Nonlinear interactions of waves in an inhomogeneous plasma

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(Submitted 10 September 1987)
Zh. Eksp. Teor. Fiz. **94**, 148–158 (July 1988)

We consider nonlinear wave processes in a weakly inhomogeneous plasma. We obtain a quasi-linear equation taking into account the effect of waves on resonant particles under conditions when the inhomogeneity appreciably affects the nature of the resonance interaction. Under the same conditions we study three-wave interactions. As an example we consider the nonlinear interaction in a relativistic plasma moving along a strong curvilinear magnetic field.

1. It has been shown in Ref. 1 that the strongest effect of an inhomogeneity in a medium is on the resonance interaction between waves and particles. Even when the characteristic size L of the inhomogeneity is much larger than the wavelength of the oscillations considered,

$$kL \gg 1, \tag{1}$$

the inhomogeneity can radically change the nature of the interaction of the charged particles in the medium with the electromagnetic waves. The fact is that the quantities which characterize the wave emission and absorption processes are the formation length l_f and, in the case of strong collective interactions, the quantity which is the inverse of the spatial growth rate $\varkappa^{-1}$, i.e., the length over which the wave amplitude changes by a factor e. These two quantities may be comparable with L. Moreover, in view of the exponential nature of the phase synchronism when there is resonance interaction even when l_f/L and $1/\varkappa L \ll 1$, it can be important to take the inhomogeneity into account.

Examples of a strong effect of an inhomogeneity on the damping and excitation of waves were given in Ref. 1. In particular, it was shown that Landau damping of Langmuir waves is appreciably larger in an inhomogeneous plasma than in a uniform one, even when the inhomogeneity is comparatively weak. Therefore, even in the linear approximation, an inhomogeneity changes the nature of the resonance interaction between waves and plasma particles considerably. We show also that it affects strongly also the nonlinear particle-wave and wave-wave interactions.

2. The plasma inhomogeneity manifests itself in two ways: first of all, an irregularity in the motion of the charged particles is caused by the forces which sustain the given nonuniformity and, furthermore, there is the effect of the inhomogeneity on the wave motion described by the geometrical-optics equations when (1) holds. It makes therefore no sense to use for the description of the wave field $\mathbf{E}(\mathbf{r})$ an expansion in plane waves:

$$\mathbf{E}(\mathbf{r}) = \int \mathbf{E}(\mathbf{k}) e^{i\mathbf{k}\mathbf{r}} d\mathbf{k}, \tag{2}$$

since they are not eigenfunctions of the field in an inhomogeneous medium. It is better to use here an expansion in the fields of wave packets:

$$\mathbf{E}(\mathbf{r}) = \sum_{\mathbf{q}} \mathbf{E}^0(\mathbf{r}, \mathbf{q}) e^{i\psi(r)}, \tag{3}$$

where $\psi(r)$ is the phase determining the local wave number $\mathbf{q} = \nabla\psi$, while $\mathbf{E}^0(\mathbf{r})$ is the slowly changing amplitude of the packet. There is in (3) summation over the different local wave numbers which characterize the wave packets. To describe the electromagnetic properties of a weakly dispersive medium in the linear approximation it is necessary to find the permittivity tensor $\varepsilon_{\alpha\beta}(\mathbf{q},\mathbf{r},\omega)$. By using the energy conservation law, the authors of Ref. 1 showed that the quantity $\varepsilon_{\alpha\beta}$ is defined as follows:

$$\varepsilon_{\alpha\beta}(\mathbf{q}, \mathbf{r}, \omega) = \frac{1}{(2\pi)^3} \iint \varepsilon^0_{\alpha\beta}(\mathbf{k}, \mathbf{r} + \boldsymbol{\eta}/2, \omega) e^{i(\mathbf{k}-\mathbf{q})\boldsymbol{\eta}} d\mathbf{k}\, d\boldsymbol{\eta}, \tag{4}$$

where $\varepsilon^0_{\alpha\beta}(\mathbf{k},\mathbf{r},\omega)$ is the response of the inhomogeneous medium to a plane electromagnetic wave:

$$\varepsilon_{\alpha\beta}^0(\mathbf{k}, \mathbf{r}, \omega) = \delta_{\alpha\beta} + \frac{4\pi i}{\omega} \sigma_{\alpha\beta}^0(\mathbf{k}, \mathbf{r}, \omega),$$

$$j_\alpha(\mathbf{r}, \omega) = \int \sigma_{\alpha\beta}^0(\mathbf{k}, \mathbf{r}, \omega) E_\beta(\mathbf{k}) e^{i\mathbf{k}\mathbf{r}} d\mathbf{k}, \tag{5}$$

$\mathbf{j}(\mathbf{r},\omega)$ is the current excited in the medium by the wave (2). We recall here that the medium is inhomogeneous but stationary, so that all wave quantities are proportional to $\exp(-i\omega t)$. The quantity $\varepsilon^0_{\alpha\beta}(\mathbf{k},\mathbf{r},\omega)$ for a collisionless inhomogeneous plasma can be easily calculated from the solution of the kinetic equation for the particles, by using functional integration. Using the transformation (4) we can write the expression for $\varepsilon_{\alpha\beta}$ in a collisionless plasma in the form[1]

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} - \frac{4\pi e^2 i}{\omega} \int d\mathbf{p}\, v_\alpha \int_{-\infty}^{t} dt' \exp[i\omega(t-t') - i\mathbf{q}\mathbf{R}^*]$$
$$\times \det{}^{-1} \left| \delta_{\mu\nu} - \frac{\partial \lambda_\mu(\mathbf{r}+\mathbf{R}^*/2)}{2\partial r_\nu} \right| \left[\left(1 - \frac{\mathbf{q}\mathbf{v}'}{\omega}\right) \delta_{\beta\sigma} + \frac{q_\sigma v_\beta'}{\omega} + \frac{i}{2\omega}\frac{\partial}{\partial r_\sigma} v_\beta' - \frac{i}{2\omega}\delta_{\beta\sigma}\frac{\partial}{\partial r_\chi} v_\chi' \right] \frac{\partial f}{\partial p_\sigma'} \Bigg|_{\mathbf{r}=\mathbf{r}+\mathbf{R}^*/2}. \tag{6}$$

