Exact solution of two-dimensional Z_N-invariant self-dual models near the critical point

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The exact solution of the quantum one-dimensional Z_N-invariant model corresponding to the two-dimensional statistical Z_N-model near the critical point is constructed. The two-particle S-matrix of the theory coincides with that found previously by Köberle and Swieca [Phys. Lett. **86B**, 209 (1979)] from phenomenological considerations.

1. INTRODUCTION

In recent years enormous advances have been achieved in our understanding of the behavior of two-dimensional statistical models at a second-order transition point. Conformal field theory, the principles of which were worked out by Belavin, Polyakov, and Zamolodchikov,[1] makes it possible, in principle, to calculate all the correlation functions at the critical point. Away from the critical point, on the other hand, the results are far more modest, although there are a number of publications devoted to the correlation functions of the Ising model away from the critical point (see, e.g., Refs. 2 and 3). However, it is not clear how to extend the results of these papers to other models. For (1 + 1)-dimensional quantum integrable theories there is another approach to the calculation of correlators, associated with the construction and solution of the quantum Gel'fand-Levitan-Marchenko (GLM) equation. For example, the form factors for the sine-Gordon model have been calculated in the framework of this formalism.[4] These form factors turned out to be rather cumbersome, and a comprehensible expression for the Green function in this case has still not been obtained.

In the present paper we propose an exact solution of models for which the form factors have the simplest possible form. These are the (1 + 1)-dimensional integrable Z_N-invariant models with a mass term. Their correlators coincide with those of the two-dimensional Z_N-invariant statistical models near the critical point.

Conformal field theory possessing Z_N-invariance has been described by Zamolodchikov and Fateev.[5]

The basic idea of our work rests on the fact that the operators of the Z_N-model at the critical point can be expressed in terms of the operators of the Wess-Zumino-Novikov-Witten (WZNW) model[6,7] and exponentials of the scalar field. A description of these models is given in Sec. 2 of the present paper. There exists a magnetic model whose Hamiltonian contains a continuous parameter—the magnetic field $\mathscr{H}$. This model is solved by means of the Bethe ansatz. For $\mathscr{H}=0$ it belongs to the universality class of the WZNW model. Excitations of this model can be broken down into two independent groups—excitations of a scalar field and excitations of a Z_N-invariant model. In a nonzero magnetic field the former become massless, while the latter become massive. From the dependence of their mass on $\mathscr{H}$ one establishes the dimension of the operator conjugate to the magnetic field: $\Delta_\varepsilon = 2/(N+2)$. This is the dimension of the energy operator of the Z_N-model. Thus, it turns out that the part of the Bethe equations of the magnet which describes the massive sector corresponds to a Z_N-model perturbed by the operator $\mathscr{H}\hat{\varepsilon}$; we have called this model the Z_N-model with a "mass term." A description of the model of the magnet is given in Sec. 3. The Bethe equations for the Z_N-model with a mass term are given in Sec. 4.

2. DESCRIPTION OF THE MODELS

The Z_N-invariant statistical model is the generalization of the Ising model to the case when the lattice variable takes the values $\omega^k = \exp(2\pi i k/N)$. In the Ising model in the continuum limit there exist order fields $\sigma(z,\bar z)$ and disorder fields $\mu(z,\bar z)$, and also a fermionic Majorana field $\chi(z)$, $\bar\chi(\bar z)$ (here and below, $z = x + it$ and $\bar z = x - it$ are complex coordinates of the plane). In the Z_N-models there are $N-1$ order fields σ_l and disorder fields μ_l. They can be understood as powers of the fundamental fields σ_1 and μ_1. The equalities

$$\sigma_l^{+}=\sigma_{N-l},\ \mu_l^{+}=\mu_{N-l}$$

hold. To the fermion field of the Ising model there correspond parafermion "currents" $\psi_l(z)$ and $\bar\psi_l(\bar z)$—operators with dimensions $[(N-l)/N, 0]$ and $[0, (N-l)/N]$, respectively. The primary fields of the Z_N-invariant statistical model and the corresponding (1 + 1)-dimensional quantum theory (below we do not stipulate the difference between these theories) are denoted by $\Phi^{l\,l}_{m\bar m}$, with

