The stability of nonhomogeneous states and path integrals

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It is shown that the Green's function of the system of linear differential equations which describe the behavior of small disturbances near a stationary state of a physical system can be represented as a path integral. The use of the asymptotic behavior of the Green's function for large times yields an effective method for the determination of the most stable eigenvalues in terms of the local dispersion equation.

INTRODUCTION

The investigation of the stability of stationary states plays an important role in plasma physics, in fluid dynamics, and in other disciplines of microscopic physics. Such investigations reduce to the determination of the eigenvalues for some system of linear equations, which is usually a quite complicated mathematical problem, which can be solved only by means of numerical methods. At the same time it is natural to assume that weakly inhomogeneous states must be amenable to a treatment similar to the WKB method for the Schrödinger equation. Such investigations were carried out in Ref. 1 for the case when the problem differs little from the self-adjoint problem, and were somewhat generalized in Ref. 2 (for the case of a second-order differential equation). These authors started from the construction of an approximate eigenfunction by means of the WKB method, and only after that they determined the corresponding eigenvalues. However, it is well known that the construction of the wave functions is considerably more complicated than the determination of the eigenvalues, and moreover in stability problems it is in fact important only to determine the most unstable eigenvalue.

In the present paper we shall not try to construct the eigenfunctions, but instead, as is customary in quantum field theory, we shall start out from some general expression for the Green's function $G(\mathbf{x},\mathbf{x},'t-t')$ of the system of linearized equations which describe the evolution of small deviations from the equilibrium position. We shall investigate the limit of $G(\mathbf{x},\mathbf{x},'t)$ as $t\to\infty$ and will indicate how to determine in the exponential asymptotic behavior the appropriate exponents which characterize the growth of the most unstable mode.

1. EXPRESSION OF THE GREEN'S FUNCTION AS A PATH INTEGRAL

For definiteness we shall assume that the system of linearized equations is a system of partial differential equations for the vector-values function ψ, which describes the deviation from the stationary state, is of the form:

$$\partial\psi/\partial t+\hat{L}(\mathbf{x},\partial/\partial\mathbf{x})\psi=0, \tag{1.1}$$

where the matrix $\hat{L}$ is a polynomial in $\partial/\partial x$ and is analytic in $\mathbf{x}$. We need to determine the retarded (matrix) Green's function

$$\hat{G}=\begin{cases}\hat{G}(\mathbf{x},\mathbf{x}',t-t'), & t>t'\\ 0, & t<t'\end{cases}$$
$$\partial G/\partial t+\hat{L}\hat{G}=\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')\hat{I} \tag{1.2}$$

(here $\hat{I}$ is the unit matrix). For this we note that for small values of $t-t'$ the Green's function must be close to $\delta(\mathbf{x}-\mathbf{x}')$. This means that in it the only important value of $\mathbf{x}$ are those close to $\mathbf{x}'$, and its Fourier components are concentrated essentially at large values of k. Therefore for small $t-t'$ the Green's function can be approximately represented by the Fourier integral

$$\hat{G}(\mathbf{x},\mathbf{x}',t-t')=\int\exp\{i\mathbf{k}(\mathbf{x}-\mathbf{x}')\}f_{\mathbf{k}}(\mathbf{x}',t-t')\frac{d^nk}{(2\pi)^n}, \tag{1.3}$$

which corresponds to the Green's function of Eq. (1.2) for frozen coefficients, i.e., coefficients evaluated at the point $\mathbf{x}=\mathbf{x}'$. Solving the corresponding equation with constant coefficients we find

$$G^{\alpha\beta}(\mathbf{x},\mathbf{x}',t-t')=\sum_j\int\exp\{i\mathbf{k}(\mathbf{x}-\mathbf{x}')-i\omega_j(\mathbf{k},\mathbf{x}')\times(t-t')\}u_j^{\alpha}(\mathbf{k},\mathbf{x}')\times\tilde{u}_j^{\beta}(\mathbf{k},\mathbf{x}')\frac{d^nk}{(2\pi)^n}, \tag{1.4}$$

where the frequencies ω_j are determined from the solution of the local dispersion equation

$$F(-i\omega,\ i\mathbf{k},\ \mathbf{x})=\det(-i\omega\delta^{\alpha\beta}+L^{\alpha\beta}(\mathbf{x}',\ i\mathbf{k}))=0, \tag{1.5}$$

and the local eigenvectors (the polarization vectors) satisfy the system of equations

$$i\omega_j u_j^{\alpha}=L^{\alpha\beta}(\mathbf{x}',\ i\mathbf{k})u_j^{\beta},\quad i\omega_j\tilde{u}_j^{\alpha}=L^{\beta\alpha}(\mathbf{x}',\ i\mathbf{k})\tilde{u}_j^{\beta},\ u_j^{\alpha}\tilde{u}_{j'}^{\alpha}=\delta_{jj'}. \tag{1.6}$$