Here $f(\mathbf{r},\mathbf{p})$ is the distribution function of particles with charge e; $\mathbf{p}' = \mathbf{p}(t')$, $\mathbf{v}' = \mathbf{v}(t')$, and $\mathbf{r}' = \mathbf{r}(t')$ are their momenta, velocities, and coordinates at time t' when they are moving along an unperturbed trajectory such that at time t they have at the point $\mathbf{r}$ a momentum $\mathbf{p}$ and a velocity $\mathbf{v}$. The quantity $\mathbf{R}^*$ in (6) is determined by the particle trajectories:

$$\mathbf{r} - \mathbf{r}' = \boldsymbol{\lambda}(\mathbf{r}, \mathbf{p}, t-t'), \quad \mathbf{R}^* = \boldsymbol{\lambda}(\mathbf{r}+\mathbf{R}^*/2, \mathbf{p}, t-t'). \tag{7}$$

The inhomogeneity manifests itself in Eq. (4) for the permit-

 0038-5646/88/071380-06$04.00

tivity $\varepsilon_{\alpha\beta}$ of a weakly inhomogeneous medium in two ways: firstly, through the quantity $\varepsilon^0_{\alpha\beta}$ in which the trajectories of the particle motion correspond to the trajectories in the inhomogeneous medium, and, secondly, through the nonlocality connected with the fact that a wave with a wave vector $\mathbf{q}$ at a given point $\mathbf{r}$ has in a neighboring point $\mathbf{r}'$ a wave vector different from $\mathbf{q}$. This corresponds to the integral nature of the transformation (4).

Expression (4) is completely equivalent to writing the permittivity in the form

$$\varepsilon_{\alpha\beta}(\mathbf{q},\mathbf{r})=\frac{1}{(2\pi)^3}\int\hat{\varepsilon}^s_{\alpha\beta}(\boldsymbol{\eta},\mathbf{r})e^{i\mathbf{q}\boldsymbol{\eta}}\,d\boldsymbol{\eta}, \tag{8}$$

where $\varepsilon^*_{\alpha\beta}$ is the kernel of the integral connection betwen the electric field and the induction vectors in an inhomogeneous medium, symmetrized with respect to their variables:

$$D_\alpha(\mathbf{r})=\int\hat{\varepsilon}_{\alpha\beta}{}^s\left(\mathbf{r}-\mathbf{r}',\frac{\mathbf{r}+\mathbf{r}'}{2}\right)E_\beta(\mathbf{r}')\,d\mathbf{r}'. \tag{9}$$

One finds easily from Eqs. (8) and (9) the equation

$$\varepsilon_{\alpha\beta}(\mathbf{q},\mathbf{r})=\varepsilon_{\alpha\beta}{}^0(\mathbf{q},\mathbf{r})+\frac{i}{2}\frac{\partial^2\varepsilon_{\alpha\beta}{}^0}{\partial\mathbf{r}\,\partial\mathbf{q}}-\frac{1}{8}\frac{\partial^4\varepsilon_{\alpha\beta}{}^0}{\partial\mathbf{r}^2\,\partial\mathbf{q}^2}+\dots, \tag{10}$$

which is obtained by expanding the kernel $\hat{\varepsilon}^s_{\alpha\beta}$ in powers of $\mathbf{r}-\mathbf{r}'$ under the condition that $|\mathbf{r}-\mathbf{r}'|\ll|\mathbf{r}|$. Equation (10) also follows directly from (4).[1] Usually[2,3] one uses a finite number of terms in the expansion (10). This is insufficient for studying the nature of the resonance wave-particle interaction in an inhomogeneous medium, as can be seen from Eqs. (6) and (7). Indeed, the dispersive properties of the medium—including the contribution from the resonance particles—are described by the functions

$$\int_{-\infty}^{t}dt'\exp[i\omega(t-t')-i\mathbf{q}\mathbf{R}^*],$$

where the value of $\mathbf{R}^*$ is not the same in an inhomogeneous medium as the value calculated using the particle trajectories (i.e., λ). We can solve Eq. (7) for $\mathbf{R}^*$ by expanding in $\frac{1}{2}\mathbf{R}^*$ (if l_f/L, $1/\varkappa L\ll1$), but taking into account a finite number of terms is equivalent to taking into account in (10) all terms of the expansion, as $\mathbf{R}^*$ occurs in the argument of the exponential. Moreover, Eq. (4) gives a rather simple prescription for calculating the permittivity tensor $\varepsilon_{\alpha\beta}$ from the quantity $\varepsilon^0_{\alpha\beta}$ which is found in the same way as in a homogeneous medium.

Changing to a consideration of the nonlinear interaction, we note to begin with the following: the dispersion relation for linear waves is determined in a weakly inhomogeneous medium by the equations

$$[q_\alpha q_\beta-q^2\delta_{\alpha\beta}+(\omega^2/c^2)\varepsilon_{\alpha\beta}(\mathbf{q},\mathbf{r},\omega)]E_\beta{}^0e^{i\psi(\mathbf{r})}=0,\quad \mathbf{q}=\nabla\psi. \tag{11}$$

At the same time, we can write the solution of the Maxwell equations in the form [see (5)]

$$\int[k_\alpha k_\beta-k^2\delta_{\alpha\beta}+(\omega^2/c^2)\varepsilon_{\alpha\beta}{}^0(\mathbf{r},\mathbf{k},\omega)]E_\beta(\mathbf{k})e^{i\mathbf{k}\mathbf{r}}\,d\mathbf{k}=0. \tag{12}$$

