What energy flux is carried away by the Kolmogorov weak turbulence spectrum?

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(Submitted 9 September 1987)
Zh. Eksp. Teor. Fiz. **94**, 172–179 (July 1988)

For media with a decreasing dispersion law it is shown that the matching of the Kolmogorov weak turbulence spectrum, realized at large **k**, to a spectrally narrow source situated at small values of **k** is realized in terms of an intermediate solution. This solution has the form of a chain of peaks of decreasing amplitude on a background which decreases more slowly. The dependence of the energy flux carried off by the turbulence spectrum on the position of the source in **k**-space is found.

Apparently, one of the most important and simplest to formulate questions which the theory of turbulence must answer is the question of how, knowing the amplitude and spectral characteristics of the energy source, one can determine the power absorbed by the system. In this paper we consider weak wave turbulence, when the evolution of the occupation numbers $n_{\mathbf{k}}$ of the plane wave states is described by kinetic equations, and the role of the sources is played by the growth rate $\gamma_{\mathbf{k}}$ of some instability:

$$\partial n_{\mathbf{k}}/\partial t-\mathrm{St}\{n_{\mathbf{k}},\ n_{\mathbf{k}'}\}=\gamma_{\mathbf{k}}n_{\mathbf{k}}-\Gamma_{\mathbf{k}}n_{\mathbf{k}}. \tag{1}$$

Here $\mathrm{St}\{n_{\mathbf{k}},n_{\mathbf{k}'}\}$ is the collision term describing the wave-wave interactions, and $\Gamma_{\mathbf{k}}$ is the wave damping decrement playing the role of an energy sink. We discuss the traditional Kolmogorov situation, when the scales of the excited waves and of the effectively damped waves are substantially different, and energy flows from the source to the sink in **k**-space.

Stationary solutions of Eq. (1) which realize a constant energy flow P were first constructed by Zakharov[1] using as an example of media exhibiting scale invariance. For a three-wave collision term

$$\begin{aligned}\mathrm{St}\{n_{\mathbf{k}}, n_{\mathbf{k}'}\}=&\int |V_{\mathbf{k}12}|^2\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,\delta(\omega_k-\omega_1-\omega_2)\\ &\times(n_1n_2-n_{\mathbf{k}}n_1-n_{\mathbf{k}}n_2)\,d\mathbf{k}_1\,d\mathbf{k}_2\\ &-2\int |V_{1\mathbf{k}2}|^2\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)\,\delta(\omega_1-\omega_k-\omega_2)\\ &\times(n_{\mathbf{k}}n_2-n_1n_{\mathbf{k}}-n_1n_2)\,d\mathbf{k}_1\,d\mathbf{k}_2\end{aligned} \tag{2}$$

the stationary Kolmogorov solution has the form

$$n_{\mathbf{k}}^0=bP^{1/2}k^{-m-d}. \tag{3}$$

Here d is the dimensionality of **k**-space, m is the homogeneity (scaling) index of the matrix element of the interaction ($V_{\lambda\mathbf{k}\lambda\mathbf{k}'\lambda\mathbf{k}''}=\lambda^m V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}$), b is a dimensionless constant of the order of unity, depending on m and on the dispersion law $a(\omega_{\lambda\mathbf{k}}=\lambda^\alpha\omega_{\mathbf{k}}$, $b=b(\alpha,m)$.