$$\sigma_l=\Phi^{ll}_{ll},\quad \mu_l=\Phi^{ll}_{-ll}.$$

The indices m, $\bar m$ indicate the law of transformation of the given field under transformations of the group $Z_N\times\tilde Z_N$:

$$\Phi^{l\bar l}_{m\bar m}\to\omega^{ms+\bar m t}\Phi^{l\bar l}_{m\bar m},$$

where s and t are integers characterizing the transformation. The numbers l, $\bar l$, together with the numbers m, $\bar m$, specify the dimension of the operator $\Phi^{l\,l}_{m\bar m}$ with even $l=2j$, $\bar l = 2\bar j$ there are fields that are neutral under transformations of the group $Z_N\times\tilde Z_N$; these are the fields $\Phi^{2j,2\bar j}_{0,0}$ with dimensions

$$\Delta=j(j+1)/(N+2),\quad \bar\Delta=\bar j(\bar j+1)/(N+2). \qquad (2.1)$$

Zamolodchikov noted[8] that if to the action of the Z_3 invariant theory at the critical point we add the perturbation

$$g\int dz\,d\bar z\,\Phi^{2,2}_{0,0}(z,\bar z),$$

the perturbed theory remains integrable while the conformal invariance is lost. In Ref. 8 several higher conservation laws for this model are constructed.

We shall call the theory with action

$$S_N=S_N^{(0)}+g\int dz\,d\bar{z}\Phi_{0,0}^{2,2}(z,\bar{z}) \quad (2.2)$$

($S_N^{(0)}$ is the action of the Z_N-theory at the conformally invariant point) a theory of parafermions with a mass term. Below we shall prove its integrability and calculate the two-particle S-matrix of the excitations.

As already stated in the Introduction, the idea of our approach stems from the close connection between the parafermion current algebra and the $SU(2)$-invariant WZNW model. We shall describe this connection.

Let $\varphi(z,\bar{z})=\varphi(z)+\varphi(\bar{z})$ be a free scalar field, normalized as follows:

$$\langle\varphi(z)\varphi(0)\rangle=-2\ln z,\ \langle\bar{\varphi}(\bar{z})\bar{\varphi}(0)\rangle=-2\ln\bar{z},\ \langle\bar{\varphi}(\bar{z})\varphi(0)\rangle=0. \quad (2.3)$$

In the combined system of the Z_N-invariant theory and the theory of a free massless field there exist the following fields of dimension (1, 0) (currents):

$$J^3(z)=N^{1/2}\partial_z\varphi(z),$$
$$J^1(z)=\frac{N^{1/2}}{2}[\psi_1(z):\exp(i\varphi(z)/N^{1/2}): +\psi_1^+(z):\exp(-i\varphi(z)/N^{1/2}):],$$
$$J^2(z)=\frac{N^{1/2}}{2i}[\psi_1(z):\exp(i\varphi(z)/N^{1/2}): -\psi_1^+(z):\exp(-i\varphi(z)/N^{1/2}):]. \quad (2.4)$$

The following operator expansion[5]:

$$J^a(z)J^b(0)=\frac{N\delta_{ab}}{z^2}+\frac{\varepsilon^{abc}J^c(0)}{z}+\ldots \quad (2.5)$$

holds. It follows from this expansion that the components of the Laurent expansion of the currents

$$J^a(z)=\sum_{n=-\infty}^{+\infty}J_n^a z^{-n-1}$$

satisfy the Kac-Moody algebra:

$$[J_n^a,\ J_m^b]=\varepsilon^{abc}J_{n+m}^c+{}^1/_2Nn\delta^{ab}\delta_{n,-m}. \quad (2.6)$$

The combined theory is the WZNW theory with energy-momentum tensor constructed from the currents:

$$T(z)=\frac{1}{N+2}:J^a(z)J^a(z): \quad (2.7)$$

The central charge C of this theory is equal to

$$C=3N/(N+2). \quad (2.8)$$

The space of the fields of the WZNW model contains the primary fields $G_{m\bar{m}}^{l\bar{l}}$, which are the m- and $\bar{m}$-components of isospins l and $\bar{l}$ (Ref. 5):

$$J_n^aG_{m\bar{m}}^{l\bar{l}}=\bar{J}_n^aG_{m\bar{m}}^{l\bar{l}}=0\quad(n>0),$$
$$(J_0^3-m)G_{m\bar{m}}^{l\bar{l}}=(\bar{J}_0^3-\bar{m})G_{m\bar{m}}^{l\bar{l}}=0, \quad (2.9)$$
$$J_0^+G_{l\bar{l}}^{l\bar{l}}=\bar{J}_0^+G_{l\bar{l}}^{l\bar{l}}=0,$$

where

$$-l\leqslant m\leqslant l,\quad -\bar{l}\leqslant\bar{m}\leqslant\bar{l},\quad \bar{l}=0,1,\ldots,N.$$