Comparing (11) and (12) we see that they are equivalent, provided the relation

$$\int\mathbf{E}(\mathbf{k})\Phi(\mathbf{k},\mathbf{r})e^{i\mathbf{k}\mathbf{r}}\,d\mathbf{k}=\sum_{\mathbf{q}}\mathbf{E}^0(\mathbf{q},\mathbf{r})e^{i\psi}$$

$$\times\frac{1}{(2\pi)^3}\iint\Phi(\mathbf{k}',\mathbf{r}+\boldsymbol{\eta}/2)e^{i(\mathbf{k}'-\mathbf{q})\boldsymbol{\eta}}\,d\mathbf{k}'\,d\boldsymbol{\eta}, \tag{13}$$

holds, where $\Phi(\mathbf{k},\mathbf{r})$ is an arbitrary function of the coordinates which varies slowly over a wavelength $\lambda=2\pi/q$. In the case of (11) and (12)

$$\Phi(\mathbf{k},\mathbf{r})=\varepsilon_{\alpha\beta}{}^0(\mathbf{k},\mathbf{r}).$$

Equation (13) effects the change from the expansion of the wave field in plane waves to an expansion in wave packets, which is more suitable for an inhomogeneous medium. Equation (13), which is obtained from the linear-approximation equation, is valid also to any order of perturbation theory—each power of the wave field must be transformed according to (13). We show this now with the quasi-linear approximation that takes into account terms quadratic in $\mathbf{E}(\mathbf{r})$ as the example. Just as in obtaining the dispersion Eq. (11), we use the energy conservation law

$$\frac{d}{dt}\left[\int f\varepsilon\,d\mathbf{p}+W\right]=0, \tag{14}$$

where f is the averaged particle distribution function, ε their energy, and W the electromagnetic energy density in the wave. Recalling the expansion (13) we can write the derivative dW/dt in the form:[1]

$$\frac{dW}{dt}=-\frac{1}{8\pi}\sum_{\mathbf{q}}\omega(\mathbf{q})\varepsilon^{aH}_{\alpha\beta}(\mathbf{q},\mathbf{r})E_\alpha{}^{0*}(\mathbf{r},\mathbf{q})E_\beta{}^0(\mathbf{r},\mathbf{q}). \tag{15}$$

The quantity $\varepsilon^{aH}_{\alpha\beta}$ is the anti-Hermitean part of the permittivity tensor (4):

$$\varepsilon^{aH}_{\alpha\beta}=\frac{\varepsilon_{\alpha\beta}-\varepsilon_{\beta\alpha}{}^*}{2i}=\frac{2\pi}{\omega}(\sigma_{\alpha\beta}+\sigma_{\beta\alpha}{}^*), \tag{16}$$

where $\sigma_{\alpha\beta}$ is the conductivity of the medium. Substituting (15) and (16) in (14) we get the equation

$$\frac{d}{dt}\int f\varepsilon\,d\mathbf{p}=-\frac{1}{4}\sum_{\mathbf{q}}(\sigma_{\alpha\beta}+\sigma_{\beta\alpha}{}^*)\langle E_\alpha{}^{0*}E_\beta{}^0\rangle, \tag{17}$$

where on the right-hand side we have the averaged work done by the electric field of the waves on the current produced by this field in the medium. On the left-hand side of (17) we have the derivative df/dt which can be determined from the solution of the kinetic equation. Expanding the electrical field in the Fourier integral (2) we can write the result in the form

$$\frac{df}{dt}\equiv\frac{\partial f}{\partial t}+\mathbf{v}\frac{\partial f}{\partial\mathbf{r}}+e\left\{\mathbf{E}_0+\frac{1}{c}[\mathbf{v}\mathbf{B}_0]\right\}\frac{\partial f}{\partial\mathbf{p}}$$

$$=\frac{e^2}{4}\iint d\mathbf{k}\,d\mathbf{k}_1\langle\exp\{-i(\omega-\omega_1)t+i(\mathbf{k}-\mathbf{k}_1)\mathbf{r}\}E_\mu{}^*(\mathbf{k}_1)E_\lambda(\mathbf{k})\rangle$$

$$\times\left[\delta_{\mu\sigma}\left(1-\frac{\mathbf{k}_1\mathbf{v}}{\omega}\right)+\frac{k_{1\sigma}v_\mu}{\omega}\right]\frac{\partial}{\partial p_\sigma}\int_{-\infty}^{t}\exp[i\omega(t-t')-i\mathbf{k}(\mathbf{r}-\mathbf{r}')]$$

$$\times\left[\delta_{\lambda\nu}\left(1-\frac{\mathbf{k}\mathbf{v}'}{\omega}\right)+\frac{k_\nu v_\lambda'}{\omega}\right]\frac{\partial f}{\partial p_\nu'}dt'+\text{c.c.} \tag{18}$$

Here $\mathbf{E}_0$ and $\mathbf{B}_0$ make up the average electromagnetic field in the plasma, and the angle brackets indicate averaging over the realizations of the field $\mathbf{E}(\mathbf{r})$. The quasi-linear equation in the form (18) cannot be used for actual calculations as it is not clear how to average the quantity $E^*_\mu(\mathbf{k}_1)E_\lambda(\mathbf{k})$. For a homogeneous medium, $\langle E_\mu^*(\mathbf{k}_1)E_\lambda(\mathbf{k})\rangle$ is proportional to $\delta(\mathbf{k}-\mathbf{k}_1)$ but this is not the case for an inhomogeneous medium, as one and the same wave field can have different Fourier harmonics when expanded in a Fourier series. The quantity $\langle E_\mu^*(\mathbf{k}_1)E_\lambda(\mathbf{k})\rangle$ depends on the magnitude of the inhomogeneity and is a functional of it. To perform the correct averaging one must change in (18) to the representation of a field of wavepackets in the form (13). To do this we substitute Eq. (18) into (17), integrate, and use the relation

$$\sigma_{\alpha\beta}{}^0(\mathbf{k},\mathbf{r},\omega)=-e^2\int d\mathbf{p}\,v_\alpha\int_{-\infty}^{t}dt'\exp[i\omega(t-t')-i\mathbf{k}(\mathbf{r}-\mathbf{r}')]\times\left[\delta_{\beta\nu}\left(1-\frac{\mathbf{kv}'}{\omega}\right)+\frac{k_\nu k_\beta'}{\omega}\right]\frac{\partial f}{\partial p_\nu'}dt'$$

to obtain Eq. (13), where $\sigma_{\alpha\beta}(\mathbf{k},\mathbf{r})$ plays the role of the quantity $\Phi(\mathbf{k},\mathbf{r})$. We see thus that in the quasi-linear approximation Eq. (13), which realizes the transition to the physically suitable representation of the electromagnetic field in a weakly inhomogeneous medium, follows from the conservation law for the electromagnetic energy and the particle energy.

