Chiral photon current and its anomaly in a gravitational field

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The concept of chirality, conserved in interactions with the gravitational field, is formulated for the electromagnetic field. The corresponding current turns out to be the one-particle analog of the Pauli-Lubanski vector. We determine the anomaly in the divergence of this current in an external gravitational field. This result is used to determine electromagnetic radiative corrections to the fermion chiral anomaly in a gravitational field.

1. INTRODUCTION

At this time the concept of chirality for massless fermions and bosons is on a quite different footing. For the massless fermion field, in interaction with electromagnetic and gravitational fields, there exists a well-defined $U(1)$ symmetry with respect to chiral rotations. A Noether axial-vector current a_μ can be associated with this symmetry and classically this current is conserved. Once the well-known triangle anomalies[1–5] are taken into account the expression for the divergence of this current becomes

$$\nabla_\mu a^\mu = \frac{Q^2\alpha}{2\pi} F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{192\pi^2} R_{\mu\nu\varkappa\lambda}\tilde{R}^{\mu\nu\varkappa\lambda}, \qquad (1)$$

where $a^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$, ψ is the massless Dirac field with electric charge Q, ∇_μ is the covariant derivative

$$\nabla_\mu a^\mu = \frac{1}{(-g)^{1/2}} \partial_\mu[(-g)^{1/2} a^\mu],$$

$F_{\mu\nu}$ is the electromagnetic field intensity tensor, $R_{\mu\nu\varkappa\lambda}$ is the Riemann tensor,

$$\tilde{F}^{\mu\nu} = \frac{1}{2(-g)^{1/2}} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \qquad \tilde{R}^{\mu\nu\varkappa\lambda} = \frac{1}{2(-g)^{1/2}} \varepsilon^{\mu\nu\rho\sigma} R_{\rho\sigma}{}^{\varkappa\lambda}.$$

In this paper we study the electromagnetic field in an external gravitational field with the aim of extending the chirality concept to the case of bosons. As is known, the free Maxwell equations, as well as those in a gravitational field, are invariant under the duality transformation

$$F_{\mu\nu}' = \cos\alpha F_{\mu\nu} + \sin\alpha \tilde{F}_{\mu\nu}. \qquad (2)$$

However, this transformation is not formulated in terms of the vector potential A_μ and, consequently, it is not possible to write down immediately the corresponding axial current.

To overcome this difficulty we make use of the light-cone formalism. In this formalism the photon is described by a complex field A, and the action is linear in A and A^*. Following this the introduction of conserved chirality is fully analogous to the fermion case.

The noncovariance of the Lagrangian density in this formalism results in noncovariance of the conserved axial current of the photons. The Lorentz-covariant current, which gives rise to the same generator, has the form

$$K^\mu = -\frac{1}{(-g)^{1/2}} \varepsilon^{\mu\nu\varkappa\lambda} A_\nu \partial_\varkappa A_\lambda. \qquad (3)$$

Although the divergence of this current is, evidently, nonzero, $\nabla_\mu K^\mu = -F_{\mu\nu}\tilde{F}^{\mu\nu}/2$, chirality conservation leads to the "naive" vanishing of the expectation value of $\nabla_\mu K^\mu$ in an external gravitational field. In this case, too, the triangle diagrams turn out to be anomalous and the result for the expectation value $\langle\nabla_\mu K^\mu\rangle$ has the form

$$\langle\nabla_\mu K^\mu\rangle = -\frac{1}{2}\langle F_{\mu\nu}\tilde{F}^{\mu\nu}\rangle = -\frac{1}{96\pi^2} R_{\mu\nu\varkappa\lambda}\tilde{R}^{\mu\nu\varkappa\lambda}. \qquad (4)$$

The existence of the anomaly (4) was established in our paper, Ref. 6, published in the form of a letter to the editor. In a different form the chiral anomaly was obtained independently by Endo and Takao.[6]

The appearance of such an anomaly is natural. Indeed, the term proportional to $R\tilde{R}$ in the fermion anomaly (1) is due to the interaction of the spin with the gravitational field, and this interaction is universal and takes place for bosons as well as for fermions.

The relation (4) constitutes the one-loop anomaly. On the other hand, as is shown in Sec. 5, it permits the calculation of the two-loop order α correction to the chiral anomaly of a fermion in a gravitational field. The answer is obtained by averaging the relation (1) and substituting for $\langle F_{\mu\nu}\tilde{F}^{\mu\nu}\rangle$ from expression (4).

This article is organized as follows. In Sec. 2 we introduce the chirality concept for the photon field in the light-cone formalism and verify the conservation of this chirality for a photon interacting with gravity. In Sec. 3 we discuss the expression for the chiral current, and in Sec. 4 we calculate the anomaly in the divergence of this current. In the short fifth section we find the electromagnetic correction to the chiral fermion anomaly in a gravitational field. In conclusion (Sec. 6) we discuss certain unsolved problems, as well as previous papers on bosonic chiral anomalies.

