Weak-localization theory: role of higher derivatives in the nonlinear sigma-model

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(Submitted 8 December 1987)
Zh. Eksp. Teor. Fiz. **94**, 255–263 (July 1988)

It is shown that additional vertices containing higher powers [$(\partial Q)^{2n}, n \geqslant 2$] of the gradients of the field Q, which appear in the microscopic derivation of the Q-functional of the nonlinear sigma model, have a positive anomalous dimensionality proportional to $n^2 - n$. By the same token, these vertices turn out to be substantial for sufficiently large n, notwithstanding their negative normal dimensionality $-2n + 2$. It turns out that it is precisely these vertices which determine the asymptotic behavior of the distribution function of the mesoscopic fluctuations, as well as the long-time asymptotic behavior of the relaxation currents in disordered conductors. In particular, allowance for these vertices leads to a change of the variance in the logarithmic normal asymptote of the distribution function of the conductivity-fluctuations.

1. INTRODUCTION

A rigorous approach to the theory of weak localization is based on the use of the nonlinear sigma-model,[1-7] the functional of which is of the form

$$F = \frac{1}{t}\int \mathrm{tr}(\partial Q)^2 \, d\mathbf{r}, \tag{1}$$

where Q is a matrix field of definite symmetry, on which are superimposed the geometric constraints $Q^2 = 1$ and $\mathrm{Tr}\, Q = 0$, while the coupling constant t is inversely proportional to the conductivity of the considered disordered system. In the framework of this approach, proof was obtained[8] for the one-parameter scaling hypothesis, the Wigner-Dyson statistics was obtained[9] for energy levels in metallic spheres, and a topological description was obtained[10,11] (with appropriate generalization of the model) of the whole-number quantum Hall effect.

Recently, however, there were raised in the theory of weak localization a number of problems that cannot be adequately treated by the usual sigma model. These include problems connected with calculation of various asymptotics in disordered conductors, viz., asymtotics of distribution functions[12] of mesoscopic fluctuations,[13,14] and of the long-time asymptotic of relaxation currents.[15] To solve such problems it was necessary to expand the functional of the sigma model and to include in it a number of additional vertices. These vertices are similar to those introduced in the analysis of the moments of the local density of states.[18] The simplest of them can be represented in the form

$$\Phi_n = \Gamma_n \int \mathrm{tr}(hQ)^n \, d\mathbf{r}, \tag{2}$$

where h is a definite spatially homogeneous matrix. Vertices of this type appear in the derivation of an effective field theory from an initial microscopic model of noninteracting electrons in a random potential, as a result of expansion in small parameters. For example, to consider the question of current relaxation[15] it was necessary to take into account all the vertices that appear[17] upon expansion in terms of the hydrodynamic parameter $\omega\tau$ (ω is the frequency and τ is the electron free path time in elastic collisions). Vertices of this type are important because of the unusually rapid (for large n) growth of the charges Γ_n under renormalization-group (RG) transformations.[17] This growth compensates for the smallness due to the fact that the unrenormalized value of the charge Γ_n is proportional to the nth power of the corresponding small parameter.

We investigate in the present paper the role of the vertices that arise in the derivation of the effective functional as a result of expansion in terms of another hydrodynamic parameter kl (k is the momentum and $l = v_F\tau$ is the mean free path). These vertices contain arbitrary powers of the field gradient ∂Q. They have been usually disregarded, since they are inessential in the sense of normal dimensionality. We shall show below, however, that just as in the case of vertices of type (2), the anomalous dimensionality of the charges at these vertices increases unusually rapidly with increase of the number of gradients in them.

It turns out that it is precisely the gradient vertices that control both the long-time asymptotic of the relaxation processes, and the asymptotic of the distribution functions of mesoscopic fluctuations. The point is that renormalization of charges at zero-gradient vertices of type (2) cannot, strictly speaking, be carried out without allowance for the vertices containing high powers of the gradients. Allowance for these vertices alters particularly strongly the asyptotics of the distribution function of the conductance fluctuations: the variance in the logarithmic normal distribution decreases by a factor of two compared with the results obtained in Ref. 12 with account taken of only zero-gradient vertices.