The energy flux carried off by the Kolomogorov spectrum is usually determined from the following considerations (Ref. 1). Let the source $\gamma_{\mathbf{k}}$ be nonzero in a small neighborhood of width Δk near the point $k=k_0$, i.e., is narrow compared to k ($\Delta k\ll k_0$). The energy flow into the medium due to the pumping ($\gamma=\int\gamma_{\mathbf{k}}\,d\mathbf{k}$)

$$P_k=\int_0^k\gamma_{\mathbf{k}}\omega_{\mathbf{k}}n_{\mathbf{k}}\,d\mathbf{k}\approx\gamma\omega_{k_0}n_{k_0}k_0^{\,d-1} \tag{4a}$$

is equated to the energy flux into the region of large k, determined by the matching conditions:

$$P_{k_0}\approx n_{k_0}{}^2k_0^{2(m+d)}b^{-2}. \tag{4b}$$

Eliminating $n_{\mathbf{k}_0}$ from (4) we obtain ($h=\alpha-m$) (Refs 1,2)

$$P\approx\gamma^2\omega_{k_0}{}^2k_0{}^{-2(m+1)}b^2\infty\gamma^2k_0{}^{2(h-1)}. \tag{5}$$

As we shall show in the present paper, the dependence (5) of the energy flux P on the position of the pumping point k_0 is incorrect. The reason is that the matching condition (4b) is based on the assumption that the occupation numbers n_k depend smoothly on k in the region $k\approx k_0$. In the case of a narrow source, when γ_k is a peaked function of the magnitude of the wave vector, the validity of this assumption is by no means obvious. One can use the estimate (4b) for $\Delta k\approx k_0$, but then (4a) needs to be replaced by $P\approx\gamma_{k_0}\omega_{k_0}n_{k_0}k_0^d$, and in place of (5) we obtain

$$P\approx\gamma_{k_0}{}^2\omega_{k_0}{}^2k_0{}^{-2m}b^2\infty\gamma_{k_0}{}^2k_0{}^{2h}. \tag{6}$$

The quantity h plays an important role in the theory of the weak-turbulence Kolmogorov spectra (see Refs. 1–3). In particular, the sign of h determines the position of the energy-containing region of the spectrum (large or small k), since the turbulence energy is

$$E=\int\omega_kn_k{}^0\,d\mathbf{k}\propto\int k^{h-1}\,dk.$$

For $h<0$ the energy integral diverges at small values of k, and the region containing the energy will be near the source $k\approx k_0$. For $h>0$ the bulk of the energy is concentrated in the region $k\approx k_d$, where k_d corresponds to the sink $\Gamma_{\mathbf{k}}$.

As we shall show presently, the dependence (6), $P\propto K_0^{2h}$, also holds in the case of a spectrally narrow source, and is related to the fact that for $k\lesssim k_0$ the spectrum is by no means a monotonically decreasing function of k. We note that the energy flows (5) and (6) differ by the factor $(k_0/\Delta k)^2$. For a narrow source this parameter is large and Eq. (5) is useless even for a rough estimate of the flow.

We consider isotropic acoustic turbulence ($\omega_{\mathbf{k}}=vk(1+a^2k^2)$, $ak_d\ll1$; $|V_{\mathbf{k}\mathbf{k}'\mathbf{k}''}|^2\propto kk'k''$) (Refs. 1–5). In the limit $ak\ll1$, after averaging over angles, the collision term (2) takes the form (up to multiplication by a constant)

$$\mathrm{St}\{n_k, n_{k'}\}\infty\int_0^k k_1{}^c(k-k_1)^c(n_1n_2-n_kn_1-n_kn_2)\,dk_1$$

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TABLE I

d	k_0	Δk	j	$\bar{s}(j)$ *	d	k_0	Δk	j	$\bar{s}(j)$ *
2	40	8	1	3.26	3	20	4	1	4.88
			2	2.96				2	4.48
			3	2.88				3	4.31
			4	2.89				4	4.23
			5	2.98				5	4.24
			6	3.19				6	4.37
			7	3.65				7	4.73
			8	4.74				8	5.63

* $\bar{s}(j) = -\ln\left(\frac{N_{j+1}}{N}\right)/\ln\left(\frac{j+1}{j}\right)$.