The summation in Eq. (1.4) is over all the branches of the dispersion equation. The formula (1.3) represents the genuine Green's function more accurately the smaller the time interval $t-t'$.

If one is interested in the Green's function in the interval $(t'+\Delta t,t'+2\Delta t)$, one can repeat the operation. Considering the Green's function (1.3) as a new initial condition for the solution of Eq. (1.1) in the interval $(t'+\Delta t,t'+2\Delta t)$, and using the property of the Green's function that it represents the solution as a convolution with the initial data, we obtain

$$G^{\alpha\beta}(\mathbf{x},\mathbf{x}',t-t')=\sum_{j\beta_1}\int\frac{d^nk_1}{(2\pi)^n}\frac{d^nk_2}{(2\pi)^n}d^nx_1\,u_{j_2}^{\alpha}(\mathbf{k}_2,\mathbf{x}_1)\times\tilde{u}_{j_2}^{\beta_1}(\mathbf{k}_2,\mathbf{x}_1)u_{j_1}^{\beta_1}(\mathbf{k}_1,\mathbf{x}')u_{j_1}^{\beta}(\mathbf{k}_1,\mathbf{x}')\exp\{i\mathbf{k}_2(\mathbf{x}-\mathbf{x}_1)+i\mathbf{k}_1(\mathbf{x}_1-\mathbf{x}')-i\omega_{j_2}(t-\tau_1)-i\omega_{j_1}(\tau_1-t')\},$$

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where we have introduced one intermediate time τ_1 and one intermediate distance $\mathbf{x}_1$. Partitioning the finite interval $t-t'$ into a large number of small subintervals, and taking the limit, we obtain a representation of the Green's function as a functional integral with respect to $D\mathbf{k}(\tau)D\mathbf{x}(\tau)$. If we had only a scalar equation and, accordingly only one branch of the dispersion equation, we would get a representation of the Feynman type, but with the action written in the Hamiltonian representation rather than the more usual Lagrangian representation. In the case of several branches of the dispersion equation, the difference is that at each instant of time, in addition to the integration with respect to $D\mathbf{k}(\tau)D\mathbf{x}(\tau)$, one must also sum over all the branches, so that the trajectories are considered on the Riemann surface corresponding to the dispersion equation.

We shall not discuss the justification of the limiting process (as the number of partition points tends to infinity), hoping that (as happens in the case of the Wiener measure, corresponding to the transition from the Schrödinger equation to the dissipative diffusion equation) the dissipative terms with higher derivatives will effectively provide a cutoff for the integration over large k and make the limiting process possible. The Green's function will then be represented by the functional integral

$$\hat{G}=\lim_{\Delta t\to 0}\prod_{i=1}^{N}\sum_{j_i}\int\frac{d^nk_i}{(2\pi)^n}d^nx_i\exp\{iS_{ij_i}\}\hat{g}_{ij_i},$$

$$\Delta t=t_i-t_{i-1},\quad t_{N+1}=t,\quad t_0=t',$$

$$S_{ij_i}=\int_{\mathbf{x}_{i-1}}^{\mathbf{x}_i}\mathbf{k}_i\,d\mathbf{x}-\int_{t_{i-1}}^{t_i}\omega_{j_i}(\mathbf{k}_i,\mathbf{x}_{i-1})\,dt, \tag{1.7}$$

$$\hat{g}_{ij_i}^{\alpha\beta}=u_{j_i}^{\alpha}(\mathbf{k}_i,\mathbf{x}_{i-1})\,u_{j_i}^{\beta}(\mathbf{k}_i,\mathbf{x}_{i-1}).$$