Here, as when we compared Eqs. (11) and (12), there occurs in Eq. (13) the function $\Phi(\mathbf{k},\mathbf{r})$, which is the linear response to the field of the electromagnetic wave. However, (11) was obtained in Ref. 1 assuming that the transfer of electromagnetic energy in an inhomogeneous medium has the same form as in a uniform medium, whereas now we start from the conservation law for the total energy—of particles and waves.

After the transformation (13) the quasi-linear Eq. (18) takes the form

$$\frac{df}{dt}=\frac{e^2}{4}\sum_{\mathbf{q}}E_\mu{}^{0*}(\mathbf{q})E_\lambda{}^0(\mathbf{q})\left[\delta_{\mu\sigma}\left(1-\frac{\mathbf{qv}}{\omega}\right)+\frac{q_\sigma v_\mu}{\omega}-\frac{i}{2}\delta_{\mu\sigma}\frac{v_\alpha}{\omega}\frac{\partial}{\partial r_\alpha}+\frac{i}{2}\frac{v_\mu}{\omega}\frac{\partial}{\partial r_\sigma}\right]\frac{\partial}{\partial p_\sigma}\int_{-\infty}^{t}dt'\exp[i\omega(t-t')-i\mathbf{q}\mathbf{R}^\cdot]$$
$$\times\left[\delta_{\lambda\nu}\left(1-\frac{\mathbf{qv}'}{\omega}\right)+\frac{q_\nu}{\omega}v_\lambda'+\frac{i}{2\omega}\frac{\partial}{\partial r_\nu}v_\lambda'-\frac{i}{2\omega}\delta_{\lambda\nu}\frac{\partial}{\partial r_\alpha}v_\alpha'\right]\times\det{}^{-1}\left|\delta_{ik}-\frac{\partial\lambda_i(\mathbf{r}+\mathbf{R}^\cdot/2)}{2\partial r_k}\right|\frac{\partial f}{\partial p_\nu'}\Big|_{\mathbf{r}+\mathbf{R}^\cdot/2}+\text{c.c.},\qquad(19)$$
$$\mathbf{R}^\cdot=\boldsymbol{\lambda}(\mathbf{r}+\mathbf{R}^\cdot/2,\,\mathbf{p},\,t-t').$$

We used here the fact that $\langle E^{0*}(\mathbf{q})E^0(\mathbf{q}_1)\rangle\propto\delta(\mathbf{q}-\mathbf{q}_1)$.

Just as the quasi-linear Eq. (19), the nonlinear equations which describe the wave-particle interaction to any order of perturbation theory can be obtained from the solution of the kinetic equation, by expanding the electromagnetic field in a Fourier series and afterwards using the transformation (13).

We now consider the three-wave interaction. Let there be three waves with frequecies $\omega_1,\omega_2,\omega_3$. Their electric fields have correspondingly the form

$$\mathbf{E}_n{}^0(\mathbf{r})\exp[i\psi_n(\mathbf{r})-i\omega_n t],\quad n=1,\,2,\,3;$$

and their local wave numbers are $\mathbf{q}_n=\mathbf{q}_n(\omega_n,\mathbf{r})=\nabla\psi_n$.

We consider the case where $\omega_1\approx\omega_2+\omega_3$. The nonlinear current $\mathbf{j}$ at the frequency $\omega_2+\omega_3$ can easily be found from the solution of the kinetic equation using the procedure described by us:

$$j_\alpha=\sigma_{N\alpha\sigma\lambda}E_{2\sigma}{}^0E_{3\lambda}{}^0\exp[i(\psi_2+\psi_3)-i(\omega_2+\omega_3)t].$$

Here $\sigma_{N\alpha\sigma\lambda}$ is the nonlinear plasma conductivity:

$$\sigma_{N\alpha\sigma\lambda}=e\int v_\alpha[\hat{M}_\sigma(\omega_2+\omega_3,\mathbf{q}_2+\mathbf{q}_3)\hat{M}_\lambda(\omega_3,\mathbf{q}_3)+\hat{M}_\lambda(\omega_2+\omega_3,\mathbf{q}_2+\mathbf{q}_3)\hat{M}_\sigma(\omega_2,\mathbf{q}_2)]f(\mathbf{p},\mathbf{r})d\mathbf{p}.\qquad(20)$$

The operators $\hat{M}_s(\omega,\mathbf{q})$ are defined as follows:

$$\hat{M}_s(\omega,\mathbf{q})=-\frac{e}{2}\int_{-\infty}^{t}dt'\exp[i\omega(t-t')-i\mathbf{q}\mathbf{R}^\cdot]\det{}^{-1}\left|\delta_{ik}-\frac{\partial\lambda_i(\mathbf{r}+\mathbf{R}^\cdot/2)}{2\partial r_k}\right|\left[\delta_{\nu s}\left(1-\frac{\mathbf{qv}'}{\omega}\right)+\frac{q_\nu v_s'}{\omega}+\frac{i}{2\omega}\frac{\partial}{\partial r_\nu}v_s'-\frac{i}{2\omega}\frac{\partial}{\partial r_i}v_i'\right]\frac{\partial}{\partial p_\nu'}\Big|_{\mathbf{r}+\mathbf{R}^\cdot/2},$$
$$\mathbf{R}^\cdot=\boldsymbol{\lambda}(\mathbf{r}+\mathbf{R}^\cdot/2,\,\mathbf{p},\,t-t').$$

The structure of the operators, $\hat{M}$ is the same as that of the quantity $\varepsilon_{\alpha\beta}$ of (6), so that the correct symmetry is assured.[1] Under time reversal ($t\to-t$) the nonlinear interaction remains the same for waves with reversed fronts ($\omega\to-\omega$, $\mathbf{q}\to-\mathbf{q}$).