$$-2\int_k^\infty k_1{}^c(k_1-k)^c(n_kn_2-n_kn_1-n_1n_2)\,dk_1, \tag{7}$$

where $n_1 = n_{k_1}$ and $n_2 = n_{k-k_1}$ (Refs. 1,3). For three-dimensional sound ($d=3$) one must assume $c=2$, and the stationary Kolmogorov solution equals $n_k^{(3)} \propto k^{-9/2}$ (Refs. 1,5). In the case $d=2$ (capillary waves on the surface of shallow water) $c=1$ and $n_k^{(2)} \propto k^{-3}$. This solution was first found by Musher, who also observed its existence in a numerical experiment (Ref. 6). The last formula differs from Eq. (3) because the average of a two-dimensional delta function in the wave vectors over the angle is proportional to $(ak)^{-1}$ (Ref. 5). This circumstance leads to the necessity of assuming here $m=1$ (for details see Ref. 3). Thus, $h=\alpha-m=-\frac{1}{2}$ for three-dimensional sound whereas $h=0$ for the two-dimensional case.

We start with the case $d=2$. If a source which is narrow compared to k generates a narrow wave-number peak n_k around $k=k_0$ then, on account of the properties of the three-wave interaction, this leads to the appearance of peaks at $k=k_j=jk_0$, j being a natural number. We first consider the interaction of the peaks with each other. Then, to first order in $\Delta k/k_0$, we obtain from Eq. (7) the following equation at $k=k_j$:

$$k_0^{-2}\frac{\partial N_j}{\partial t}=\sum_{l=1}^{j} l(j-l)\,[N_lN_{j-l}-N_j(N_{j-l}+N_l)]$$
$$-2\sum_{l=j}^{\infty} l(l-j)\,[N_jN_{l-j}-N_l(N_{l-j}+N_j)]+\gamma\Delta_{j1}. \tag{8}$$

Here N_j is the number of waves in the jth peak

$$N_j=\int_{k_j-\Delta k}^{k_j+\Delta k} n_k\,dk.$$

Equation (8) has a stationary solution of the Kolmogorov type, with asymptote $N_j \propto j^{-3}$ for $j \gg 1$ (see Ref. 4). It is understood that within the framework of Eqs. (1) and (7) such a solution does not have external stability: waves outside the peaks ($k \neq kj$) must also be generated. It is nevertheless possible that the occupation numbers n_k for $k \neq k_j$ grow to values which are substantially smaller than the amplitudes of the peaks, i.e., the spectrum which is established represents a chain of peaks on a low background. If such a solution exists, its properties can be found by perturbation theory in the small parameter $\Delta k/k_0$.

We write down the kinetic equation for $k \neq k_j$, considering to first order the interaction with a chain of peaks ($n=[k/k_0]$ is the integer part of k/k_0):

$$\frac{\partial n_k}{\partial t}=-4n_k\sum_{j=1}^{\infty} kk_jN_j+2\sum_{j=1}^{n} k_j(k-k_j)N_jn_{k-k_j}$$
$$+2\sum_{j=n}^{\infty} k_j(k_j-k)N_jn_{k_j-k}+2\sum_{j=1}^{\infty} k_j(k_j+k)N_jn_{k_j+k}. \tag{9}$$

The equation (9) has the stationary solution

$$n_k=g(k)k^{-1}, \tag{10}$$

where $g(k)$ is an even periodic function of period k_0. Thus, the background of the spectrum must decrease with increase of k slower than the amplitude of the peaks. As a consequence, the solution under discussion is realized at an intermediate scale interval $k_0 < k \ll k_m$, where k_m is determined by the ratio of the amplitude of the first peak to the amplitude of the background.

$$k_m{}^2 \approx 2N_1/[\Delta k_1 n(k_0/2)].$$

In the region $k > k_m$ the usual monotonically decreasing Kolmogorov spectrum $n_k \propto k^{-3}$ must be realized.