2. THE SADDLE-POINT METHOD

The general expression (1.7) for the Green's function allows one to displace the original integration contour (the real axis) in each of the variables $\mathbf{k}_i=\mathbf{k}(t_i)$ and $\mathbf{x}_i=\mathbf{x}(t_i)$ where t_i is the running time on the integration path if the action S_{i,j_i} and the matrices $\hat{g}_{ij_i}$ are analytic functions of their arguments. By displacing the integration contour one can arrange things so that the integrand becomes a monotonically decreasing function in of the distance of each variable from the saddle point, in the same manner as it is done in the usual saddle-point method for integrals of a single complex variable. On account of the dispersion equation (1.5) there are branch points of $\omega(\mathbf{k},\mathbf{x})$ as well as branch points and other singularities in x due to the singularities in the quantities $L^{\alpha\beta}$, which are determined by the structure of the unperturbed stationary state. In this case one always chooses a definite "physical" branch of $L^{\alpha\beta}$, if these quantities are not single-values in the complex plane. The branch points of $\omega(\mathbf{k},\mathbf{x})$ from the dispersion equation are of little consequence, since all the branches are represented in the integral (1.7), and if in displacing the contour we cross a branch cut, this simply means going from one branch to the other. In the present section we shall assume that the coefficients $L^{\alpha\beta}$ are entire functions of $\mathbf{x}$, i.e., the have no singularities at any finite point of the complex x plane and are polynomials in k.

In order to find the saddle trajectories we must vary both $\mathbf{x}$ and $\mathbf{k}$. In doing this we assume that only a large magnitude of the action is important and that the matrix multipliers $\hat{g}_{ij_i}$ make an unimportant contribution to the determination of the saddle-point trajectory. A justification for this assumption can be found *a posteriori*.

As a result we are led to the complex Hamilton equations

$$d\mathbf{k}/dt=-\partial\omega_j/\partial\mathbf{x},\quad d\mathbf{x}/dt=\partial\omega_j/\partial\mathbf{k}; \tag{2.1}$$

the index j may be omitted if one considers the Riemann surface $F=0$ and considers the equations which determine the trajectory as local Hamilton equations on this surface.

We shall be interested in the asymptotic behavior of $G(\mathbf{x},\mathbf{x}',t)$ for large t, and therefore, the saddle-point trajectory must return after a real time t to a real point x. In the sequel we shall consider one degree of freedom, i.e., $\mathbf{x}$ and $\mathbf{k}$ are one-dimensional vectors. The reason for this restriction is that the classification of recurrent solutions can be quite complicated in the multidimensional case. In the simplest cases there is no difference of principle from the one-dimensional situation.

Hamilton's equations admit of an energy integral which completely determines the trajectories in the one-dimensional case: $\omega(k,x)=\omega=\text{const}$, and we must select this constant in such a way that for real t the trajectory should return to the real point x. There are two types of such trajectories: fixed points and periodic solutions.

A. Trajectories of the fixed-point type

In the simplest case the equations which determine a fixed point of the equation

$$\partial\omega/\partial k=\partial\omega/\partial x=0 \tag{2.2}$$

are satisfied for a real value x_0 and some complex k_0, and it is these values which determine the whole trajectory. In this case the only important points can be the ones for which e^{iS} decreases monotonically as one goes away from the point (x_0, k_0). It is easy to show that if k_0 is also real these points correspond to a maximum of $\operatorname{Im}\omega(k,x)$ on the real axis. To ensure the possibility of the deformation of the integration contour from the real axis to a contour with monotonic decrease of e^{iS}, we shall assume that the saddle points (x_0,k_0) can be obtained from points of the real axis by means of continuous deformation of the function $\omega(k,x)$. We note that the conditions (2.2) are the natural generalization to the nonhomogeneous case of the conditions for absolute instability (Ref. 3).

When x_0 and k_0 are displaced into the complex plane, the behavior of the trajectories in the vicinity of these points will be determined by the linear system of equations

$$\frac{d\delta k}{d\tau}=-\frac{\partial^2\omega}{\partial x\,\partial k}\delta k-\frac{\partial^2\omega}{\partial x^2}\delta x,\qquad \frac{d\delta x}{d\tau}=\frac{\partial^2\omega}{\partial x\,\partial k}\delta x+\frac{\partial^4\omega}{\partial x^2}\delta k$$

with the solutions

$$\delta k = C_1 k_+ e^{i\lambda t} + C_2 k_- e^{-i\lambda t},$$
$$\delta x = C_1 x_+ e^{i\lambda t} + C_2 x_- e^{-i\lambda t}, \tag{2.3}$$

where

$$\lambda^2 = -\left(\frac{\partial^2\omega}{\partial x\,\partial k}\right)^2 + \frac{\partial^2\omega}{\partial k^2}\frac{\partial^2\omega}{\partial x^2},$$