One can similarly find the nonlinear response at the frequencies $\omega_1-\omega_2$ and $\omega_1-\omega_3$. Under the conditions $\Delta\omega=\omega_1-\omega_2-\omega_3\ll\omega_1$ and $|\varepsilon_{\alpha\beta}^{aH}|\ll|\varepsilon_{\alpha\beta}^H|$ (ε^H is the Hermitean part of the permittivity tensor, $\varepsilon=\varepsilon^H+\varepsilon^{aH}$) the equation for the evolution of the wave $\mathbf{E}_i$ can be written in the form

$$\frac{1}{\omega_1}\frac{\partial}{\partial\omega_1}[\omega_1{}^2\varepsilon_{\alpha\beta}^H]\frac{\partial}{\partial t}E_{1\alpha}^{0*}E_{1\beta}^0-\frac{c^2}{\omega_1}\frac{\partial}{\partial r_i}\left[\left(q_{1\beta}\delta_{i\alpha}+q_{1\alpha}\delta_{i\beta}-2q_1\delta_{\alpha\beta}+\frac{\omega_1{}^2}{c^2}\frac{\partial\varepsilon_{\alpha\beta}^H}{\partial q_{1i}}\right)E_{1\alpha}^{0*}E_{1\beta}^0\right]+\omega_1\frac{\partial}{\partial q_{1i}}$$
$$\times\left[\frac{\partial\varepsilon_{\alpha\beta}^H}{\partial r_i}E_{1\alpha}^{0*}E_{1\beta}^0\right]=-2\omega_1\varepsilon_{\alpha\beta}^{aH}E_{1\alpha}^{0*}E_{1\beta}^0-4\pi(\sigma_{N\alpha\sigma\lambda}+\overset{\bullet}{\sigma}_{N\alpha\lambda\sigma})E_{1\alpha}^{0*}E_{2\sigma}^0E_{3\lambda}^0\exp[i\Delta\omega t-i(\psi_1-\psi_2-\psi_3)].\qquad(21)$$

Equation (21) is obtained from the dispersion Eq. (11) by using the nonlinear current (20) in the right-hand side. As in a uniform medium, we choose instead of a wave with a fixed $\mathbf{q}$ a set of waves with a wave-vector spread $\delta\mathbf{q}\ll\mathbf{q}$. This corresponds to an addition $\delta\psi-\mathbf{q}\delta\mathbf{r}$ to the phase, where $\delta\mathbf{q}=\nabla\delta\psi$ (just $\delta\psi-\mathbf{q}\delta\mathbf{r}$, because $\delta\psi=(\partial\psi/\partial\mathbf{q})\delta\mathbf{q}+\mathbf{q}\delta\mathbf{r}$ and we are interested in a set with different wave vectors $\mathbf{q}$). Moreover, expanding the dispersion relation in the small quantities $\delta\omega$, $\delta\mathbf{q}$, $\delta\mathbf{r}$ and making the substitution

$$\delta\omega=i\partial/\partial t,\quad\delta\mathbf{q}=-i\partial/\partial\mathbf{r},\quad\delta\mathbf{r}=i\partial/\partial\mathbf{q},$$

we are led to the left-hand side of Eq. (21). In the right-hand side we have taken into account the anti-Hermitean part of the tensor $\varepsilon_{\alpha\beta}$, a part satisfying the estimate $|\varepsilon^{aH}|/|\varepsilon^H|\sim\delta\omega/\omega$ and corresponding to damping or excitation of waves by interactions with the particles of the medium.

The wave amplitude $\mathbf{E}_1^0$ varies slowly in space, so that it can be assumed to be constant over distances comparable with the wavelength. We average Eq. (21) over a volume which contains many wavelengths, but, on the other hand, has a dimension $\mathscr{L}$ which is much smaller than the characteristic length L of the inhomogeneity and the length L_N over which the waves appreciably interact with one another. As a result, the second term on the right-hand side of Eq. (21) becomes proportional to the function

$$\frac{1}{\mathscr{L}^3}\int \exp\left[-i(\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)\mathbf{r}-i\frac{\partial(\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)}{2\partial\mathbf{r}}\mathbf{rr}\right]d\mathbf{r}, \quad (22)$$

with a characteristic width of the order of $(\partial q/\partial r)^{1/2}\approx(q/L)^{1/2}$ relative to $\Delta\mathbf{q}=\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3$. If the characteristic spread of the waves over the spectrum is larger than this quantity, we can replace the function (22) by a δ-function. As a result we have

$$\frac{1}{\omega_1}\frac{\partial}{\partial\omega_1}[\omega_1{}^2\varepsilon^H_{\alpha\beta}]\frac{\partial}{\partial t}E^{0*}_{1\alpha}E^0_{1\beta}-\frac{c^2}{\omega_1}\frac{\partial}{\partial r_i}\left[\left(q_{1\beta}\delta_{i\alpha}\right.\right.$$

$$\left.\left.+q_{1\alpha}\delta_{i\beta}-2q_i\delta_{\alpha\beta}+\frac{\omega_1{}^2}{c^2}\frac{\partial\varepsilon^H_{\alpha\beta}}{\partial q_{1i}}\right)E^{0*}_{1\alpha}E^0_{1\beta}\right]$$

$$+\omega_1\frac{\partial}{\partial q_i}\left[\frac{\partial\varepsilon^H_{\alpha\beta}}{\partial r_i}E^{0*}_{1\alpha}E^0_{1\beta}\right]=-2\omega_1\varepsilon^{aH}_{\alpha\beta}E^{0*}_{1\alpha}E^0_{1\beta}-4\pi(\sigma_{N\alpha\sigma\lambda}$$

$$+\dot{\sigma}_{N\alpha\lambda\sigma})\left(\frac{2\pi}{\mathscr{L}}\right)^3e^{i\Delta\omega t}E^{0*}_{1\alpha}E^0_{2\sigma}E^0_{3\lambda}\delta(\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3). \quad (23)$$

Equation (23) describes the interaction between three waves with fixed phases in an inhomogeneous medium. For waves with random phases we must average (23), separating in the wave E^0_1 the random component $\mathscr{E}^0_1$ and the component e^0_1 which is correlated with the waves E^0_2 and E^0_3. This procedure is completely analogous to the one known for a uniform medium.[4]