These fields have dimensions

$$\Delta=l(l+2)/4(N+2),\quad \bar{\Delta}=\bar{l}(\bar{l}+2)/4(N+2). \quad (2.10)$$

All the fields of the WZNW model can be obtained from the primary fields by the action of the operators J_n^a with $n<0$.

There exists a simple relation between a primary field of the WZNW theory and a primary field of the Z_N-theory (Ref. 5):

$$G_{m\bar{m}}^{l\bar{l}}(z,\bar{z})=\Phi_{m\bar{m}}^{l\bar{l}}(z,\bar{z}):\exp\left[\frac{im\varphi(z)}{2N^{1/2}}+\frac{i\bar{m}\varphi(\bar{z})}{2N^{1/2}}\right]: \quad (2.11)$$

This, for us, is the fundamental relation.

3. INTEGRABLE HEISENBERG MAGNET WITH SPIN $S=N/2$

Our subsequent arguments are based on the remarkable fact that there exists a simple integrable model belonging to the universality class of the WZNW model on the group SU(2). This is the Heisenberg antiferromagnet with spin S. The Hamiltonian of the model has the form

$$H=J\sum_{n=1}^{N_0}P_{2S}(\mathbf{S}_n\mathbf{S}_{n+1}), \quad (3.1)$$

where $P_{2S}(x)$ is a polynomial of degree $2S$ of full defined form. We shall not need its concrete form, which can be found in Ref. 9. On scales much larger than the lattice constant this model possesses conformal symmetry. Affleck has shown[10] that the central charge of the model (3.1) is given by formula (2.8) with $N=2S$. In Ref. 9 Bethe equations were obtained for the model (3.1). The energy of the system is expressed in terms of a set $\{\lambda_\alpha\}$ of rapidities satisfying the following system of equations:

$$\left(\frac{\lambda_\alpha-iN/2}{\lambda_\alpha+iN/2}\right)^{N_0}=\prod_{\beta=1}^{M}\left(\frac{\lambda_\alpha-\lambda_\beta-i}{\lambda_\alpha-\lambda_\beta+i}\right),$$
$$E=-\frac{4J}{\pi}\sum_{\alpha=1}^{M}\frac{N}{4\lambda_\alpha^2+N^2}, \quad (3.2)$$

where the integer M can be expressed in terms of the z-component of the spin of the system:

$$M=N_0N/2-S^z.$$

As we should expect, in the equations for the rapidities of the excitations, which can be obtained from Eqs. (3.2), we can see a separation of the parafermion degrees of freedom and the degrees of freedom of the scalar field φ (2.3) (see, e.g., Refs. 11 and 12). Central for our arguments is the fact that switching on the magnetic field $\mathscr{H}$ does not affect the degrees of freedom of the field φ, while at the same time making the excitations in the Z_N-sector massive. We shall now prove this, and also the fact that the magnetic field provides the action of the Z_N-model with an additional term (2.2) $g\sim\mathscr{H}^2$, i.e., the dimension of the square of the magnetic field is equal to $\Delta_{\mathscr{H}}=N/(N+2)$.

These two facts enable us to obtain from the Bethe equations for the magnet described by (3.1) the Bethe equations for the Z_N-model with a mass term, described by (2.2).

We shall consider the thermodynamic equations for the model (3.1). We have taken them from Ref. 9. The free energy of the model can be expressed in terms of the functions $\varepsilon_n(\lambda)$ $(n = 1,2,\dots)$—the excitation energies, which satisfy the following system of equations:

$$\varepsilon_n = Ts*\ln(1+\exp(\varepsilon_{n-1}/T))(1+\exp(\varepsilon_{n+1}/T)) - 2s\delta_{n,N}, \tag{3.3a}$$

$$\lim_{n\to\infty} \varepsilon_n/n = \mathscr{H}, \tag{3.3b}$$

$$F = -T\int_{-\infty}^{+\infty} s(\lambda)\ln[1+\exp(\varepsilon_N(\lambda)/T)]\,d\lambda, \tag{3.3c}$$

where $s(\lambda) = (2\cosh \pi\lambda)^{-1}$, and the symbol $*$ denotes convolution:

$$f*g(\lambda) = \int_{-\infty}^{+\infty} f(\lambda-\lambda')g(\lambda')\,d\lambda'.$$

Here and below we take $J = 1$, so that the magnon velocity $v = 1$.