The presence of the background in the spectrum leads to the appearance of an additional damping for waves with $k=k_j$, with a decrement

$$4k_jk_m\bar{g}=4k_0{}^{-1}k_jk_m\int_0^{k_0} g(k)\,dk,$$

on account of which, as k_j increases, the chain of peaks deviates more and more from the Kolmogorov law (there is a steeper fall-off, see Table I below). Considering the next orders in perturbation theory and taking into account terms which are quadratic in the amplitude of the background, one can derive other properties of the solution under discussion (deviations from the Kolmogorov law $n_k \propto j^{-3}$, fine structure of the peaks, etc.).

It is curious to note when two-dimensional acoustic turbulence is excited by an external force rather then by the increment, i.e., when the right-hand side of Eq. (1) contains a term F_k in place of $\gamma_k n_k$, the stationary solution can be obtained analytically in closed form:

$$n_k=\frac{1}{k}\int_{-\infty}^{\infty} e^{-i\omega k}\{[F(0)]^{1/2}-[F(0)-F(\omega)]^{1/2}\}\,d\omega,$$

$$F(\omega)=\int_0^\infty k^{-1}e^{i\omega k}F_k\,dk. \qquad (11)$$

Setting $F_k = \delta(k-k_0)$ we obtain from Eq. (11) a solution in the form of a chain of peaks, decreasing according to the Kolmogorov law:

$$n_k=\frac{1}{k}\sum_{m=1}^\infty \frac{\delta(k-mk_0)}{4m^2-1}.$$

Incidentally, the energy flow is obtained in this case trivially:

$$P_k=\int_0^\infty k^2F_k\,dk$$

and does not depend on n_k. The considerations about the structure of the spectrum stated above were verified by a numerical simulation of Eq. (1) with the collision term (7) ($c=1$) and a narrow source of the form

$$\gamma_k=\gamma\exp\{-[(k-k_0)/\Delta k]^2\}.$$

To ensure the existence of an energy sink in the region of large k, we have assumed than $n_k\equiv 0$ for $k>k_d$ (see also Refs. 2 and 4). The equation was solved on discrete grids of size $N=200$ and 400 points in k. The initial state was chosen in the form $n_k(0)=\text{const}=10^{-5}$, $\gamma=100$. An explicit difference scheme was used, with first approximation of the order $O(\tau)$, which was stable under the condition that the time step was 100 times smaller than the reciprocal of the increment, i.e., $\gamma\tau<10^{-2}$. The calculation was carried out on the "conveyor" processor A-12, Ref. 7. One should note the anomalously large time intervals for the establishment of the spectra, which were of the order of hundreds of reciprocal increments.

We considered a solution as steady-state when the relative rate of change of the occupation numbers became smaller than 10^{-3} per unit of dimensionless time ($1/\gamma$). The transient time increased as the relative width $\Delta k/k_0$ of the source decreased: $t\approx 60$ for $\Delta k/k_0=\frac{2}{5}$, $t=70$ for $\Delta k/k_0=\frac{1}{5}$, $t\approx 100$ for $\Delta k/k_0=\frac{1}{10}$. The steady-state spectrum for $k_0=40$, $\Delta k=8$ is shown in Fig. 1. As can be seen, the amplitude of the first peak exceeds by two orders of magnitude the amplitude of the background for $k\approx k_0/2$. The first peak is substanially narrower than the source its half-width at a height which is e times smaller than the maximum is $\Delta k_1\approx 2$. The other peaks get wider according to the law $\Delta k_j\propto k_j$, see Fig. 2 (the dashed lines are straight). The number of waves in the jth period

$$N_j=\sum_{k_j-k_0/2}^{k_j+k_0/2} n_k$$

decreases approximately according to the Kolmogorov law $N_j\propto j^{-3}$ (See Table 1, which lists the values of the average spectral index $\bar{s}(j)$). On account of the two latter circumstances ($\Delta k_j\propto j$, $N_j\propto j^{-3}$) the peak amplitudes decrease according to the law $n(k_j)\approx N_j/\Delta k_j\propto j^{-4}$. In Fig. 1, the upper dashed line has a slope -4, and the lower slope -1 (in a logarithmic scale). It should be noted that the decay law $n_k\propto k^{-1}$ governs not only the amplitudes of the minima, but also the background as a whole, with the exception of a narrow neighborhood of the peaks. This can be seen by defining the quantity