(k_+, x_+) and (k_-, x_-) are the eigenvectors, and C_1 and C_2 are arbitrary complex constants. In the general case λ^2 is not real, and therefore there exists one unstable and one stable complex direction. If the point (x_0, k_0) deviate little from the real axis, one may choose constants in such a manner that the trajectory should return to a real point x. For this it is necessary that

$$x - x_0 = C_1 x_+ + C_2 x_- = C_1 x_+ e^{\lambda t} + C_2 x_- e^{-\lambda t},$$

which can always be realized. Obviously, for $t \to +\infty$ the trajectory will spend more and more time near the fixed point and $\omega \to \omega(x_0, k_0)$. Obviously, such trajectories can exist also in the nonlinear approximation.

B. Periodic trajectories

Periodic trajectories must have a real period (t is real) defined by the integral

$$T = \oint dk \Big/ \frac{\partial\omega}{\partial x} = \oint dx \Big/ \frac{\partial\omega}{\partial k} \tag{2.4}$$

along a contour surrounding the branch points of the integrand. The dispersion equation $F(-i\omega, ik, x) = 0$ has real coefficients because the real functions must satisfy the original equations (1.1) and can be real solutions of Eq. (2.1) for real $\varkappa = ik$, $\Omega = -i\omega$ and for real x. In this case the expressions (2.4) will also be real, so that, in general, solutions with a real period exist. Assume that we have a solution with the real period $T(\omega)$ on some curve $\omega(\sigma)$ (σ is a parameter) in the complex plane. We consider solutions for σ_1 and σ_2 such that $T(\omega(\sigma_1))/T(\omega(\sigma_2)) = r_2/r_1$ (r_1 and r_2 are real numbers), and we compare $S_1(x,x,t)$ and $S_2(x,x,t)$ after a time interval $t = r_1 T_1 = r_2 T_2$:

$$i(S_1 - S_2) = ir_1 \oint k_1\, dx - i\omega_1 t - ir_2 \oint k_2\, dx + i\omega_2 t.$$

Assuming that

$$|r_2 - r_1| \ll r_1, \qquad T_1 = \oint \partial k/\partial\omega|_{\omega_1}\, dx,$$
$$i(k_1 - k_2) \approx \partial k/\partial\omega|_{\omega_1}(\omega_1 - \omega_2),$$

we obtain

$$i(S_1 - S_2) \approx i(r_1 - r_2)\oint k_1\, dx = \frac{i(T_2 - T_1)}{T_1} r_1 \oint k_1\, dx, \tag{2.5}$$

and if $\operatorname{Im}\oint k_1\, dx \neq 0$, which is generally true, it is advantageous from the point of view of decreasing Im S to move away from the curve $\omega(\sigma)$ all the way to the final point for which real-periodic solutions exist. Such points could be points where $T \to \infty$ or singular points of the oscillator type when, as the branch points approach each other without limit, the period in the integral (2.4) approaches a finite limit, and afterwards these points separate in the perpendicular direction and the period becomes imaginary. The limiting point obtained from the confluence of the branch points is a fixed point and the exponents λ in Eq. (2.3) must be purely real, which is a very special case for a complex fixed point. If the point corresponds to real x one can show, passing to the real variables k and Ω, that it corresponds to max Im S and therefore must be discarded.

If $\oint x dx$ is purely real for some $\omega_0 = \omega(\sigma_0)$, then ω_0 and the corresponding trajectory must be included among the competing saddle points. However, this case is quite complicated for the investigation of a possible displacement of the contours and rather difficult to realize (simple examples of the dispersion equation do not lead to such a situation).

The case of a real period which tends to infinity is possible only when one of the turning points goes to infinity and the whole trajectory gets longer and longer as t increases. This obviously corresponds to a continuous spectrum, which in our case is tied only to large values of x.

As was the case for the Schrödinger equation, such an approach, i.e., the determination of the steepest-descent trajectory, corresponds to the first term in an expansion with respect to a small parameter of "quasiclassicity," determined subsequent comparison with the following terms of this expansion. Generally speaking, even in the presence of such a small parameter, the zeroth term may be insufficient for the determination of stability of the original state. For example, one might have Im $\omega_0 = 0$, and then it becomes necessary to determine the terms of the following order.