We shall also need equations for the particle densities ρ_n and hole densities $\tilde{\rho}_n$. In the thermoydnamic limit they have the form

$$\rho_n + \tilde{\rho}_n = s*(\tilde{\rho}_{n-1} + \tilde{\rho}_{n+1}). \tag{3.4}$$

The equations (3.4) make sense not only in the state of thermodynamic equilibrium, when $\tilde{\rho}_n/\rho_n = \exp(\varepsilon_n/T)$, but also for an arbitrary distribution of particles and holes. In the ground state, $\rho_N = s(\lambda)$ and $\rho_n = 0 (n \neq N)$. A magnetic field gives rise to holes in ρ_N: $\tilde{\rho}_N \neq 0$. An adequate description of the excitations is given in the language of holes with label N and particles with all other labels. We rewrite Eqs. (3.4) in such a way that the kernels in them act on the functions $\tilde{\rho}_N$ and ρ_N $(n \neq N)$. After the calculations we obtain three series of equations:

$$\tilde{\rho}_n + \mathscr{A}_{nm}*\rho_m = \mathscr{A}_{n,N-1}*s*\tilde{\rho}_N \quad (n,\ m \leqslant N-1), \tag{3.5a}$$

$$\tilde{\rho}_{n+N} + A_{nm}*\rho_{m+N} = a_n*\tilde{\rho}_N \quad (n,\ m = 1,\ 2,\dots), \tag{3.5b}$$

$$\rho_N + (A_{N,N})^{-1}*\tilde{\rho}_N + \sum_{n=1}^{\infty} a_n*\rho_{n+N} + \sum_{n=1}^{N-1} s*\mathscr{A}_{n,N-1}*\rho_n = s, \tag{3.5c}$$

where the Fourier transforms of the kernels are equal to

$$A_{nm}(\omega) = \mathrm{cth}\frac{|\omega|}{2}\{\exp[-|n-m|\,|\omega|/2] - \exp[-(n+m)|\omega|/2]\},$$

$$a_n(\omega) = \exp(-n|\omega|/2),$$

$$\mathscr{A}_{nm}(\omega) = \frac{2\,\mathrm{cth}(\omega/2)\,\mathrm{sh}[[N-\max(n,m)]\omega/2]\,\mathrm{sh}[\min(n,m)\omega/2]}{\mathrm{sh}(N\omega/2)}.$$

It can be seen from Eqs. (3.5) that the excitations with $n < N$ and those with $n > N$ can be separated from each other. They are coupled only through the density $\tilde{\rho}_N$.

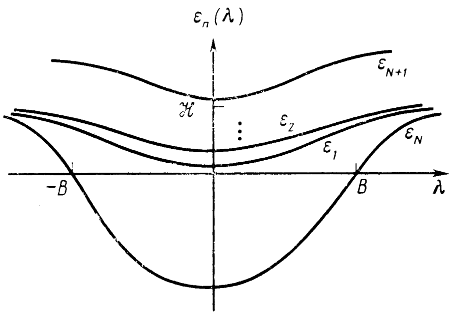

FIG. 1

We shall see what happens when a magnetic field is switched on. Let $T\to 0$. then from Eqs. (3.3) it follows that

$$\varepsilon_{n+N} = n\mathscr{H} + A_{n1}*\varepsilon_N^{(+)} \qquad (n = 1, 2, \dots), \tag{3.6a}$$

$$\varepsilon_n = \mathscr{A}_{n,N-1}*s*\varepsilon_N^{(+)} \qquad (n = 1,\dots N-1), \tag{3.6b}$$

$$\varepsilon_N^{(-)}(\lambda) + \int_{|\lambda'|>B} (A_{NN})^{-1}(\lambda-\lambda')\varepsilon_N^{(+)}(\lambda')\,d\lambda' = -2s(\lambda) + \mathscr{H}/2, \quad \varepsilon_N(\pm B) = 0, \tag{3.6c}$$

where $\varepsilon_N^{(\pm)}(\lambda)$ are the positive and negative parts of the function $\varepsilon_N(\lambda)$.