2. CHIRALITY OF THE PHOTON FIELD

We begin the discussion with the case of a free photon field. Since the free Maxwell equations have the form

$$\partial_\mu F_{\mu\nu} = 0, \quad \partial_\mu \tilde{F}_{\mu\nu} = 0, \qquad (5)$$

the symmetry with respect to the duality transformations (2) is obvious.

The fields of definite duality are the combinations

$$F_{\mu\nu}^{\pm} = F_{\mu\nu} \mp i\tilde{F}_{\mu\nu}, \qquad (6)$$

which may be viewed as fields with positive and negative chirality.

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The problem lies in the circumstance that the transformation (2) is not formulated as a transformation of the vector potential A_μ which is the object of quantization. More than that, although relation (2) viewed as an equation expressing A'_μ in terms of A_μ, has solutions for A_μ satisfying Maxwell's equations (i.e., on the mass shell), for arbitrary A_μ solutions don't exist.

We would like to introduce chirality in the form of a $U(1)$ invariance of the action as a functional of A_μ, i.e., for arbitrary A_μ. This turns out to be possible in the light-cone formalism. As we shall see, in this formalism chiral transformations of the vector potential lead to the duality transformation (2) for the field intensity only on the mass shell.

The light-cone gauge has been discussed in the literature in great detail. We shall present here the needed formulas, using spinor notation. Each vector index is replaced by a pair of spinor indices with the help of the relation

$$b_{\alpha\dot{\alpha}}=(\sigma^\mu)_{\alpha\dot{\alpha}}b_\mu,\quad \sigma^\mu=(1,\boldsymbol{\sigma}).$$

The indices are raised and lowered with the help of the tensors $\varepsilon_{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$. In particular, the coordinate four-vector x^μ is replaced by $x^{\alpha\dot{\alpha}}$. As the "new" time we choose

$$x^{1\dot{1}}/2^{1/2}=(x^0-x^3)/2^{1/2}.$$

The equation for A_μ

$$\partial_\nu\partial_\nu A_\mu-\partial_\mu\partial_\nu A_\nu=0, \tag{7}$$

becomes in terms of spinor indices

$$\partial_{\beta\dot{\beta}}\partial^{\beta\dot{\beta}}A_{\alpha\dot{\alpha}}-\partial_{\alpha\dot{\alpha}}\partial^{\beta\dot{\beta}}A_{\beta\dot{\beta}}=0. \tag{8}$$

It is not hard to see that in these equations the time derivatives of the fields $A_{1\dot{1}}$ and $\partial_{1\dot{1}}A_{1\dot{1}}$ are absent, so that in the canonical quantization formalism the field $A_{1\dot{1}}$ plays the role of a Lagrange multiplier and is expressible in terms of other fields.

The gauge condition is chosen in the form

$$A_{2\dot{2}}=0. \tag{9}$$

Thus only the fields $A_{1\dot{2}}$ and $A_{2\dot{1}}$ remain. The $2\dot{2}$ component of Eq. (8) gives

$$\partial_{2\dot{2}}\partial^{\beta\dot{\beta}}A_{\beta\dot{\beta}}=\partial_{2\dot{2}}(\partial_{2\dot{2}}A_{1\dot{1}}-\partial_{2\dot{1}}A_{1\dot{2}}-\partial_{1\dot{2}}A_{2\dot{1}})=0. \tag{10}$$

On the assumption that the inverse operator $\partial_{2\dot{2}}^{-1}$ exists we obtain

$$\partial^{\beta\dot{\beta}}A_{\beta\dot{\beta}}=0,\quad A_{1\dot{1}}=\partial_{2\dot{2}}^{-1}(\partial_{2\dot{1}}A_{1\dot{2}}+\partial_{1\dot{2}}A_{2\dot{1}}). \tag{11}$$

The equations for A_{12}, $A_{2\dot{1}}$ take the form

$$\Box A_{1\dot{2}}=0,\quad \Box A_{2\dot{1}}=0,\quad \Box=\partial_\mu\partial_\mu=\partial_{\beta\dot{\beta}}\partial^{\beta\dot{\beta}}/2, \tag{12}$$

and the action S has the following form:

$$S=-{}^1/_4\int d^4x F_{\mu\nu}F_{\mu\nu}$$
$$={}^1/_2\int d^4x\,[A_\nu\Box A_\nu+(\partial_\nu A_\nu)^2]={}^1/_2\int d^4x A_{1\dot{2}}\Box A_{2\dot{1}}. \tag{13}$$