The term of next order is determined by the quadratic terms in the action:

$$S - S_0 = \delta S = \int_0^t \delta k \frac{d\,\delta x}{dt} - \int_0^t \Big[\frac{1}{2}\frac{\partial^2\omega}{\partial x^2}(\delta x)^2 + \frac{\partial^2\omega}{\partial k\,\partial x}\delta x\,\delta k + \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}(\delta k)^2\Big] dt,$$

with all the derivatives and the action S_0 taken on the steepest descent trajectory. In order to carry out the corresponding Gaussian quadrature in the functional integral (1.7), it is necessary to reduce the quadratic form to principal axes, for which, however, it is necessary to transform the integral to the Lagrangian form, carrying out the Gaussian integration with respect to δk:

$$\delta S = -\int_0^t d\tau \Big[\Big(\frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\Big)^{1/2}\delta k + \Big(2\frac{\partial^2\omega}{\partial k^2}\Big)^{-1/2}\Big(\frac{\partial^2\omega}{\partial x\,\partial k}\delta x - \frac{d\,\delta x}{d\tau}\Big)\Big]^2 + \int_0^t d\tau\Big[\frac{1}{2\,\partial^2\omega/\partial k^2}\Big(\frac{\partial^2\omega}{\partial x\,\partial k}\delta x - \frac{d\,\delta x}{d\tau}\Big)^2 - \frac{1}{2}\frac{\partial^2\omega}{\partial x^2}(\delta x)^2\Big].$$

To reduce the Lagrangian quadratic form to its principal axes, we utilize the usual technique based on the extremal properties of the form, i.e., minimizing it with the additional condition that it be normalized

$$\frac{1}{2}\int_0^t (\delta x)^2\, d\tau = 1.$$

This leads to an eigenvalue problem of the Sturm-Liouville type:

$$\frac{d}{d\tau}\left[\frac{1}{\partial^2\omega/\partial k^2}\left(\frac{d\delta x_n}{d\tau}-\frac{\partial^2\omega}{\partial x\,\partial k}\delta x_n\right)\right]+\frac{\partial^2\omega}{\partial x^2}\delta x_n=\lambda_n\delta x_n. \tag{2.6}$$

The corresponding Gaussian quadrature for G is easily effected in terms of λ_n and we shall have

$$G(x,x,t)=J\prod_n\left(\frac{\lambda_n^0}{\lambda_n}\right)^{1/2}e^{iS_0}. \tag{2.7}$$

Here λ_n^0 are some normalizing factors such that the product should have a limit. Everything else is included in the factor J, including the Jacobian of the transformation from $x(\tau)$ to the a_n

$$\left(x(\tau)=\sum_n a_n x_n(\tau)\right),$$

as well as a quadrature with respect to δk. We select as normalizing factors λ_n those corresponding to

$$\partial^2\omega/\partial x^2=\partial^2\omega/\partial x\partial k=0,$$

which may be considered as the case of the free particle, since if $\partial^2\omega/\partial k^2\neq 0$ all along the steepest-descent trajectory, then by the substitution

$$\varphi=\int_0^\tau\frac{\partial^2\omega}{\partial k^2}d\tau$$

the functional δS reduces to the functional of a free particle of unit mass, and the factor J is easily computed. For the determination of the eigenvalues which characterize the instability, the quantity J does not play any role, since we are only interested in factors which are exponential in t. In order to determine them we note that the magnitude of the product $\Pi_n(\lambda_n^0/\lambda_n)$ determines the factors which are substantially different from unity. For a large time interval t the spectrum of the equation (2.6) is basically dense, with intervals of the order of $1/t$. Introducing the new variable $\nu=n/t$ we obtain

$$\Pi=\exp\sum_n\ln\frac{\lambda_n^0}{\lambda_n}\approx\exp t\int_0^\infty\ln\frac{\lambda^0(\nu)}{\lambda(\nu)}d\nu,$$

and thus

$$G\sim e^{iS_0}\exp\frac{t}{2}\int_0^\infty\ln\frac{\lambda^0(\nu)}{\lambda(\nu)}d\nu,$$

which determines the eigenvalue with the largest increment:

$$\omega=\omega^0+\frac{i}{2}\int_0^\infty\ln\frac{\lambda^0(\nu)}{\lambda(\nu)}d\nu. \tag{2.8}$$

It should be noted that, in the case of a periodic trajectory with real period tending to infinity, S_0 has a term $\oint p\,dq$ with nonvanishing imaginary part. The integral may have a finite limit for $t\to\infty$, corresponding to finite transparency (the amplified waves come in freely from infinity), or it may diverge for $t\to\infty$, when the amplified waves do not cross the barrier, which modifies somewhat the asymptotic behavior of $G(x,x,t)$.