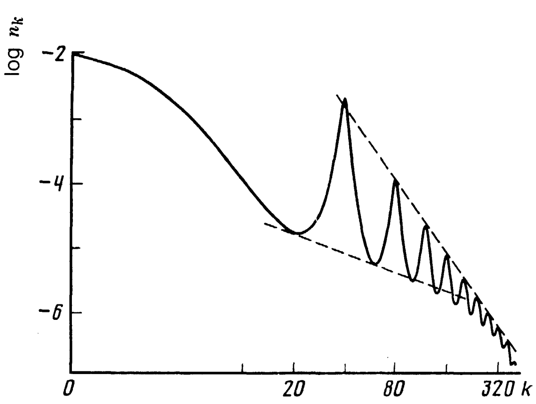

FIG. 1.

$$s_0(k)=-\frac{\ln(n_{k+k_0}/n_k)}{\ln[(k+k_0)/k]},$$

which, for instance, differs from unity by less than $\frac{1}{10}$ in the intervals $13\leqslant k\leqslant 28$ and $56\leqslant k\leqslant 65$ ($k_0=40$, $\Delta k=8$). Furthermore $s_0(k)$ increases, in agreement with the fact that the spectrum goes over into the smooth Kolmogorov solution. The decrease of the oscillations of n_k with increase of k and the transition to the smooth spectrum are illustrated in Fig. 3, which corresponds to $k_0=20$, $\Delta k=4$, $N=400$. The dashed straight line in this figure has the slope -3. For $k>240$ the influence of the energy sink already manifests itself and leads to a rapid decrease of the spectrum.

To verify that the structure of the spectrum observed in the numerical experiment does not owe its existence to insufficiently fine discretization of space, we have compared the results for two cases differing only in the mesh of the discretization ($\Delta k=4$, $k_0=20$, $N=200$ and $\Delta k=4$, $k_0=20$, $N=400$). The energy fluxes from the sources

$$P=\sum_{k=1}^{k_d} k\gamma_k\omega_k n_k,$$

for these two cases agreed within 2×10^{-3} in relative magnitude. The fine structure of the spectrum (ratio of occupation numbers at the minima and maxima, the mean index, etc). also agreed within several percent. In absolute magnitude the occupation numbers at corresponding points ($k\to 2k$)

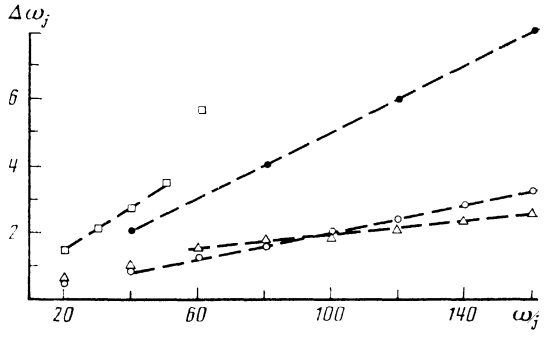

FIG. 2. The dependence of the widths of the peaks $\Delta\omega_j=[(n_{j-1}-2n_j+n_{j+1})/4n_j]^{-1/2}$ on their positions: ●—two-dimensional sound, $\omega_0=k_0=40$, $\Delta\omega=\Delta k=8$; ○—two-dimensional sound, $\omega_0=k_0=20$, $\Delta\omega=\Delta k=4$; △—three-dimensional sound, $\omega_0=k_0=20$, $\Delta\omega=\Delta k=4$; □—capillary waves on deep water, $\omega_0=10$, $\Delta\omega=2$.