2. RENORMALIZATION OF VERTICES WITH A LARGE NUMBER OF GRADIENTS

The effective Q-functional obtained from the model of non-interacting electrons in a random potential contains, besides the functional (1) of the usual sigma model, additional zero-gradient vertices (2) and gradient vertices of the form

$$\Phi_n = Z_n s_{\alpha_1 \ldots \alpha_{2n}} \int \mathrm{tr}(\partial_{\alpha_1} Q \ldots \partial_{\alpha_{2n}} Q)\, d^d r. \tag{3}$$

Here, just as in (1) and (2), Q is a $2N \times 2N$ quaternion-real field[2] (N is the number of replicas, assumed equal to zero in the final results), α_i are vector indices in d-dimensional space, and $s_{\alpha_1 \ldots \alpha_{2n}}$ is a tensor symmetric in any pair of indices. We shall not describe the procedure of deriving the vertices (3), which is fully analogous to the corresponding procedure for the zero-gradient vertices (2).[12,17] The unrenormalized value of the charge at the vertex (3) is

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$Z_n(0) \propto t_0^{-1} l^{2n-2}$, where the unrenormalized value of the coupling constant $t_0 \sim (\pi\varepsilon_F \tau)^{-1}$ plays the role of the weak-disorder parameter.

Of course, the derivation gives rise also to vertices that contain higher field derivatives $\partial^l Q$. It can be shown, however, that the most substantial are precisely the vertices (3), which contain the maximum possible number of fields Q at a specified number $(2n)$ of derivatives. (The situation is similar to the case of zero-gradient vertices, where the most substantial turned out to be[12] the vertices containing the maximum possible number of fields Q for a given number of "external fields" h.) We confine ourselves to consideration of the vertices (3) in a space of dimensionality $d = 2$.

To carry out a renormalization-group (RG) analysis of the vertices (3) for arbitrary n, it is necessary to transform to conformal coordinates:

$$\partial_{\pm} Q \equiv (\partial_x \pm i\partial_y) Q. \tag{4}$$

As a result of this transformation, the vertex (3) goes over into a sum of all possible vertices of the type $\mathrm{Tr}(\partial_{\mu_1} Q \ldots \partial_{\mu_{2n}} Q)$, in which the subscripts μ_i take on values $+$ or $-$, and the number of derivatives ∂_+ coincides with the number ∂_-.

The functional (3) is supplementary, and it must be renormalized jointly with the functional (1) of the sigma model. The procedure of RG transformations of gradient vertices is in the main similar to the corresponding procedure[12,17] for zero-gradient vertices of type (2), although it does turn out to be noticeably more cumbersome. We use the usual scheme[2] of resolving the fields Q into "fast" and "slow" components:

$$Q = U^+ Q_0 U, \tag{5}$$

where Q_0 is the "fast" component of the field Q, and satisfies the same conditions ($Q_0^2 = 1$, $\mathrm{Tr}\, Q_0 = 0$) as Q, while U^+ and U are unitary slow fields that determine the slow component of the field Q. It is convenient in what follows to express ∂Q in terms of the gauge field $\mathbf{A}$:

$$\partial_\mu Q = U^+ [\partial_\mu - iA_\mu,\ Q_0] U, \quad A_\mu = iU\partial_\mu U^+. \tag{6}$$

The RG transformation procedure consists now of averaging over the fast components of the field Q_0. The averaging is carried out with a weight $\exp(-F_0)$, where F_0 is that part of the vertex (1) which contains only fast variables. Renoralization of any additional gradient vertex gives rise to the diagrams shown in the figure. We note that the renormalization of zero-gradient vertices of type (2) is described by the one diagram a only. The solid lines on the diagrams correspond to diffusion propagators t/q^2, where $\lambda l^{-1} < q < l^{-1}$, $0 < \lambda < 1$, q is the fast momentum in terms of which the integration is carried out in RG transformations, and λ is a scale factor. A square denotes a renormalized vertex containing $2n > 2$ gradients of the field Q, while a circle denotes the part of the vertices (1) containing both fast and slow variables. The power of q inside a square or a circle indicates the number of gradients of the fast variables. Diagrams a–d are obviously logarithmic for $d = 2$. Diagram e differs from zero only after one of the propagators is expanded in terms of a slow momentum. It becomes then logarithmic and acquires derivatives of the gauge field in the form of the combination

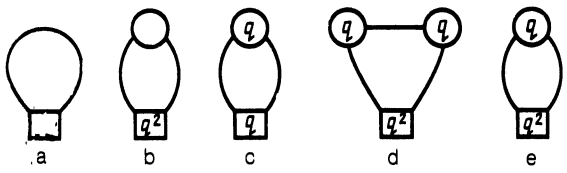

FIG. 1.