To clarify the accuracy of the obtained formulas (determine the effective small parameter) it is necessary to do a perturbative calculation of the corrections to G due to fourth-order terms in δk and δx (the third order corrections vanish from parity considerations) and require that these terms be small. Roughly speaking, the effective expansion parameter is determined by the ratio of the second term to the first in Eq. (2.8). The calculation of the integral in (2.8) is quite complicated in the general case. In the case of trajectories of the fixed-point type, things become much simpler, since for large t the spectrum of (2.6) is determined by the vicinity of the fixed point where the steepest-descent trajectory spends most of the time. Equation (2.6) reduces to one with constant coefficients, i.e., to the usual oscillator equation. One can make direct use of the known result (Ref. 4):

$$\omega=\omega(k_0,x_0)+\frac{1}{2}\lambda,$$

where the oscillator frequency is

$$\lambda=\left[\frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{\partial k^2}-\left(\frac{\partial^2\omega}{\partial x\,\partial k}\right)^2\right]^{1/2}$$

(the branch of the root is fixed by the requirement $\operatorname{Im}\lambda<0$), so that the oscillator corrections decrease the increment. It is understood that in this case one can determine terms in the asymptotic behavior of the Green's function the exponents of which differ from $\omega(k_0,x_0)$ by $i\lambda(\frac{1}{2}+n)$, i.e., we have a discrete spectrum extending in the direction where $\operatorname{Im}\omega$ decreases.

To conclude this section we consider a simple example. Let the dispersion equation have the form $-i\omega=k^2(f(x)-\varepsilon^2k^2)$, corresponding to an instability in the interval $k^2<f/\varepsilon^2$ (ε is a small parameter). Assume that the function f converges to a constant limit as $t\to\pm\infty$ and attains its maximum on the real axis. The equations which determine the fixed point have a real solution $x=x_0$, $k=k_0$ corresponding to the maximum of f on the real axis and we have an unstable discrete spectrum starting at $-i\omega=f^2(x_0)4\varepsilon^2$ (we don't write out the oscillator corrections which are small in the parameter ε). If instead of a maximum the function f has only one minimum, then the fixed point corresponds to the minimum of iS on the real axis and must be discarded. To determine the periodic solutions we solve the dispersion equation with respect to k^2:

$$k^2=\frac{f}{2\varepsilon^2}\pm i\left(\frac{\Omega}{\varepsilon^2}-\frac{f^2}{4\varepsilon^4}\right)^{1/2}\qquad(\Omega=-i\omega).$$

For the sequel the branch points of k^2 are important, where $\Omega=f^2/4\varepsilon^2$. As to the point $k=0$, it has no bearing on instabilities. Computing the appropriate period

$$T=\oint\frac{dx}{2ik(f-2\varepsilon^2k^2)}$$
$$=\oint\left[\frac{f^2}{2\varepsilon^2}+i\left(\frac{\Omega}{\varepsilon^2}-\frac{f^2}{4\varepsilon^4}\right)^{1/2}\right]^{-1/2}\frac{dx}{(\Omega\varepsilon^2-f^2/4)^{1/2}},$$

we see that this quantity is real since on the opposite sides of the cut k^-k^*. The limiting value $\Omega=f_\infty/4\varepsilon^2$ corresponds to $T\to\infty$ and an absolutely unstable spectrum for $x\to\pm\infty$. It is easy to see that

$$\operatorname{Im}\oint k\,dx\neq 0,$$

so that, in agreement with the general reasoning, we must necessarily hit upon the limiting value of the real period $T(\omega)$. It is easy to verify that all the conditions that allow a displacement of the integration contour (in k) off the real axis are satisfied. The integral

$$-\int_{x}^{x_1} \operatorname{Im} k\, dx$$

at the turning point $x_1 \to \infty$ characterizes the transparency of the barrier and will diverge if f does not converge rapidly enough to f_∞. In this case the asymptotic behavior of the Green's function at the fixed point will not be a simple exponential $e^{-i\omega_0 t}$, but will contain an additional small factor related to the fact that the amplified waves do not penetrate easily back to the point. This factor can be enhanced if one chooses for each t a sufficiently large $x(t)$.