Introducing the notation

$$A_{2\dot{1}}=A\,(2)^{1/2}, \tag{14}$$

we obtain

$$S=\int d^4x\,\bar{A}\Box A, \tag{15}$$

where the bar denotes complex conjugation. This expression exhibits in an obvious manner the chiral $U(1)$ invariance of the theory with respect to the global transformations

$$A\to Ae^{i\varphi},\quad \bar{A}\to\bar{A}e^{-i\varphi}. \tag{16}$$

We present in this gauge the expressions for the intensities $F_{\mu\nu}$ in terms of A and $\bar{A}$

$$\begin{gathered}F_{\alpha\dot{\alpha},\beta\dot{\beta}}=(\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}F_{\mu\nu},\\ F_{\alpha\dot{\alpha},\beta\dot{\beta}}^{+}=\varepsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta},\ F^{-}_{\alpha\dot{\alpha},\beta\dot{\beta}}=\varepsilon_{\alpha\beta}\bar{f}_{\dot{\alpha}\dot{\beta}},\\ f_{\alpha\beta}=(\partial_\alpha{}^{\dot{\alpha}}A_{\beta\dot{\alpha}}+\partial_\beta{}^{\dot{\alpha}}A_{\alpha\dot{\alpha}})/2,\\ f_{11}={}^1/_2\partial_\zeta^{-1}\partial^2A-\partial_\zeta^{-1}\Box\bar{A},\\ f_{22}=\partial_\zeta A/2\\ f_{12}=f_{21}=\bar{\partial}A/2.\end{gathered} \tag{17}$$

Here we used the notation

$$\begin{gathered}\partial=2^{1/2}\partial_{2\dot{1}},\quad \bar{\partial}=2^{1/2}\partial_{1\dot{2}},\quad \partial_\zeta=2^{1/2}\partial_{2\dot{2}},\quad \partial_\tau=2^{1/2}\partial_{1\dot{1}},\\ \Box=\partial_\mu\partial_\mu=(\partial_\zeta\partial_\tau-\partial\bar{\partial})/2.\end{gathered} \tag{18}$$

The expressions for $\bar{f}_{\dot{\alpha}\dot{\beta}}$ are obtained from (17) by complex conjugation. It is seen from the relations (17) that the correspondence between chiral rotations of the field A and the intensities $F_{\mu\nu}$ holds only for fields A obeying the equations of motion.

Of course, the introduction of chirality for the free photon field carried out above is a rather banal exercise. More complicated from the technical point of view is its generalization to interacting fields. A nontrivial example of conservation of the above introduced chirality is provided by the interaction of photons with the gravitational field.

To begin with we consider the action

$$S=-{}^1/_4\int d^4x(-g)^{1/2}g^{\mu\varkappa}g^{\nu\lambda}F_{\mu\nu}F_{\varkappa\lambda}. \tag{19}$$

The symmetric matrix g will be represented in the form $g=e^H$, where H is also a symmetric real matrix. Since $(-g)^{1/2}=\exp(-\mathrm{Tr}\,H/2)$, it follows that the action (19) contains only the traceless part of the matrix H, $h=H-\mathrm{Tr}\,H/4$ (this corresponds to conformal invariance of the action)

$$S={}^1/_4\int d^4x\,\mathrm{Tr}(e^hFe^hF). \tag{20}$$

All contractions in (20) proceed via the flat matrix $\eta_{\mu\nu}=(1,-1,-1,-1)$. We make use of perturbation theory and expand S in a series in h

$$S={}^1/_4\int d^4x\,\mathrm{Tr}(F^2+2hFF+h^2F^2+hFhF+\ldots). \tag{21}$$

By going over to intensities with definite duality and using the spinor notation we obtain for terms of zeroth, first, and second order in h the following expressions:

$$S^{(0)} = -{}^1/_8 \int d^4x \,(f^{\alpha\beta} f_{\alpha\beta} + \bar{f}^{\dot\alpha\dot\beta} \bar{f}_{\dot\alpha\dot\beta}),$$

$$S^{(1)} = {}^1/_8 \int d^4x h^{\alpha\dot\alpha,\,\beta\dot\beta} f_{\alpha\beta} \bar{f}_{\dot\alpha\dot\beta},$$

$$S^{(2)} = -\int d^4x \,\{{}^1/_{48}(\mathrm{Tr}h^2)(f_{\alpha\beta} f^{\alpha\beta} + \bar{f}_{\dot\alpha\dot\beta}\bar{f}^{\dot\alpha\dot\beta}) + {}^1/_{64} h^{\alpha\dot\alpha,\,\beta\dot\beta} h^{\gamma}{}_{\dot\beta,}{}^{\delta}{}_{\dot\alpha} f_{\{\beta\gamma} f_{\delta\alpha\}} + {}^1/_{64} h^{\alpha\dot\alpha,\,\beta\dot\beta} h_{\beta}{}^{\dot\gamma},{}_{\alpha}{}^{\dot\delta} \bar{f}_{\{\dot\beta\dot\gamma} \bar{f}_{\dot\delta\dot\alpha\}}\}. \tag{22}$$

The expression for $S^{(0)}$ appears as if it didn't conserve chirality. However on the mass shell $S^{(0)}$ vanishes, while off-shell it conserves chirality in terms of A and $\bar{A}$, as can be seen from the expression (15).