$\partial_\mu A_\nu - \partial_\nu A_\mu$. It follows from the definition (6) that the field $\mathbf{A}$ satisfies the pure calibration condition

$$\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = 0, \tag{7}$$

so that these combinations only cancel out the field commutators $[A_\mu, A_\nu]$ that appear in other diagrams.

In the one-loop approximation, as can be seen from the diagrams, there are separated in each renormalized vertex two fast components of the field Q. It is convenient to sum all the diagrams, fixing at first in each vertex a pair of fields $\partial_\mu Q$, $\partial_\nu Q$, from which are separated two fast components. We designate the operator of this partial RG transformation by $\hat{R}_{\mu\nu}$. We represent the complete RC operator in the form

$$\hat{R} = \frac{t}{16\pi} \sum_{(i,j)} \hat{R}_{\mu_i \mu_j}, \tag{8}$$

where the summation is carried out over all the selections of pairs $\partial_{\mu_i} Q$, $\partial_{\mu_j} Q$ in the renormalized vertex. The action of the operator $\hat{R}_{\mu\nu}$ on the gradient vertex $\mathrm{Tr}(\partial_\mu Q C \partial_\nu Q D)$, where C, $D = \Pi_i \partial_{\mu_i} Q$, depends on whether each of the matrices C and D contains an even or odd number of field gradients ∂Q. In the case when this number is odd we obtain

$$\begin{aligned}
\hat{R}_{\mu\nu}\, \mathrm{tr}(\partial_\mu Q C \partial_\nu Q D) &= \frac{1}{2} \mathrm{tr}(\partial_\mu Q C \partial_\nu Q D^+ + \partial_\nu Q C \partial_\mu Q D^+ \\
&- \delta_{\mu,-\nu} \partial_\nu Q \partial_\mu Q \{C, D^+\}) \\
&- \frac{1+\delta_{\mu,-\nu}}{2} [\mathrm{tr}(C\partial_\mu Q)\, \mathrm{tr}(D\partial_\nu Q) + \mathrm{tr}(C\partial_\nu Q)\, \mathrm{tr}(D\partial_\mu Q)] \\
&+ \frac{\delta_{\mu\nu}}{2} \mathrm{tr}(CQ\partial_\mu Q)\, \mathrm{tr}(DQ\partial_\mu Q).
\end{aligned} \tag{9}$$

If both matrices C and D contain an even number of ∂Q, we have

$$\begin{aligned}
\hat{R}_{\mu\nu}\, \mathrm{tr}(\partial_\mu Q C \partial_\nu Q D) &= \frac{1}{2} \mathrm{tr}(\delta_{\mu,-\nu} (\partial_\mu Q C \partial_\nu Q D^+ + \partial_\nu Q C \partial_\mu Q D^+) \\
&- \partial_\nu Q \partial_\mu Q \{C, D^+\}] \\
&+ \frac{1+\delta_{\mu,-\nu}}{2} [\mathrm{tr}(C\partial_\nu Q \partial_\mu Q)\, \mathrm{tr}\, D + \mathrm{tr}(D\partial_\mu Q \partial_\nu Q)\, \mathrm{tr}\, C] \\
&+ \frac{\delta_{\mu\nu}}{4} [\mathrm{tr}(CQ\partial_\mu Q \partial_\mu Q)\, \mathrm{tr}(DQ) + \mathrm{tr}(DQ\partial_\mu Q \partial_\mu Q)\, \mathrm{tr}(CQ)].
\end{aligned} \tag{10}$$

Both equations are given here in the replica limit $N = 0$. Note that Eqs. (9) and (10) are applicable only to renormalization of the supplementary vertices. One cannot obtain a renormalization of the vertex (1) by putting in (1) $C = D = 1$, since its renormalization is described only by diagrams a and b. The diagrams c–e, in which the renormalized vertex contains two gradients from the fast component

of the field, do not arise in this case, since the corresponding part of the vertex (1) is a zeroth-approximation functional.