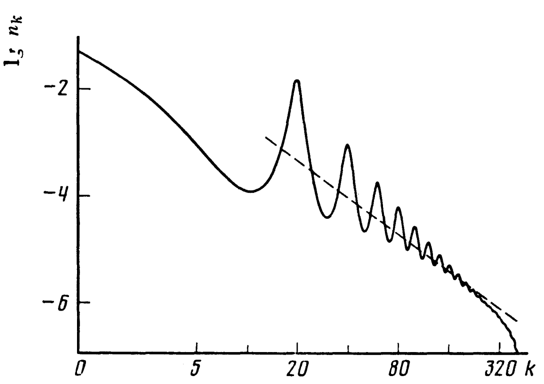

FIG. 3.

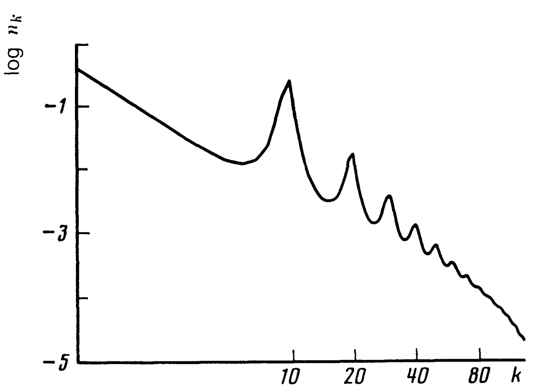

FIG. 5.

differ approximately by a factor of eight (in agreement with the scaling $n_{\lambda k} \to \lambda^q n_k$, $q = 2m + d - \alpha = 3$).

It is interesting to note that the energy flow decreases as the relative width of the source increases: $P = 1534$ for $\Delta k / k_0 = \frac{1}{10}$, $P = 1442$ for $\Delta k / k_0 = \frac{1}{5}$, and $P = 1362$ for $\Delta k / k_0 = \frac{2}{5}$. Thus, when the integrated strength of the source, $\gamma = \int \gamma_k \, dk$, increases the energy flow P decreases (cf. Eq. (5)). On the other hand, the energy flow is practically independent of k_0 (for constant $\Delta k / k_0$) to within one percent, owing to the discretization.

We now turn to three-dimensional sound. In this case one can also obtain a solution having the form of a chain of peaks: $N_j \propto j^{-9/2}$, $j \gg 1$; but now the equation for the background (the analog of Eq. (9)) does not admit of any power-law solutions. The results of the numerical solution of Eqs. (1) and (7) with $c = 2$ is shown in Fig. 4, which corresponds to $k_0 = 20$, $\Delta k = 4$; the dependence of the mean index on j is given in Table I. The number of waves in the jth period decreases aproximately according to the Kolmogorov law $N_j \propto j^{-9/2}$, the width of the peaks increases linearly with the label $\Delta k_j \propto k_j$, and the amplitudes of the peaks behave as $n(k_j) \propto j^{-11/2}$. The dashed straight line in Fig. 4 has the slope -5.5.

We note that for $k_0 = 1$ a monotonically decreasing spectrum is established, which for $1 < k \ll k_d$ is close to the solution $n_k \propto k^{-9/2}$. It should be stressed that stationary spectra having the form of chains of narrow peaks on a more slowly decreasing background occur not only in weak acoustic turbulence. It appears that this type of distributions should be generated by a spectrally narrow source in the arbitrary case of weak turbulence of waves with a scale-invariant decaying dispersion law. We have carried out the numerical modeling of weak turbulence of capillary waves over deep water ($\omega_k \propto k^{3/2}$, for the expression of the matrix element see Refs. 1,8). Solving the kinetic equation in ω-space (see also Ref. 8) on a discrete lattice with $N = 128$ and a source of width $\Delta\omega = 2$ placed at $\omega_0 = 10$ we have obtained a steady-state solution containing five well-pronounced narrow peaks at $\omega = j\omega_0, j = 1,...,5$. The sixths and seventh peaks are weak; for $\omega > 77$ the spectrum decreases monotonically as k increases, Fig. 5.