To first order in h one may substitute into $S^{(1)}$ expressions of the form (17) for $f_{\alpha\beta}, \bar{f}_{\dot\alpha\beta}$, and within the accuracy considered terms proportional to $\Box A$, $\Box\bar{A}$ may be ignored. Then $f \sim A, \bar{f} \sim \bar{A}$ and chirality conservation takes place.

The action $S^{(2)}$ manifestly does not conserve chirality. However nonmanifest chirality nonconservation is also present in $S^{(1)}$, since to this order in h the quantity $\Box\bar{A}$ is already different from zero and in $f_{\alpha\beta}$ is present the field $\bar{A}$ of the "other" chirality.

To demonstrate the cancellation of chirality nonconserving terms we use the S-matrix approach. We make use of the interaction picture. The action $S^{(0)}$ determines the propagator for the field A and therefore the Wick contractions for $f_{\alpha\beta}, \bar{f}_{\dot\alpha\beta}$. Although we are using a noncovariant formalism, it can be verified in the usual way that the noncovariant terms cancel in the evaluation of the S-matrix and one may use covariantized Wick contractions, corresponding to the following definition:

$$\langle 0|TF_{\mu\nu}(x)F_{\varkappa\lambda}(y)|0\rangle = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\varkappa}\langle 0|TA_\nu(x)A_\lambda(y)|0\rangle + (\mu\leftrightarrow\nu, \varkappa\leftrightarrow\lambda) - (\mu\leftrightarrow\nu) - (\varkappa\leftrightarrow\lambda), \tag{23}$$

where the A_μ propagator has the usual form

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = i\eta_{\mu\nu}\Box^{-1}\delta(x-y).$$

On going over to $f_{\alpha\beta}, \bar{f}_{\dot\alpha\dot\beta}$ we obtain

$$\langle 0|Tf_{\alpha\beta}(x)\bar{f}_{\dot\alpha\dot\beta}(y)|0\rangle = (\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta} + \partial_{\alpha\dot\beta}\partial_{\beta\dot\alpha})\, i\Box^{-1}\delta(x-y),$$
$$\langle 0|Tf_{\alpha\beta}(x)f_{\gamma\delta}(y)|0\rangle = -i(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma})\delta(x-y),$$
$$\langle 0|T\bar{f}_{\dot\alpha\dot\beta}(x)\bar{f}_{\dot\gamma\dot\delta}(y)|0\rangle = -i(\varepsilon_{\dot\alpha\dot\gamma}\varepsilon_{\dot\beta\dot\delta} + \varepsilon_{\dot\alpha\dot\delta}\varepsilon_{\dot\beta\dot\gamma})\delta(x-y). \tag{24}$$

The contractions ff and $\bar{f}\bar{f}$, evidently, do not describe propagation of particles. As a result diagram lines are contracted corresponding to taking into account the fact that $f_{\alpha\beta}$ contains the $\Box A$-field of the "other" chirality. Taking this contraction explicity into account we obtain to second order in h the following addition to the action:

$$\Delta S^{(2)} = \frac{i}{2}\int d^4x\, d^4y \times h^{\alpha\dot\alpha,\,\beta\dot\beta} f_{\alpha\beta} \overbrace{\bar{f}_{\dot\alpha\dot\beta}(x)\, h^{\gamma\dot\gamma,\,\delta\dot\delta} f_{\gamma\delta}\bar{f}}_{\dot\gamma\dot\delta}(y) + (f\leftrightarrow\bar{f}). \tag{25}$$

Substitution of the contact contractions (24) yields

$$\Delta S^{(2)} = -S^{(2)}. \tag{26}$$

In this manner all chirality nonconserving vertices cancel to the order under consideration. We could not find a simple method to show that this cancellation happens to all orders, although this result is undoubtedly true.

In this connection one should note the discussion, going back to Ref. 7, of reconstruction of vertices with the help of various transversality conditions. We consider, say, photon scattering in an external gravitational field in second Born approximation. The amplitude is given by a sum of pole diagrams and contact vertices of the form h^2F^2. If the initial and final photons have different chirality, then the diagrams with the photon pole become contact diagrams and the diagram with the graviton pole vanishes. But without pole terms in the amplitude it is, evidently, impossible to satisfy the transversality requirement with respect to the gravitational field. Consequently the amplitude equals zero.