Equations (9) and (10) show that a functional containing only the supplementary vertex (3) is not closed with respect to RG transformations. Firstly, the vector structure of the initial equation is not conserved, and secondly, vertices appear containing products of matrix traces. All the additional vertices generated by RG transformations must also be renormalized, assuming the unrenormalized values of the corresponding charges to be zero. In this case, if two fast fields are chosen in the one-loop calculations under the sign of one matrix trace, the action of the operator $\hat{R}_{\mu\nu}$ on the corresponding vertex is described as before by Eqs. (9) and (10). The two fast fields can be chosen also from different matrix traces. In that case the action of the operator $\hat{R}_{\mu\nu}$ is described by the equation

$$\hat{R}_{\mu\nu}[\operatorname{tr}(C\partial_\mu Q)\operatorname{tr}(D\partial_\nu Q)]$$
$$=\frac{\delta_{\mu,-\nu}}{2}\operatorname{tr}[\partial_\nu Q\partial_\mu Q(C+C^+)(D+D^+)$$
$$-\partial_\mu Q(C+C^+)\partial_\nu Q(D+D^+)]. \qquad (11)$$

Since the trace of an odd number of ∂Q is equal to zero, the matrices C and D can contain here only an odd number of ∂Q. Equations (9)–(11) make it possible to carry out RG transformation of an arbitrary gradient vertex.

3. ANOMALOUS DIMENSIONALITY OF GRADIENT VERTICES

The complete set of gradient vertices, with respect to which the RG transformations are closed, includes vertices with arbitrary placement of the vector indices, containing all possible products of matrix traces. The general vertex of this type takes the form

$$\Phi_n^{\,s}=Z_n^{\,s}\int d^2r(\varphi_{\mu_1\mu_2})^{s_1}\dots(\varphi_{\nu_1\dots\nu_{2l}})^{s_l}\dots, \qquad (12)$$

$$\varphi_{\nu_1\dots\nu_{2l}}\equiv\operatorname{tr}(\partial_{\nu_1}Q\dots\partial_{\nu_{2l}}Q), \qquad (13)$$

$$\sum_l ls_l=n. \qquad (14)$$

Equation (12) contains equal numbers of derivatives ∂_+ and ∂_-. This circumstance, just as the condition (14), is due to the fact that all these vertices are generated by RG transformations from the initial vertex (3), to which there corresponds in the set (12) a combination of vertices with $s_n=1$ and $s_l=0$ with $l\neq n$.

Equations (9)–(11) make it possible to determine the action of the complete renormalization-group operator $\hat{R}$ (8) on any of the vertices (12). The RG equations for the charges on these vertices, the number of which is equal, for fixed n, to the number of subdivisions of n into sums of positive integers, is of course not independent. The problem consists therefore of finding the eigenvectors of the operator R, i.e., vertex combinations that are self-reproducing under RG transformations. Denoting these vectors by $|\Phi_{nj}\rangle$ (n is the number of the fields ∂Q and j is the serial number of the vector), we write the RG equation in the form

$$\frac{d|\Phi_{nj}\rangle}{d\ln\lambda^{-1}}=\left[(2-2n)+\hat{R}\right]|\Phi_{nj}\rangle=\left[(2-2n)+\frac{t}{16\pi}E_{nj}\right]|\Phi_{nj}\rangle. \qquad (15)$$

Here the first term in the square brackets is the normal dimensionality of any of the vertices Φ_n, and the second is the anomalous dimensionality of the "vectors" $|\Phi_{nj}\rangle$ which must be determined. It turns out to be significant for vectors whose eigenvalues E_{nj} are so large that they cancel out the negative normal dimensionality, the smallness of the coupling constant notwithstanding. It is convenient to represent the charges at all the supplementary vertices in the form

$$Z_n=z_n(\lambda l)^{2n-2t_0-1}, \qquad (16)$$

where the scale and dimensional factors are explicitly separated. Using next the Gell-Mann–Low equation for the coupling constant $\tilde{t}\equiv t/16\pi$ of the sigma model (1)[1–7]

$$\frac{d\tilde{t}^{-1}}{d\ln\lambda^{-1}}=-1, \qquad (17)$$

we rewrite the RG equations for the charges, which follow from (15), in the form

$$d\ln z_{nj}/du=E_{nj}\Rightarrow z_{nj}\propto\exp(E_{nj}u), \qquad (18)$$

where $u=\ln(t/t_0)$, and t_0 and t are respectively the unrenormalized values of the coupling constant and of those renormalized in accordance with (17).