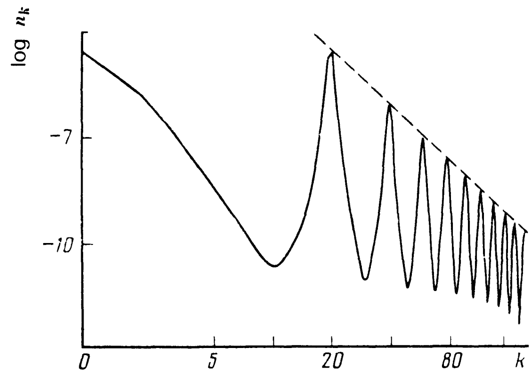

FIG. 4.

The number of waves N_j in the peaks decreases as j increases approximately according to the Kolmogorov law; in this case the Kolmogorov exponent is $s_0 = 17/6 \approx 2.83$ (Ref. 1). The widths of the peaks increase linearly with their order in the interval $k_0 \ll k_j \ll k_m$, just as for acoustic turbulence (see Fig. 2). Thus, in the transition interval $k_0 < k < k_m$ the amplitudes of the peaks decrease with their order according to a power law, with exponent (in ω-space is larger by one than the Kolmogorov exponent (see Table II, which lists the exponent $s_0(\omega)$).

Having an idea about the structure of the solution generated by a spectrally narrow source, we return to the energy flow question posed at the beginning of the article. In the presence of a narrow peak the estimate (4a) retains its form, all that is needed is to substitute N_1 for n_{k_0}:

$$P \approx \gamma_{k_0} \omega_{k_0} N_1 k_0^{d-1}. \tag{12}$$

And now in place of (4b) we obtain

$$N_j \approx P^{1/2} k_0 / k_j^{m+d}, \quad N_1 \approx P^{1/2} / k_0^{m+d-1}. \tag{13}$$

Substituting Eq. (13) into (12) we obtain Eq. (6). Indeed, it was already mentioned that for the two-dimensional case the energy flow does not depend on $k_0 (h = 0)$, according to the results of the numerical simulation. In the three-dimensional case, the dependence is close to a reciprocal: $P \propto k_0^{-1}$

TABLE II

ω_0	$\Delta\omega$	ω	$s_0(\omega)$ *
10	2	10	3.85
		20	3.91
		30	3.81
		40	3.58
		50	3.36

* $s_0(\omega) = \ln\left(\frac{n_\omega}{n_{\omega+\omega_0}}\right) \Big/ \ln\left(\frac{\omega+\omega_0}{\omega}\right)$.

($k_0 = 1$, $P \approx 713$; $k_0 = 10$, $P \approx 71.7$; $k_0 = 20$, $P \approx 34.3$) as it should for $h = -\frac{1}{2}$. For capillary waves we have $\omega_0 = 1$, $P \approx 110$; $\omega_0 = 10$, $P \approx 8$; $\omega_0 = 20$, $P \approx 4.5$, which approximately corresponds to a reciprocal dependence of the energy flow on the frequency:

$$P \propto k_0^{2h} = k_0^{-3/2} = \omega_0^{-1}$$

($h = {}^3/_2 - {}^9/_4 = -{}^3/_4$, see Ref. 1.)·

Thus, independently of the spectral width of the source, the energy flow carried off by the Kolmogorov spectrum of decaying turbulence is proportional to k_0^{2h}, i.e., it increases as the source is moved into the energy-containing region.

We are grateful to V. E. Zakharov and V. S. L'vov for useful discussions of the problems touched upon here.

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Translated by M. E. Mayer