We also note Ref. 8, where the effective Lagrangian for the photon field, resulting from graviton exchange, is calculated. The authors of Ref. 8 emphasize the fact that their Lagrangian contains no terms that would violate chirality conservation for the photon field.

3. CHIRALITY CURRENT FOR THE PHOTON FIELD

We have demonstrated, in the light-cone gauge, chirality conservation of the photon field in interaction with gravity. It is not hard to write down the charge operator that measures chirality ($Q = +1$ for the left-handed state and $Q = -1$ for the right-handed state)

$$Q = \int d^3x \frac{i}{4}\bar{A}(x)\overleftrightarrow{\partial}_\zeta A(x), \tag{27}$$

where $d^3x = d^2x_\perp d\zeta$ and $\overleftrightarrow{\partial} = \vec{\partial} - \overleftarrow{\partial}$. The charge Q contains the ζ-component of the conserved current

$$j_\mu(x) = \frac{i}{4}\bar{A}(x)\overleftrightarrow{\partial}_\mu A(x). \tag{28}$$

The current (28), however, is not a Lorentz vector because A and $\bar{A}$ are not scalars.

It is not hard to verify that K_μ [see formula (3)] is the Lorentz covariant current, whose ζ-component coincides with j_ζ. The price paid for its covariance is the loss of the equation of continuity, since K_μ fails to be conserved already at the claasical level:

$$\nabla_\mu K^\mu = -{}^1/_2 F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{29}$$

Nevertheless, matrix elements of $\nabla_\mu K^\mu$ in an external gravitational field are naively equal to zero. Indeed, the operator

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{i}{2}(f_{\alpha\beta}f^{\alpha\beta} - \bar{f}_{\dot\alpha\dot\beta}\bar{f}^{\dot\alpha\dot\beta}) \tag{30}$$

changes chirality by ± 2, while the interaction with gravity conserves chirality. Therefore a nonzero expectation value of $\nabla_\mu K^\mu$ in an external gravitational field may be referred to as chiral anomaly.

We shall evaluate this anomaly later and now make a number of remarks with respect to the introduction of the chiral current. First, the current K_μ depends explicitly on the vector potential A_μ and is therefore gauge-noninvariant. It is known, however, that the corresponding charge

$$Q=\int d^3x\, K^0$$

is gauge-invariant. Moreover, as will be seen from the calculation in the following section, matrix elements of K_μ in a gravitational field do not depend on the choice of gauge of the photon field.

In the papers of Refs. 9–11 an explicitly gauge-invariant current is proposed, proportional to the difference between the number of left-handed and right-handed photons

$$S^\mu=\varepsilon^{\mu\nu\varkappa\lambda}\partial_\nu F_{\varkappa\rho}F_\lambda{}^\rho.$$

This current is conserved for the free field. However it has a noncanonical dimension and the corresponding charge is defined not by chirality alone but also by the energy of the photon.

The last remark has to do with the analogy between the current K^μ and the Pauli-Lubanski vector.[12] This vector is defined in its most general form as an object constructed out of the generators of the Poincare group

$$\Gamma^\mu=\varepsilon^{\mu\nu\varkappa\lambda}P_\nu M_{\varkappa\lambda}, \tag{31}$$

where P_μ and $M_{\varkappa\lambda}$ are the generators of translations and Lorentz rotations. For massless single-particle states of spin s one has

$$\Gamma_\mu|p_\mu,\lambda\rangle=\lambda p_\mu|p_\mu,\lambda\rangle, \tag{32}$$

where p_μ is the four-momentum, $p^2=0$, and $\lambda=\pm s$ is the helicity of the state.

We wish to construct a current such that the integral over its zeroth component gives the helicity generator Q, $Q\,|p_\mu,\lambda\rangle=\lambda\,|p_\mu,\lambda\rangle$. It is natural to take for such a current

$$j^\mu=-\frac{i}{3!}\varepsilon^{\mu\nu\varkappa\lambda}S_{\nu\varkappa\lambda}, \tag{33}$$

where $S_{\nu\varkappa\lambda}$ is the spin part of the angular momentum density

$$S_{\nu\varkappa\lambda}=-\frac{\partial L}{\partial(\partial^\nu\varphi^a)}(\Sigma_{\varkappa\lambda})_{ab}\varphi^b, \tag{33a}$$

$\Sigma_{\varkappa\lambda}$ being a representation of the generators of Lorentz rotations $M_{\varkappa\lambda}$ for the field φ^a. We have $\Sigma_{\varkappa\lambda}=(i/2)\sigma_{\varkappa\lambda}$ for the spinor field ψ, while for the vector field

$$(\Sigma_{\varkappa\lambda})_{ab}=i(g_{\varkappa a}g_{\lambda b}-g_{\varkappa b}g_{\lambda a}).$$

The current j_μ may be viewed as the one-particle analog of the Pauli-Lubanski vector. It is not hard to see that the definition (33), (33a) gives $j_\mu=a_\mu/2$ for the Dirac field and $j_\mu=2K_\mu/3$ for the vector field A_μ. Using these definitions one may construct chiral currents for other spins as well.