To find E_{nj} it is necessary to classify the action of the operator $\hat{R}$ in the basis (12). This action admits of the following illustrative representation. We depict the trace of the product of the matrices ∂_+Q in the form of a ring of black and white spheres. We set an assembly of such rings in correspondence with vertex (12). The action of the operator R reduces then to a sum of operations of three types: a) inversion (reversal of the order in which the spheres follow one another) of a definite section of one of the rings; b) separation of one ring into two; c) joining together two rings into one. Operation a) reduces, obviously, to permutation of the fields ∂_+Q. Operations b) and c) are also operations of a permutation group, since they correspond to permutation of matrix indices in some pair of fields ∂Q. By the same token, the problem of diagonalizing the operator $\hat{R}$ in the basis (12) reduces to expansion of some representation of a group of permutations into irreducible ones. Only the largest of the eigenvalues (18) is significant for physical applications. Just this eigenvalue determines the anomalous dimensionality of the supplementary gradient vertices. It suffices therefore to find only the eigenvector corresponding to the largest eigenvalue. This eigenvector corresponds to the most symmetric representation of the permutation group. It is convenient to represent it in the form

$$|\Phi_{n0}\rangle=\sum_{\{s\}}\left[\prod_m(2m)^{s_m}s_m!\right]^{-1}|s\rangle, \qquad (19)$$

$$|s\rangle\equiv|s_1\dots s_m\dots\rangle\equiv(\varphi_1)^{s_1}\dots(\varphi_m)^{s_m}\dots. \qquad (20)$$

The "occupation numbers" s_m satisfy here the condition (14), while φ_m is the matrix trace (13) in which the fields ∂_+Q and ∂_-Q are in strict alternation:

$$\varphi_m\equiv\operatorname{tr}[(\partial_+Q\partial_-Q)^m]. \qquad (21)$$

Using Eqs. (9)–(11) and obvious combinatorial considerations, we can express the action of the renormalization-group operator R on the vector (19) as follows [1)]:

$$\tilde{t}^{-1}\hat{R}=\sum_{m,l\geqslant 1}\left\{(m+l)a^{+}_{m+l}a_{m+l}b^{m}_{m+l}+\frac{m+l}{2}a_m{}^{+}a_l{}^{+}a_{m+l}\right.$$
$$\left.+2mla^{+}_{m+l}a_m a_l(1-b^{m}_{m+l})\right\}. \qquad (22)$$

Here a_m^+ and a_m are creation and annihilation operators acting on the states (20) [the contributions in (22) that are cubic in these operators correspond to the operations b) and c) described above]:

$$a_m{}^{+}|s_1\ldots s_m\ldots\rangle=|s_1\ldots s_m+1\ldots\rangle,$$
$$a_m|s_1\ldots s_m\ldots\rangle=s_m|s_1\ldots s_m-1\ldots\rangle. \qquad (23)$$

When the inversion operator b^{m}_{m+l} [corresponding to operation a)] acting on the states (20) serves to eliminate from the class of states in which the gradients ∂_+ and ∂_- alternate:

$$b^{m}_{m+l}\varphi_{m+l}\equiv b^{m}_{m+l}\varphi_{(+-\ldots+-)(+-\ldots+-)}=\varphi_{(-+\ldots-+)(+-\ldots+-)}.$$

(The parentheses contain here m and l pairs of indices, respectively.) A direct check shows, however, that when the operator (22) acts on the vector (19) all the "extra" states cancel out and the vector (19) turns out to be the eigenvector of the operator $\hat{R}$:

$$\tilde{t}^{-1}\hat{R}|\Phi_{n0}\rangle=(n^2-n)|\Phi_{n0}\rangle. \qquad (24)$$

Thus, the largest eigenvalue is $E_{n0}=n^2-n$. Both the initial vertex (3) and those resulting from the RG transformation of the vertices (12) and are needed for further applications contain also the vector (19) in the expansion in terms of the eigenvector of the operator (19). Their anomalous dimensionality, i.e., the growth of the corresponding charges, is therefore determined by this largest eigenvalue.

