7\alpha)} \quad (x_1>r_0).$$

One distinctive feature of the $d=2$ case is that the ballistic phonons correspond to a small phase volume, $\sim d/r$. For $r \gg r_0$, $r>(dw/\beta)^{1/2}$, however, the phonons which are incident on the surface move off comparatively easily into the reservoir. An exceptional case is constituted by the small group of phonons which are incident on the surface at angles $\pi/2-\varphi(\varphi \to 0)$. Since the specular reflection coefficient p increases as $\varphi \to 0$, propagation of these phonons is approximately ballistic. If $(1-p) \propto \varphi^a$, the critical value of φ for this group is $\varphi_{cr} \sim (d/r)^{1/(a+1)}$. We now see that the relative number of phonons which arrive at the observation point diffusively falls off with increasing r, and an estimate of the type in (46) remains valid. For $n>4(T^* \sim T_d = \text{const})$, as at $d=3$, the value of N_ω for the phonons of fairly high frequencies $[x_1(\omega)>x_r]$ does not change up to the time that the spot disappears. At $n=4$, however, we have $T^* \propto t^{1/4}$. If the heat removal is sufficiently rapid $(D_d/\beta r_0^2 \ll (T^*/T_d)^{4/3}, B=2/3)$, the temperature decay for $r>r_0$ is extremely sharp ($\propto r^{-3/2}$). As a result, it becomes possible to satisfy conditions (48); for the corresponding frequencies we have $N_\omega \propto t^{1/4}$.

In case c), which may occur in a one-dimensional geometry or at $r \sim r_0$, during the lifetime of the spot there is a growth of N_ω due to the temperature rise near the observation point. If the relation $x_1 \ll x$ initially holds, then $N_\omega(t)$ initially decreases and then (at $x_1 \sim x$) begins to increase.

CONCLUSION

This analysis shows that, despite the comparative simplicity of both the experimental situation and the mathematical formulation of the problem, the physical picture associated with a hot spot is extremely rich. The space-time distribution of the temperature is determined by both the surface density of injected energy, E, and the experimental geometry. The general nature of this distribution is shown in Fig. 2. The region $0<x<x_r$ is described by Eqs. (4)–(6). The properties of the region $x \gtrsim x_r$ are listed in Table I for the three- and two-dimensional cases (Fig. 1).

With regard to the frequency spectrum of the filling numbers N_ω of the nonequilibrium phonons emitted by a spot, we find that most of the energy in the $N=3$ case is concentrated at frequencies $\hbar\omega \sim T_b$ [T_b is given by (24)], while at $d=2$, we have $\hbar\omega \sim T_d$ (T_d was introduced in Sec. 5). The shape of the spectrum, however, is quite different from a Planckian spectrum; in particular, the low-frequency region [corresponding to $l_N(\omega, T=T_m)>x_s$] is depleted. As was mentioned earlier, this situation agrees with experimental results.[12,13]

The time evolution of N_ω during the lifetime of a spot $[x_s(t)<x_r(t)]$ according to the results in Sec. 7 is also distinguished by a diversity of forms. The basic conclusions are presented in Table II. We see that a typical result is a decay of N_ω with increasing t, which reflects a gradual cooling of the spot. At $n=4$ in the two- and three-dimensional cases, however, in a certain region of values of ω and t there can be a growth of $N_\omega(t)$ as a result of the increase in the temperature of that region of the spot which serves as a source of phonons of the corresponding frequencies, if the outer part of the atmosphere is sufficiently narrow and does not prevent the escape of phonons. Growth of $N_\omega(t)$ also occurs if the region in which the phonons are produced corresponds to a three-dimensional geometry, since the area of the emitting surface increases over time in this case (the atmosphere "swells"). These events can occur, in particular, in the case of a nonlocal heat conduction. Finally, $N_\omega(t)$ may grow if the observation point itself is in a fairly hot part of the atmosphere (e.g., in the one-dimensional geometry). Furthermore, $N_\omega(t)$ may have a local minimum, if this growth is prevented by a decay region. Finally, after the condition $x_s(t) \sim x_r(t)$ is attained, the spot decays.

Akimov *et al.*[8] have recently carried out an experimental study of the hot-spot situation in a thin wafer of pure GaAs. In the region $r>r_0$, a nonmonotonic dependence $N_\omega(t)$ was observed, with a maximum for a group of ballistic

TABLE I. Characteristics of the temperature distributon in the region of the runaway (quasi-steady) asymptotic behavior for $d=2$ and 3.

		$n>4$	$n=4$
$d=3$	$x_r<x<r_0$	$x_r=r_0(t/t^*)^{(4-n)/(8-n)}$ $T_r \approx T_0^*(t/t^*)^{1/(8-n)}$ $T(x) \sim T_r(x_r/x)^{1/(n-4)}$ $T^*=T(r_0) \sim T_0^*$	$T_r \sim T_0^*(t/t^*)^{1/4}$ $x_r \sim r_0[1+{}^1/_8 \ln(t/t^*)]^{-1} \sim r_0$ $T \propto e^{-x/x_r}$
	$r>r_0$	Concept of a temperature becomes meaningless (ballistic regime)	Concept of a temperature becomes meaningless at $r>r_b$ $\sim r_0[1+1/8\ln(t/t^*)] \sim r_0$
$d=2$, $\beta \to \infty$	$x_r<x<r_0$	Similar to $d=3$	$x_r \sim r_0$, $T_r \sim T_0^*(t/t^*)^{1/4}$
	$r>r_0$	Concept of a temperature becomes meaningless for $r>r_d$	$T(r) \sim T_r(r_0/r)^{3/2}$