The qualitative pattern of the behavior of the functions $\varepsilon_N(\lambda)$ is given in the Figure; $\varepsilon_N(\lambda) > 0$ for $|\lambda| > B(\mathscr{H}) \cong \pi^{-1}\ln(1/\mathscr{H})$. From Eq. (3.6b) we find, by making use of properties of the kernel, that for $|\lambda| \ll B$ the equality

$$\varepsilon_n(\lambda) = \frac{4}{N}\sin\frac{\pi n}{N}\,\mathrm{ch}\frac{2\pi\lambda}{N}\int_B^{\infty}\exp(-2\pi\lambda'/N)\,\varepsilon_N^{(+)}(\lambda')\,d\lambda' \tag{3.7}$$

holds. Thus, by taking into account the estimate of $B(\mathscr{H})$ and $\varepsilon_N(\infty) \sim \mathscr{H}$, we obtain

$$\varepsilon_n(0) = A\sin\frac{\pi n}{N}\mathscr{H}^{1+2/N}, \tag{3.8}$$

where A is a constant that does not depend on n or $\mathscr{H}$.

Thus, at temperatures $T \ll \mathscr{H}^{1+2/N}$ the degrees of freedom with $n < N$ are not excited, just like those with $n > N$, since $\varepsilon_{n=N} \gtrsim n\mathscr{H} \gg T$. There remains just the single mode ε_N. For $T = 0$ it is described by Eq. (3.6c). This mode is gapless. Therefore, at energies much smaller than $\mathscr{H}^{1+2/N}$ the theory possesses conformal symmetry. It is not difficult to understand that this is the conformal symmetry of the free scalar-field theory. The magnetic field has broken the $SU(2)$ symmetry but has preserved the $U(1)$ symmetry, leaving the field φ (2.3) massless. One can prove rigorously that the low-energy fixed point corresponds to central charge $C = 1$. For this it is necessary to calculate the specific heat C_v and make use of Cardy's formula[13]

$$C_v/T = \pi C/3. \tag{3.9}$$

From the dependence of the mass of the excitations ε_n $(n \leqslant N-1)$ on $\mathscr{H}^2$ we establish the dimension of the field $\mathscr{H}^2$: $\Delta_{\mathscr{H}} = N/(N+2)$, and the dimension of the operator conjugate to this field: $\Delta_\varepsilon = 1 - \Delta_{\mathscr{H}} = 2/(N+2)$. This, of course, is the operator $G_{0,0}^{2,2}$ in the classification of the WZNW model. Specifically, it has the necessary dimensions [see (2.1) and (2.10)] and does not depend[14] on the field φ. Since $G^{2;2}_{0,0} = \Phi^{2;2}_{0,0}$, the integrability of the theory (2.2) is proved.

4. THE BETHE EQUATIONS FOR THE Z_N-INVARIANT MODEL WITH A MASS TERM

We return to Eqs. (3.5a), which describe the distributions of the rapidities of the excitations of the Z_N-sector. According to formula (3.7), the minima of the energies of these excitations lie at $\lambda = 0$, very far from the points $\lambda = \pm B$ at which the energy $\varepsilon_N(\lambda)$ vanishes. Therefore, for small energies we can simply neglect the mutual influence of these degrees of freedom and, in Eqs. (3.5a), replace $\tilde{\rho}_N$ by the vacuum density $\tilde{\rho}_N^{(0)}$, described by the equation

$$\rho_N^{(0)}(\lambda)+\int\limits_{|\lambda'|>B}(A_{NN})^{-1}(\lambda-\lambda')\tilde{\rho}_N^{(0)}(\lambda')d\lambda'=s(\lambda). \quad (4.1)$$

Then for $|\lambda| \ll B$ Eqs. (3.5a) acquire the following form:

$$\tilde{\rho}_n+\mathscr{A}_{nm}*\rho_m=N^{-1}M_n\,\mathrm{ch}\,(2\pi\lambda/N), \quad (4.2)$$

where, according to formula (3.8),

$$M_n=A\sin(\pi n/N)\mathscr{H}^{1+2/N}. \quad (4.3)$$

The energy of the excitations is

$$N_0^{-1}E=\sum_{n=1}^{N-1}M_n\int d\lambda\,\mathrm{ch}\,(2\pi\lambda/N)\rho_n(\lambda). \quad (4.4)$$