Unfortunately, the solution obtained does not admit a $2+\varepsilon$ expansion in the dimensionality of space. The eigenvector (19) is made up of states (21), which can be naturally expressed in the conformal coordinates (4), i.e., they are purely two-dimensional objects. For small n, however, the anomalous dimensionality of the vertices (3) and (12) can be calculated directly in Cartesian coordinates. For $n=2$, the eigenvalues depend substantially on ε even in the single-loop approximation and turn out at $\varepsilon>0$ (including $d=3$) to be irrational numbers. The feasibility of a group analysis of the problem for $d>2$ seems therefore doubtful. For $d=1$ the problem admits of an exact solution, but we shall not present it here.

We shall show below how the growth (18) of the charges on the supplementrary vertices influences the asymptote of the distribution function of the mesoscopic fluctuations and other physical quantities.

4. GAUGE INVARIANCE IN AN EXTERNAL FIELD AND MESOSCOPIC FLUCTUATIONS

We shall show here how to modify the functional of the expanded nonlinear sigma model to calculate the moments of the mesoscopic fluctuations of the conductivity. To derive an effective field theory from the initial microscopic model it is convenient to introduce a source such that differentiation with respect to it yields the conductivity and its moments.[18,12] This source is a matrix "external field" $\mathbf{A}^{\rm ext}$ conjugate to the electron current and constitutes a matrix in the replica indices [the matrix structure of $\mathbf{A}^{\rm ext}$ coincides with the structure of the field $\mathbf{A}$ in Eq. (6)]. On going over to the Q-functional it is necessary to carry out, besides the aforementioned expansion in the hydrodynamic parameters $\omega\tau$ and kl, also expansion in powers of the source $\mathbf{A}^{\rm ext}$ recognizing that to calculate the moments of the mesoscopic fluctuations it is necessary to retain all the powers of $\mathbf{A}^{\rm ext}$ in this functional.

It turns out that the effective functional containing all the powers of $\mathbf{A}^{\rm ext}$ is of the form

$$\mathcal{F}[\mathbf{A}^{\rm ext}]=F[\mathbf{A}^{\rm ext}]+\sum_{n=2}^{\infty}\Phi_n[\mathbf{A}^{\rm ext}], \qquad (25)$$

where $F[\mathbf{A}^{\rm ext}]$ and $\Phi_n[\mathbf{A}^{\rm ext}]$ are obtained from the functionals (1) and (3) with the aid of the minimal substitution

$$\partial_\alpha Q\to\nabla_\alpha Q\equiv\partial_\alpha Q-[i\mathbf{A}_\alpha{}^{\rm ext},\,Q]. \qquad (26)$$

This functional is invariant to the gauge transformations

$$Q\to U^{+}QU,\ \mathbf{A}^{\rm ext}\to U^{+}\mathbf{A}^{\rm ext}U-iU^{+}\partial_\alpha U. \qquad (27)$$

The corresponding invariance is inherent in the initial fermion field theory and is naturally preserved in the derivation of the Q functional in all orders in ∇Q.

Using the generating functional (25), we obtain the expectation value of the nth power of the conductivity of a sample with dimensions L^2, in the form [12]

$$\langle\sigma^n\rangle=\left(-\frac{e^2}{16\pi\hbar N^2L^2}\right)^n$$
$$\times\left[\prod_{j=1}^{n}\operatorname{tr}\frac{\partial^2}{(\partial\mathbf{A}_j^{\rm ext})^2}\right]\frac{\int\mathcal{D}Q\exp(-\mathcal{F}[\mathbf{A}^{\rm ext}])}{\int\mathcal{D}Q\exp(-\mathcal{F}[0])}\Bigg|_{\substack{\mathbf{A}^{\rm ext}=0\\ N=0}}. \qquad (28)$$