Note. Here t^* and T_0^* are determined by the conditions $l(T_0^*) \sim \min(r_0, r_0^2/l_N(T_0^*))$, $t^*=\max(r_0/w, \tau_N(T_0^*))$. In the $d=2$ case, we assume rapid heat removal: $D_d/\beta r_0^2 \lesssim 1$. In the case of a nonlocal heat conduction [regime 1, $r_0/w\tau_N<(\tau_N/\tau_i)^{1/3}$] $x_r \sim r_0$. For $x<r_0$ we have $T \approx T_3[D(T_3)t/x^2]^{4/11}(t/\tau_N(T_3))^{1/11}$; for $r>r_0$ we have $T \sim T_D(r_0/r)^{7/9}(t/\tau_N(T_D))^{5/11}$ where $D(T_D)\tau_N(T_D) \sim r_0^2$.

TABLE II. Time dependence $N_\omega(t)$ in various stages of the existence of a spot $[x_s(t) < x_r(t)]$ at $r \gg r_0$ for various parameters of the problem.

$\frac{x_s}{w\tau_N(T_m)} \ll 1$	$\frac{x_s}{w\tau_N(T_m)} \gg 1$, $\frac{x_r}{w\tau_N(T_r)} \ll 1$	$x_r/w\tau_N(\omega, T_r) \gtrsim 1$				
		$n > 4$	$n = 4,\ d = 3$			$n = 4,\ d = 2,\ \beta \to \infty$
			$\frac{x_r}{w\tau_N(T_r)} \sim 1$	$\frac{x_r}{w\tau_N(T_r)} \gg 1$		
				$w\tau_N(T^*) \gg r_0$	$w\tau_N(T^*) \ll r_0$	
$t^{-1/(8-n)}$	$t^{-1/(10-2\alpha-n)}$	const	$t^{1/4}$	$\xi^{1/(5-\alpha)}$	$\sim \xi^2$	$t^{1/4}$

Note. $\xi \sim 1 + 1/8 \ln(t/t^*)$. If $r \sim r_0$ ($d = 2, 3$ and $r_0/w\tau_N(\omega, T^*) \gtrsim 1$, then $N_\omega \propto t^{1/n}$; under conditions of a nonlocal heat conduction we would have $N_\omega \propto t^{5/[11-8(5-\alpha)]}$, if $w\tau_N(\omega, T^*) > r_0$ or $N_\omega \propto t^{45(10-2\alpha)/11(45-7\alpha)}$, if $w\tau_N(\omega, T^*) < r_0$.

phonons with energies of 0.3 and 0.6 THz. The position of the maximum did not depend on r (ruling out a diffusion nature for the corresponding effect). This behavior could be explained on the basis of the model proposed above, under the assumption that the growth in $N_\omega(t)$ stems from the factors listed above and that the position of the maximum corresponds to the time at which the spot begins to decay. Order-of-magnitude estimates of the spot lifetime and also of the temperature of the edge of a spot, $T^*(r \sim r_0)$, based on the data of Ref. 8 agree with this assumption. In the one-dimensional case (in which the phonon detector is positioned opposite the excitation region), a local minimum was observed at early times. The presence of such a minimum can also be explained by our model, as we have already seen.

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APPENDIX

Let us find which regions of the spot are primarily responsible for the heat transfer to a given point x. We assume a quasisteady temperature distribution describable by a power law $T \propto x^{-m}$. If phonons with an energy $\hbar\omega$ arrive at the point x primarily from a region (x_1, x), the meaning is that as these phonons diffuse in the region (x_1, x) they are scattered at a probability ~ 1 by thermal phonons. Hence

$$\frac{x}{D(\omega)} \int_{x_1}^{x} \frac{dx'}{\tau_N(\omega, T(x'))} \sim 1. \tag{A1}$$

Since we have $\tau_N = \tau_N(\omega)$, the value of x_1 and thus that of the temperature $T_1 = T(x_1)$ depend on ω. Phonons with relatively low frequencies may arrive from hotter regions. From (A1) we find an estimate of the typical frequency of the phonons which can arrive at point x after being produced near point x_1 with temperature T_1:

$$\omega(T_1) \leqslant \omega_{\rm ph}(T/T_1)^{(5-\alpha-1/m)/5}, \tag{A2}$$

where $\omega_{\rm ph}$ is given by (35). Since the occupation numbers for these phonons satisfy $\sim T_1/\omega(T_1)$, the component of the heat flux at point x is proportional to $\omega^2(T_1)T_1$. The total heat flux at point x, on the other hand, is proportional to

$$\int^{\omega(T)} d\omega\,\omega(T_1)T_1 \sim \int_T^{T_m} dT_1 \omega^2(T_1) T_1 \frac{d\omega}{dT_1} \sim \int_T^{T_m} dT_1 \omega^3(T_1). \tag{A3}$$

Using (A3), we easily see that if

$$m > 3/(10-3\alpha) \tag{A4}$$

then the integral is determined by its lower limit [in the single-mode model which we are considering here, we would have $\alpha = 1$, and (A4) would take the form $m > 3/7$]. This result means, in particular, that under condition (A4) the heat flux at point x is determined in order of magnitude by the temperature in the hotter regions. It is easy to see that for law (37) condition (A4) holds, i.e., that our calculation is self-consistent.

[1)]For many semiconductors, the temperature dependence $\varkappa(T)$ is sharper than T^{-1} at $T \lesssim \Theta$. As we will see below, this circumstance may prove important.

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