The right-hand side of Eq. (23) is proportional to the total time derivative of the electromagnetic energy density of the wave

$$W=\frac{1}{16\pi\omega}\frac{\partial}{\partial\omega}(\omega^2\varepsilon^{aH}_{\alpha\beta})E^{0*}_{1\alpha}E^0_{1\beta}$$

namely:

$$\frac{dW_1}{dt}=-\frac{\omega_1}{8\pi}\varepsilon^{aH}_{\alpha\beta}E^{0*}_{1\alpha}E^0_{1\beta}-\frac{1}{4}(\sigma_{N\alpha\sigma\lambda}$$

$$+\dot{\sigma}_{N\alpha\lambda\sigma})\left(\frac{2\pi}{\mathscr{L}}\right)^3e^{i\Delta\omega t}E^{0*}_{1\alpha}E^0_{2\sigma}E^0_{3\lambda}\delta(\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3).$$

Writing down the same equation for the two other waves with energy densities W_2 and W_3, we verify that the total energy is conserved:

$$\frac{d}{dt}\left(W_1+W_2+W_3+\int\varepsilon f\,d\mathbf{p}\right)=0.$$

The quantity df/dt is here the change in the particle distribution function caused not only by effects which are quadratic in the field, i.e., quasi-linear, but also by cubic effects which are connected with the nonlinear Landau damping. The quantity df/dt is calculated from the solution of the kinetic equation in the same way as in the quasi-linear approximation (18). We see thus that also for the three-wave interaction the transformation (13) conserves energy.

Concluding this section, we establish the connection between the nonlinear conductivity tensor $\sigma_{N\alpha\sigma\lambda}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{r})$ obtained by us and the kernel of the integral relation between the electrical fields and the electrical current caused by them:

$$j_\alpha(\mathbf{r})=\iint\hat{\sigma}_{N\alpha\sigma\lambda}(\mathbf{r},\mathbf{r}',\mathbf{r}'')E_\sigma(\mathbf{r}')E_\lambda(\mathbf{r}'')\,d\mathbf{r}'\,d\mathbf{r}''.$$

As in (9), we determine the symmetrized kernel $\hat{\sigma}^s_{N\alpha\sigma\lambda}$:

$$j_\alpha(\mathbf{r})=\iint\hat{\sigma}^s_{N\alpha\sigma\lambda}\left(\mathbf{r}-\mathbf{r}',\mathbf{r}-\mathbf{r}'',\mathbf{r}-\frac{\mathbf{r}-\mathbf{r}'}{2}-\frac{\mathbf{r}-\mathbf{r}''}{2}\right)$$

$$E_\sigma(\mathbf{r}')E_\lambda(\mathbf{r}'')\,d\mathbf{r}'\,d\mathbf{r}''. \quad (24)$$

The nonlinear response to plane waves $\sigma^0_N(\mathbf{k}_1,\mathbf{k}_2,\mathbf{r})$ can then be expressed in terms of $\hat{\sigma}^s_N$ as follows:

$$\sigma_N{}^0(\mathbf{k}_1,\mathbf{k}_2,\mathbf{r})=\frac{1}{(2\pi)^6}\iint\hat{\sigma}_N{}^s\left(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\mathbf{r}-\frac{\boldsymbol{\eta}_1+\boldsymbol{\eta}_2}{2}\right)$$

$$\exp[-i(\mathbf{k}_1\boldsymbol{\eta}_1+\mathbf{k}_2\boldsymbol{\eta}_2)]\,d\boldsymbol{\eta}_1\,d\boldsymbol{\eta}_2. \quad (25)$$

On the other hand, according to (13),

$$\sigma_N(\mathbf{q}_1,\mathbf{q}_2,\mathbf{r})=\frac{1}{(2\pi)^6}\int\sigma_N{}^0\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{r}\right.$$

$$\left.+\frac{\boldsymbol{\eta}_1+\boldsymbol{\eta}_2}{2}\right)\exp\{i[(\mathbf{k}_1-\mathbf{q}_1)\boldsymbol{\eta}_1+(\mathbf{k}_2-\mathbf{q}_2)\boldsymbol{\eta}_2]\}\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\boldsymbol{\eta}_1\,d\boldsymbol{\eta}_2. \quad (26)$$

Substituting expression (25) into (26) we get

$$\sigma_N(\mathbf{q}_1,\mathbf{q}_2,\mathbf{r})=\frac{1}{(2\pi)^6}\iint\hat{\sigma}_N{}^s(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\mathbf{r})$$

$$\exp(-i\mathbf{q}_1\boldsymbol{\eta}_1-i\mathbf{q}_2\boldsymbol{\eta}_2)\,d\boldsymbol{\eta}_1\,d\boldsymbol{\eta}_2.$$

We see thus that, as in the linear approximation, the nonlinear conductivity is the Fourier transform of the symmetrized kernel (24). One also checks easily that the transformation (13) corresponds, in any order of perturbation theory, to the following symmetrization of the kernel $\hat{\sigma}^n_N$ (n is the order of the nonlinearity):

$$\mathbf{j}=\int\hat{\sigma}_N{}^n\left[\mathbf{r}-\mathbf{r}',\ldots,\mathbf{r}-\mathbf{r}^{(k)},\ldots,\mathbf{r}-\frac{1}{2}\sum_{k=1}^{n}(\mathbf{r}-\mathbf{r}^{(k)})\right]$$

$$\prod_{k=1}^{n}\mathbf{E}(\mathbf{r}^{(k)})\,d\mathbf{r}^{(k)}.$$

3. We consider now the interaction of waves in a plasma which is a stationary stream of relativistic charged particles with a Lorentz factor $\gamma\gg1$ moving along a very strong curvilinear magnetic field. As their velocity is close to the light velocity, practically all particles are resonant, and the inhomogeneity of the medium affects most strongly just the resonant particles. Therefore, the waves in such a plasma are most strongly exposed to the effect of the inhomogeneity. The inhomogeneity is caused by the curvature of the magnetic field lines when the anisotropy vector of the plasma is directed along the magnetic field. Moreover, the problem about the emission of electromagnetic waves by such a plasma is of principal interest for the understanding of the mechanism of the radio-emission of pulsars.