4. TRIANGLE DIAGRAMS

The anomaly calculation, given below, is based on the dispersion approach of Ref. 13. To demonstrate the similarity between the bosonic and fermionic cases we discuss them in parallel.

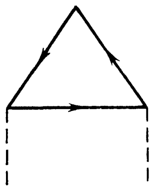

FIG. 1.

First, a few words about the description of triangle diagrams in general (Fig. 1). We note that in the presence of contact vertices one must also consider the diagram shown in Fig. 2. We discuss the production of two photons and two gravitons by the currents a_μ and K_μ. Each matrix element is determined by one form-factor

$$\begin{aligned}\langle 0|a_\mu|2\gamma\rangle&=f_1(q^2)\,q_\mu F_{\varkappa\lambda}\tilde F^{\varkappa\lambda},\\ \langle 0|a_\mu|2g\rangle&=f_2(q^2)\,q_\mu R_{\varkappa\lambda\rho\sigma}\tilde R^{\varkappa\lambda\rho\sigma},\\ \langle 0|K_\mu|2g\rangle&=f_3(q^2)\,q_\mu R_{\varkappa\lambda\rho\sigma}\tilde R^{\varkappa\lambda\rho\sigma},\end{aligned} \tag{34}$$

where q_μ is the four-momentum transferred by the corresponding current. The fact that there is only a single structure is a consequence of gauge invariance with respect to external lines. We emphasize here that the external photons and gravitons lie on the mass shell.

We make use of dispersion relations to find f_{1-3}. The imaginary parts are given by tree diagrams, which at first sight agree with all invariances of the classical theory. It is not hard to see, however, that if that were the case then we would have $\mathrm{Im} f_{1-3}=0$. Indeed, the four-dimensional longitudinal parts of the currents produce in Born approximation intermediate particles with total chirality ± 2. But annihilation into photons and gravitons is only possible for states with zero chirality.

In the dispersion approach the anomaly manifests itself in the need for infrared regularization. With the masses of the intermediate particles set equal to zero from the very beginning we run into the problem of the imaginary part being ill-defined, because the singularities due to the exchange of massless particles come up to the boundary of the physical region. To regularize this infrared region we introduce infinitesimal masses m for the intermediate particles. Such a regularization respects gauge invariance with respect to the external photon and gravitational fields, but gives rise to nonconservation of the chiral current. In the language used here the anomaly manifests itself in $\mathrm{Im}\, f_i(q^2)$ being different from zero; moreover, it is clear from the above con-

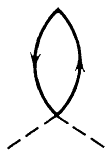

FIG. 2.

siderations that the imaginary parts of $f_i(q^2)$ can differ from zero only in the region $q^2 \sim m^2$, which tends to zero.

The evaluation of the imaginary parts of the form-factors f_{1-3} is carried out in the usual manner.

The final result for the imaginary parts looks as follows:

$$\mathrm{Im}\, f_1(q^2) = \lim_{m\to 0}\left(-\frac{\alpha}{4q^2}\right)(1-v^2)\ln\frac{1+v}{1-v} = -\frac{\alpha}{2}\,\delta(q^2),$$
$$\mathrm{Im}\, f_2(q^2) = \lim_{m\to 0}\frac{1}{128\pi q^2}(1-v^2)^2\ln\frac{1+v}{1-v} = \frac{1}{192\pi}\,\delta(q^2),$$
$$\mathrm{Im}\, f_3(q^2) = \lim_{m\to 0}\frac{1}{128\pi q^2}v^2(1-v^2)\ln\frac{1+v}{1-v} = \frac{1}{96\pi}\,\delta(q^2), \tag{35}$$

where $v = (1 - 4m^2/q^2)^{1/2}$ is the velocity of the intermediate particles in the center-of-mass frame.

Using dispersion relations to obtain the real parts of f_{1-3} we arrive at Eqs. (1) and (4). In this manner, from the technical point of view, the fermionic and bosonic chiral anomalies arise in a completely analogous way.

We next make a few remarks regarding the evaluation of the relations (35). It is convenient to consider directly matrix elements of the divergences, and not of the currents themselves. Since in all cases we have to deal with a single form-factor, the relation between the imaginary parts of the matrix elements of the currents and of their divergences is trivial.

The next remark consist in the observation that in the case of the form-factor f_2 the contact term (Fig. 2) gives a nonvanishing contribution.