Equations (4.2) and (4.4) are the Bethe equations for the Z_N-model with a mass term (2.2) in the continuum limit. These equations can be obtained as a limit of the following discrete equations:

$$E=\sum_{j=1}^{m}M_1\,\mathrm{ch}\,\theta_j, \quad (4.5a)$$

$$\exp(iM_1L\,\mathrm{sh}\,\theta_j)=\prod_{k=1}^{m}\frac{\mathrm{sh}\,(^1/_2\theta_j-{}^1/_2\theta_k-i\pi/N)}{\mathrm{sh}\,(^1/_2\theta_j-{}^1/_2\theta_k+i\pi/N)}. \quad (4.5b)$$

The equations (4.5) describe a system of m scalar particles of mass M_1, situated on a segment of length L. Periodic boundary conditions are imposed on the wave function of the particles. As follows from Eq. (4.5b), the two-particle S-matrix of these particles is equal to

$$S_{1,1}(\theta)=\mathrm{sh}\,(\theta/2+i\pi/N)/\mathrm{sh}\,(\theta/2-i\pi/N). \quad (4.6)$$

this S-matrix has a pole on the physical sheet $0<\mathrm{Im}\,\theta<\pi$; correspondingly, Eq. (4.5b) admits solutions of the form

$$\theta_j^{(n,p)}=\lambda_j^{(n)}+i(n+1-2p)(\pi/N)+O[\exp(-LM_1)],\quad p=1,\dots n;\ n=1,\dots N-1, \quad (4.7)$$

which describe bound states of fundamental particles with masses

$$M_n=M_1[\sin(\pi n/N)/\sin(\pi/N)].$$

In order to obtain Eqs. (4.2) from Eqs. (4.5), it is necessary to represent the number m of particles in the form $m=m_1+2m_2+\dots+(N-1)m_{N-1}$, where m_j is the number of rapidities of the bound state with label j, rewrite Eqs. (4.5) in terms of these rapidities [$\lambda^{(j)}$ in (4.7)], and go over to the continuum limit $L\to\infty$, $m_j/L=\mathrm{const}$.

We have restored the discrete equations from the continuum equations. This procedure is not rigorous, but we shall convince ourselves of the correctness of the results obtained by ascertaining that the two-particle S-matrix satisfies the requirements of unitarity and crossing symmetry.

The requirement of crossing symmetry in the given case implies

$$S_{j,k}(\theta)=S_{N-j,k}(i\pi-\theta), \quad (4.8)$$

where $S_{j,k}$ is the two-particle S-matrix of the jth and kth bound states.

By rewriting the formula (4.5b) in terms of the rapidities $\lambda^{(j)}$ and $\lambda^{(k)}$ from (4.7), we obtain

$$S_{j,k}=S_{|j-k|}\prod_{l=1}^{\min(j,k)-1}S^2_{|j-k|+2l}S_{j+k},$$

$$S_j(\theta)=\frac{\mathrm{sh}\,(\theta/2+i\pi j/2N)}{\mathrm{sh}\,(\theta/2-i\pi j/2N)}. \quad (4.9)$$

It is easy to verify that the condition (4.8) is fulfilled. The unitarity is obvious. The S-matrices (4.6), (4.9) were found earlier[15] from a purely phenomenological approach.

5. CONCLUSION

We note first of all a characteristic feature of our approach: Throughout in this paper, nowhere have we needed the Hamiltonian of the Z_N-models under consideration. The solution that we have obtained pertains to a whole class of lattice Z_N-theories having the same critical behavior.

The approach described in the paper can be generalized to more complicated models, in which the order and disorder parameters are labeled not by integers, as in the Z_N-models, but by the weight vectors of the groups $SU(N)$, $O(2N)$, E_6, E_7, and E_8. For the WZNW models on these groups factorization of the primary fields, analogous to that described by us in Sec. 2, occurs. The inclusion of a magnetic field leads in the corresponding models of magnets to a separation of the massive and massless modes in the excitation spectrum.

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