Moving along a curved trajectory, the charged particles emit electromagnetic waves. This radiation, called curvature radiation, has a characteristic frequency $\omega \approx c\gamma^3/\rho$ (ρ is the radius of curvature of the magnetic field line) and is directed in a narrow cone of aperture $\theta \approx 1/\gamma$ along the direction of the particle motion. To describe this radiation, it is of principal importance to take the inhomogeneity into account.

It was shown in Ref. 5 that a plasma moving along a curvilinear magnetic field is unstable to excitations of so-called curvature-plasma oscillations. The instability growth rate, which has a hydrodynamic character, is so large that one must take into account the nonlinear effects for both particle-wave and wave-wave interactions.

It is convenient to introduce in each point in space a set of coordinates fixed to the field lines. The vector **b** is directed along the magnetic field (the z-axis), **n** is the normal vector (the x-axis), and **l** the binormal vector (the y-axis). It is convenient to introduce the parameter $p_\parallel$ which is the particle momentum along the vector **b**. We write the particle distribution function in the form

$$f(\mathbf{p},\mathbf{r}) = \int F(p_\parallel)\,\delta\left[\mathbf{p}-p_\parallel\mathbf{b}(\mathbf{r}) - \frac{v_\parallel}{\rho\omega_B} p_\parallel \mathbf{l}(\mathbf{r})\right] dp_\parallel. \tag{27}$$

The last term in (27) describes the drift of the particles when they move along the trajectory ($\omega_B = eB/mc\gamma$ is the cyclotron frequency). The magnetic field is so strong that the angle between the direction of motion of the particle and the direction of the magnetic field, when we take drift into account, is less than the quantity γ^{-1} which characterizes the directivity of the emission. We can thus neglect the drift term in (27). Moreover, the trajectories can be assumed to be planar, since the radius of torsion of the magnetic-field line is much larger than its radius of curvature. We can also neglect the change in the radius of curvature along the magnetic field line if $\mathbf{b}d\rho/d\mathbf{r} \ll \gamma$, as is the case.

It is necessary for what follows to know the particle trajectory:

$$\mathbf{r}-\mathbf{r}' = \mathbf{b}v_\parallel(t-t') - \mathbf{n}v_\parallel^2(t-t')^2/2\rho - \mathbf{b}v_\parallel^3(t-t')^3/6\rho^2 + \dots . \tag{28}$$

We need here not the whole trajectory, but only that part of it near **r** on which the radiation is produced. We have therefore carried out in (28) an expansion in the ratio of the production length ρ/γ to the inhomogeneity length ρ, i.e., in the quantity $\gamma^{-1} \ll 1$. Since

$$\begin{aligned}\mathbf{b}(\mathbf{r}+\mathbf{R}^*/2) &= \mathbf{b}(\mathbf{r}) + \mathbf{n}(\mathbf{b}\mathbf{R}^*)/2\rho - \mathbf{b}(\mathbf{b}\mathbf{R}^*)^2/8\rho^2 + \dots,\\ \mathbf{n}(\mathbf{r}+\mathbf{R}^*/2) &= \mathbf{n}(\mathbf{r}) - \mathbf{b}(\mathbf{b}\mathbf{R}^*)/2\rho + \dots,\end{aligned} \tag{29}$$

the quantity $\mathbf{R}^*$ in Eqs. (7) and (19) is equal to

$$\mathbf{R}^* = \mathbf{b}v_\parallel(t-t') - \mathbf{b}v_\parallel^3(t-t')^3/24\rho^2. \tag{30}$$

With the same accuracy

$$\det\left|\delta_{\mu\nu} - \frac{\partial\lambda_\mu(\mathbf{r}+\mathbf{R}^*/2)}{2\partial r_\nu}\right| = 1.$$

Here and elsewhere $v_\parallel$ is the longitudinal particle velocity and is a function of $p_\parallel$:

$$v_\parallel = cp_\parallel/(m^2c^2+p_\parallel^2)^{1/2}.$$

The cyclotron frequency ω_B is much larger than the frequency ω of the observed radiation. In such a plasma, the oscillations considered are thus the low-frequency limit for which $\omega_B \to \infty$. This means that the possible cyclotron rotation is instantaneously forgotten and the quantity $\partial f'/\partial \mathbf{p}'$ equals

$$\begin{aligned}\frac{\partial f}{\partial p_\nu'} &= \int \frac{\partial F(p_\parallel)}{\partial p_\parallel}\frac{p_\nu'}{p_\parallel}\,\delta(\mathbf{p}-p_\parallel\mathbf{b})\,dp_\parallel,\\ \mathbf{p}' &= p_\parallel[\mathbf{b} - \mathbf{n}v_\parallel(t-t')/\rho - \mathbf{b}v_\parallel^2(t-t')^2/2\rho^2].\end{aligned} \tag{31}$$

Substituting (29)–(31) into Eq. (6) we find the magnitude of the permittivity of the plasma:[1]

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta} - \frac{4\pi ie^2}{\omega m}\int dp_\parallel\, mv_\parallel \frac{\partial F}{\partial p_\parallel}\int_0^\infty d\tau\,\exp\left\{i[\omega - v_\parallel\mathbf{q}\mathbf{b}]\tau + \frac{i}{24}\frac{v_\parallel^3}{\rho^2}\tau^3(\mathbf{q}\mathbf{b})\right\}\left[b_\alpha b_\beta + (n_\alpha b_\beta - n_\beta b_\alpha)\frac{v_\parallel}{2\rho}\tau - n_\alpha n_\beta\frac{v_\parallel^2}{4\rho^2}\tau^2\right]. \tag{32}$$

It follows from (32) that in an inhomogeneous field there exists in the limit $\omega \ll \omega_B$ a response not only to the longitudinal electric field (along the magnetic field), but also to the field at right angles to the external field. This is connected with the nonlocal nature of the particle-wave interaction when the response at a given point **r** is determined also by its vicinity where the electrical field has a nonvanishing component along the magnetic field direction.