Some discussion is needed regarding the introduction of a photon mass. A direct addition to the action (19) of the mass term

$$S_m = \frac{m^2}{2}\int d^4x(-g)^{1/2}g^{\mu\nu}A_\mu A_\nu \tag{36}$$

corresponds to the use of the Proca formalism for the description of a massive vector field. Of course, in the process one introduces into the theory an additional dynamical degree of freedom, which describes quanta of the vector field with zero chirality.

We want to emphasize, however, the fact that the quantity Im f_3, determined by formula (35), turns out to be gauge-invariant, i.e., the same answer as is obtained in the Proca formulation is also obtained by an addition to the action of the following type

$$\Delta S = -\frac{\xi}{2}\int d^4x(-g)^{1/2}(\nabla_\mu A^\mu)^2 + S_m,$$

where S_m is the mass term (36) and ξ is a gauge parameter. We remark that the same answer is obtained also in gauges that are noncovariant with respect to the external field

$$\Delta S = -\frac{1}{2}\int d^4x[\xi(\partial_\mu A^\mu)^2 - m^2 A_\mu A^\mu]. \tag{37}$$

Gauge invariance of the calculation shows that the introduction of the mass ensures a redefinition of the infrared singularity without introducing new degrees of freedom.

Finally, the last remark refers to the establishment of a direct connection between the answers (35) for the fermionic current Im f_2 and the current for the vector field Im f_3. Let us introduce a current S_μ for the fermions, analogous to the Pauli-Lubanski vector,

$$S_\mu = -\frac{1}{4m}\varepsilon_{\mu\nu\varkappa\lambda}\bar\psi\overleftrightarrow{\partial}_\nu\sigma_{\varkappa\lambda}\psi. \tag{38}$$

Making use of the equations of motion we can express this current in the form

$$S_\mu = \bar\psi\gamma_\mu\gamma_5\psi + \frac{i}{2m}\partial_\mu(\bar\psi\gamma_5\psi). \tag{39}$$

We show first that the matrix elements $\langle 0|S_\mu|2g\rangle$ and $\langle 0|K_\mu|2g\rangle$ differ only in sign. Using perturbation theory we suppose that the gravitons have the same, say left, helicity. Then the Riemann tensor for each graviton is anti-self-dual,[1] and the right-hand connections $\omega_\mu{}^{\dot\alpha}{}_{\dot\beta}$ equal zero. The indices $\dot\alpha, \dot\beta$ are tetrad indices. We shall describe the photon by the field $A_{\alpha\dot\alpha} = (\sigma^a)_{\alpha\dot\alpha}e^\mu_a A_\mu$, where e^μ_a is the frame field. Since $\omega^{\dot\alpha}_{\mu\dot\beta} = 0$, it follows that the dotted indices are sterile with respect to interaction with left-handed gravitons.

Therefore the expression for $\langle K_\mu\rangle$, describing propagation in the loop of the fields $A_{\alpha 1}$ and $A_{\alpha\dot 2}$, looks the same as in the case of the loop in which two Weyl fields are propagating. But the evaluation of $\langle S_\mu\rangle$ corresponds precisely to the latter case. As a result matrix elements of the currents K_μ and S_μ differ only by an overall sign due to anticommutation of fermion operators.

On the other hand, direct calculation of the imaginary part of $\langle\partial_\mu S_\mu\rangle$ yields

$$\mathrm{Im}\int d^4x\, e^{-iqx}\langle\partial_\mu S_\mu(x)\rangle$$
$$= (1-q^2/4m^2)\,\mathrm{Im}\int d^4x\, e^{-iqx}\langle\partial_\mu a_\mu(x)\rangle$$
$$= -\frac{v^2}{1-v^2}\,\mathrm{Im}\, f_2(q^2). \tag{40}$$

Making use of the explicit expression for Im f_2, and changing the sign in accordance with what was said above, we arrive at the previous answer for Im f_3 [see formulas (35)].

5. ELECTROMAGNETIC CORRECTION TO FERMION CHIRAL ANOMALY IN A GRAVITATIONAL FIELD

Relation (4) permits an easy evaluation of the electromagnetic correction to the usual fermion chiral anomaly in an external gravitational field. To this end we construct the average of relation (1) in the external gravitational field. Using formula (4) for $\langle F\tilde F\rangle$ we obtain

$$\nabla_\mu a^\mu = -\frac{1}{192\pi^2}\left(1 - \frac{2\alpha Q^2}{\pi}\right)R_{\mu\nu\varkappa\lambda}\tilde R^{\mu\nu\varkappa\lambda}, \tag{41}$$

where Q is the electric charge of the fermion.

This result corresponds to a particular infrared regularization of the two-loop diagrams, namely when the infrared photon mass m_γ is assumed to be much larger than the fermion mass m_f

$$m_\gamma \gg m_f. \tag{42}$$

In that case only the two-photon intermediate state (Fig. 3) contributes to the two-loop result for Im f_2.