One should note that even if the subsequent terms in the expansion (1.7) yield corrections of the order of unity, the displacement of the integration contour in k and x into the complex plane may turn out to be useful. In view of the monotone decay of the function e^{iS} on the displaced contour one can, e.g., estimate G by means of the method of majorants, or make use of the Feynman variational principle (Ref. 4). Sometimes one can find a simpler effective equation near the fixed point, which replaces sufficiently accurately the original system (1.1).

3. BRANCH POINTS

In the presence of branch points in the coefficients $L^{\alpha\beta}$ as functions of x it may happen that when the integration contour is displaced from the real x axis (in the direction where $|e^{iS}|$ decreases) we encounter a branch point in x and the steepest-descent trajectory goes off onto another "unphysical" sheet of the Riemann surface of the coefficients $L^{\alpha\beta}$. In this case, as happens for usual contour integrals, the value of G (in the presence of an appropriate large parameter) will be determined directly by the branch point. At the same time the extremals of the corresponding action in the expression for the function G will satisfy the condition that the first variations with respect to k vanish, a condition which, on account of the fact that $x = x_*$, reduces to the equalities:

$$\partial x/\partial \tau = \partial \omega/\partial k|_{x_*, k_*} = 0.$$

Thus, in this case the extermal trajectory consists of a portion described by the complete Hamilton equations, from the point x to the point x_* where at the same time the velocity dx/dt vanishes. After that the trajectory remains for some time at that point, and then again, according to the complete Hamilton equations, returns to the point x. The trajectory reaches the point x_* and returns to x after a finite time interval, so that for large times the trajectory spends almost all of its time at the point x_*, k_*, and the corresponding eigenvalue $\omega(x_*, k_*)$ will characterize the instability. The determination of the corrections to $\omega(x_*, k_*)$ due to the trajectories which are near to the steepest-descent trajectory differs in this case from those of the preceding section and reduces to a special eigenvalue problem. Of course, such trajectories may not even exist, in spite of the existence of a branch point for $L^{\alpha\beta}$, and in this case the while asymptotic behavior of the Green's function will be determined exactly as in the preceding section.

CONCLUSION

We have shown that for an arbitrary system of differential equations describing a physical system (a "correct" system, i.e., one where the solutions which oscillate rapidly in space are damped in time) one can write a functional integral representing the Green's function for such a system. The main distinction of this representation from the usual Feynman integral representation of the Green's function of the Schrödinger equation consist in the many-sheeted nature of the Hamiltonian, corresponding to the different branches of the dispersion equation (1.5), as well as in the complex-valued nature of all the variables on the trajectories. The introduction of such a representation allows one to consider complex values for the differential operator $i\partial/\partial x$ and of the independent variable x and to construct the asymptotic behavior of the green's function for large real times, by solving Hamilton's equations with $\omega(\mathbf{k},\mathbf{x})$ playing the role of Hamiltonian. For the case of a single spatial variable the possible types of trajectories reduce to three: a) the fixed point, b) periodic trajectories with real period and real $\oint \mathbf{k}\, d\mathbf{x}$; both these cases yield a discrete spectrum, and c) periodic trajectories with real period tending to infinity, yielding a continuous spectrum. It should be stressed once again that each trajectory needs to be checked for its adequacy in representing the asymptotic behavior of the original functional integral.

The results permit a simple investigation of the stability of weakly inhomogeneous states, if the dispersion equation for linear disturbances is known in the local approximation. The approach itself remain unchanged for the case of a large number of dimensions, but the behavior of the steepest descent trajectories may be quite complicated and the problem of finding the correct saddle-points is quite difficult. The confinement to differential equations does not seem essential, since the final results are formulated only in terms of the local dispersion equation. We note that Hamilton's equations with small imaginary terms have been used in Ref. 5 for the solution of the problem of propagation of wave packets in unstable media. In the book by Maslov[6] Hamiltonian equations which are close to real ones have been utilized for the construction of solutions of linear equations.

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