We get similarly from the quasi-linear Eq. (19) an equation for the evolution of the longitudinal distribution function $F(p_\parallel)$:

$$\begin{aligned}\frac{dF_\parallel}{dt} = \frac{e^2}{4}\sum_{\mathbf{q}} E_\mu^{0*}(\mathbf{q})E_\lambda^0(\mathbf{q})\int_0^\infty d\tau\left(b_\mu + n_\mu\frac{v_\parallel}{2\rho}\tau\right)\\ \times\frac{\partial}{\partial p_\parallel}\left\{\exp\left[i(\omega - v_\parallel\mathbf{q}\mathbf{b})\tau + i\mathbf{q}\mathbf{b}\frac{v_\parallel^3}{24\rho^2}\tau^3\right]\right.\\ \left.\left(b_\lambda - n_\lambda\frac{v_\parallel}{2\rho}\tau\right)\frac{\partial F_\parallel}{\partial p_\parallel}\right\} + \text{cc}.\end{aligned} \tag{33}$$

Since the emission of waves occurs along the direction of the particle motion with a small angular spread $\Delta\theta \sim 1/\gamma$, the distribution function $f(\mathbf{p},\mathbf{r},t)$ also acquires a spread in transverse momenta p_x, but it is small, $p_x/p_\parallel \sim \gamma^{-1}$, and in the derivation of Eq. (33) we integrated over the transverse momenta. As in (32), the evolution of the distribution function is determined by fields with a component both along the magnetic field and at right angles to it—along the normal vector **n**.

We now find the magnitude of the nonlinear conductivity $\sigma_{N\alpha\sigma\lambda}$ of (20). Using the same Eqs. (29)–(31) we get

$$\begin{aligned}\hat{M}_s(\omega,\mathbf{q}) = -\frac{e}{2}\int_0^\infty d\tau\int_{-\infty}^\infty dp_\parallel\exp\left[i(\omega - v_\parallel\mathbf{q}\mathbf{b})\tau + i\mathbf{q}\mathbf{b}\frac{v_\parallel^3}{24\rho^2}\tau^3\right]\\ \times\left(b_s - n_s\frac{v_\parallel}{2\rho}\tau\right)\delta\left(\mathbf{p} - p_\parallel\mathbf{b} - \mathbf{n}p_\parallel\frac{v_\parallel}{2\rho}\tau\right)\frac{\partial}{\partial p_\parallel}.\end{aligned} \tag{34}$$

Substituting (34) into (20) we find

$$\sigma_{N\alpha\sigma\lambda}=\frac{e^3}{4}\int_{-\infty}^{\infty} dp_\parallel \int\int_0^{\infty} d\tau\, d\tau_1\, v_\parallel \left[b_\alpha+n_\alpha \frac{v_\parallel}{2\rho}(\tau+\tau_1)\right]$$

$$\left[b_\sigma-n_\sigma \frac{v_\parallel}{2\rho}(\tau_1-\tau)\right]$$

$$\times \exp[i(\omega_2+\omega_3)\tau_1-i(\mathbf{q}_2+\mathbf{q}_3)\mathbf{R}_1^\cdot]\frac{\partial}{\partial p_\parallel}\left\{\left[b_\lambda-n_\lambda\frac{v_\parallel}{2\rho}(\tau-\tau_1)\right]\cdot\right.$$

$$\left.\times[\exp(i(\omega_3\tau-\mathbf{q}_3\mathbf{R}^\cdot))+\exp(i(\omega_2\tau-\mathbf{q}_2\mathbf{R}^\cdot))]\frac{\partial F}{\partial p_\parallel}\right\},$$

$$\mathbf{R}^\cdot_1=\mathbf{b}(v_\parallel\tau_1-v_\parallel^3\tau_1^3/24\rho^2), \qquad (35)$$

$$\mathbf{R}^\cdot=(\mathbf{b}+\mathbf{n}v_\parallel\tau_1/2\rho-\mathbf{b}v_\parallel^2\tau_1^2/8\rho^2)v_\parallel\tau-\mathbf{b}v_\parallel^3\tau^3/24\rho^2.$$

We note that Eqs. (32), (33), (35) contain instead of the dispersion function $1/(\omega-q_\parallel v_\parallel+i0)$, which corresponds to a uniform plasma, a function which can be expressed in terms of the integral

$$-\frac{i}{\pi}\int_0^{\infty}\exp\left[i\xi\tau+\frac{i\tau^3}{3}\right]d\tau,\qquad \xi=2(\omega-q_\parallel v_\parallel)\frac{\rho^{2/3}}{q_\parallel^{1/3}v_\parallel},$$

the imaginary part of which is the Airy function $\mathrm{Ai}(\xi)$ which describes the emission and absorption process of electromagnetic waves in a "curved" magnetic field. In vacuo when $\omega=qc$, the argument ξ of the Airy function becomes equal to

$$\xi=\left(\frac{\rho\omega}{c\gamma^3}\right)^{2/3}[1+\gamma^2\theta^2]>0,\qquad \cos\theta=\frac{\mathbf{qv}}{qv_\parallel},\qquad \gamma\gg1,$$

which corresponds to the usual curvature radiation. The case $\xi=0$ is possible when the refractive index n of the waves becomes larger than unity: $n>1+2\gamma^2$ which corresponds to the simultaneous existence of the curvature and Cherenkov emission mechanisms. It is just then that the emission of electromagnetic waves is most efficient, so that a new oscillation mode which does not exist in a uniform field appears, named by the curvature-plasma mode.[6] The instability growth rate is so large that it is necessary to take nonlinear effects into account. Analysis shows that saturation is due not to the quasi-linear relaxation of the charged-particle distribution function, but to three-wave decay interactions when the waves are removed from the narrow cone of angles $\theta\sim1/\gamma$ where their emission occurs. However, all these problems belong to the theory of the radio-emission of pulsars and are outside the scope of the present paper.

The consideration of an inhomogeneous medium given here can easily be extended to nonstationary processes.

The author is grateful to V. S. Beskin and A. V. Gurevich for fruitful discussions.

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Translated by D. ter Haar