Indeed, let us consider the three-particle intermediate state (Fig. 4). The corresponding imaginary part is unam-

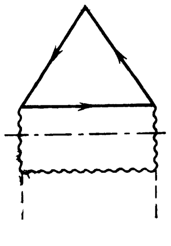

FIG. 3.

biguously determined for $m_\gamma \neq 0$ and $m_f = 0$ and equals zero, since the finite photon mass has no effect on the conservation of the current a_μ for $m_f = 0$. This duscussion is an obvious analog of the Adler-Bardeen theorem.[14]

And so it remains to deal with two-particle cuts. As regards the fermionic intermediate state of Fig. 5, its contribution vanishes too. Indeed, the relevant region in the dispersion integral is $q^2 \sim m_f^2 \ll m_\gamma^2$. Therefore to determine the imaginary part it is necessary to in fact evaluate the renormalization of the current vertex for $q = 0$ (accurate to corrections of order $\sim q^2/m_\gamma^2$). In our regularization, when $m_f = 0$ and $m_\gamma \neq 0$, this renormalization coincides with the renormalization of the vector current vertex and cancels against the Z-factors of the external lines.

Lastly, the two-photon intermediate state (Fig. 3) contributes to the anomaly. It is relevant that in the divergence $\partial_\mu a^\mu$ the first loop gives rise to a polynomial in the momenta, $\partial_\mu a^\mu = (\alpha Q^2/2\pi) F\tilde{F}$, and therefore the two-loop calculation reduces to a consistency check of the one-loop calculation.

6. CONCLUSION

Although the existence of the boson chiral anomaly is established, there remain some questions that need further study.

First, it cannot be said that the definition of the chiral current is as satisfactory as in the fermion case. The problem lies in our inability to come up with a field-theoretical formalism, in which manifest Lorentz covariance is compatible with chiral invariance of the action. In the framework of the light-cone formalism the chiral current is conserved, but manifest Lorentz covariance is lacking. On the other hand in the usual covariant description the current K_μ is not conserved.

It is natural to seek a way out of this situation by means of increasing the number of auxilliary fields, as is done, for example, in the case of supersymmetry.

An indication that such a reformulation might be natural is provided by the interpretation[15] of the anomaly (4) in

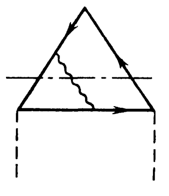

FIG. 4.

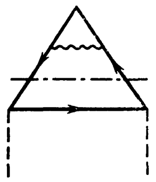

FIG. 5.

terms of zero modes. In a Euclidean space with nontrivial topology, such that the invariant $\int d^4x g^{1/2} R\tilde{R}$ is different from zero, the coefficient in the anomaly (4) can be expressed as the difference of left and right zero modes of the antisymmetric tensor field $\varphi_{\mu\nu}$. We recall that the field A_μ has no such zero modes.[16]

We note one more circumstance, called to our attention by Duff. The chiral anomaly for massless Weyl fermion fields of spin s is described by the single formula[17]

$$\int d^4x\, g^{1/2} \nabla_\mu J^\mu = (2s^3 - s) \frac{1}{96\pi^2} \int d^4x\, g^{1/2} R\tilde{R}. \qquad (43)$$

It turns out that the photon anomaly (4) also satisfies this relation if it is modified by the natural factor $(-1)^{2s}$.

To conclude we say a few words about previous work on chiral anomalies for bosons. The first example of this kind was discussed in Ref. 4, which dealt with the antisymmetric tensor field.

A second example is the non-Abelian vector field. The generalization of expression (3) for K_μ to the non-Abelian case is well-known:

$$K_\mu = -\varepsilon_{\mu\nu\varkappa\lambda}(A_\nu^a \partial_\varkappa A_\lambda^a + {}^2/_3 f^{abc} A_\nu^a A_\varkappa^b A_\lambda^c).$$

Further, as was noted in Ref. 19, the triangle diagrams arising in the evaluation of the average value of K_μ in the external non-Abelian field agree [up to a factor (-2)] with the corresponding fermion diagrams for the current a_μ in the same external field. In this way the calculation demonstrates the presence of the anomaly in $\partial_\mu K_\mu$. It should be noted that in the external field of an instanton this anomaly counts the number of zero modes of the boson field, analogously to the case of $\partial_\mu a_\mu$. The status of the anomaly is, however, not entirely clear since the "naive" Ward identities have not been formulated.

The anomaly for the non-Abelian vector field is interesting also because an analogous situation is to be expected for the chiral current of gravitons. The anomaly of this current has not been obtained as yet.

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[1)]This procedure is analogous to the choice of the anti-self-dual external field in the Euclidean approach. As is well known, however, such a field as a result of the reality condition cannot be weak. But for gravitons there is no such restriction.

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