In diagram language, $\langle\sigma^n\rangle$ comprises n current electron loops joined by impurity lines in all possible manners. In expression (28), on the other hand, there appear also diagrams that do not break up into current loops, and to exclude these diagrams it is necessary to expand the matrix structure of the fields Q and $\mathbf{A}^{\rm ext}$ and ascribe to them additional indices (the field $\mathbf{A}^{\rm ext}$ is diagonal in these indices: $\mathbf{A}^{\rm ext}_{ij}\equiv\mathbf{A}^{\rm ext}_j\delta_{ij}$).[12]

We shall use the RG method to calculate the fluctuation moments (28). It is necessary here to take into account all the vertices of type (12), which are generated in the renormalization process (the $\partial_\alpha Q$ are also replaced in these vertices by the covariant derivatives $\nabla_\alpha Q$). Owing to the special matrix structure of the differentiation in (28), a direct contribution to $\langle\sigma^n\rangle$ is made only by vertices having the structure

$$\gamma_n\int[\operatorname{tr}(\mathbf{A}^{\rm ext}Q)^2]^n\,d^2r. \qquad (29)$$

These vertices are contained in the gauge-invariant vertices (12) of the form $[\varphi_{\alpha\alpha}]^n$, which are transformed in the conformal coordinates (4) into $[\varphi_{+-}]^n$. In Ref. 12 the vertices (29) were renormalized without allowance for the contribution made by the gradient vertices; the largest eigenvalue in the growth rate (18) turned out then to equal $2n^2-3n$, i.e., larger than the largest eigenvalue (24) obtained by renormalizing the gradient vertices. With allowance for the contribution of the charges of the gradient vertices, the RG equations

for the charges γ (29) can be symbolically written in the form

$$(d/du-G)\boldsymbol{\gamma}=H\mathbf{z}, \tag{30}$$

where G and H are definite matrices independent of u. For arbitrary relations between the unrenormalized values of the charges γ and $\mathbf{z}$, the renormalization of γ would be determined both by the eigenvalues of the matrix G and by the induced solutions proportional to the charges z. The gauge invariance of the functional (25), however, connects the unrenormalized values of the charges γ and z and ensures their proportionality in the course of renormalization. Therefore no eigensolutions (30) can appear: their coefficients must vanish.

Thus, the growth of the charges γ_n and hence the u-dependence of the moments of the mesoscopic fluctuations $\langle\sigma^n\rangle$, is determined by the largest eigenvalue (24) of the problem of gradient vertices: at large n we get $\langle\sigma^n\rangle \propto \exp(un^2)$ and not $\exp(2un^2)$, as obtained in Ref. 12 with account taken of only the eigensolutions (30). This means that the variance $4u$ must be replaced by $2u$ in the logarithmically normal asymptote of the distribution function of the mesoscopic fluctuations of the conductivity.[12]

It can be shown that the gradient vertices determine also the behavior of the high moments (and consequently also the asymptotes of the distribution functions) of mesoscopic fluctuations of both the density of states[12] and the local density of states (the "participation ratio," Ref. 16). The renormalization of the charges on the corresponding vertices can also be symbolically represented in the form (30). In these cases, however, there is no gauge connection between the charges γ and z, so that both natural and induced solutions appear. Although the largest eigenvalues of the charges at the zero-gradient vertices coincide in these cases with $E_{n0}=n^2-n$, the growth of the fluctuation moments is determined nonetheless by the induced solution, which turns out to be proportional to $\exp(un^2)$. A similar mechanism of the influence of the gradient vertices on the long-time asymptote of the relaxation currents is described in detail in Ref. 17. We note, however, that in all these cases the only varying contribution in the growth exponent (18) is the one linear in u, and this leads only to a change of the preexponential factor in the corresponding logarithmically normal asymptotic expressions, in contrast to the conductivity-fluctuations case, considered in the present section, in which the argument of the exponential changes.

We note in conclusion the possible existence of other physical situations in which account must be taken of the supplementary gradient vertices considered here. The contribution of such vertices can turn out to be particularly important in the strong-localization region, since their growth (18) is determined by the quantity $u=\ln(t/t_0)$, which increases when this region is approached.

The authors are deeply grateful to B. L. Al'tshuler for numerous discussions of the work, and also to P. B. Wiegmann and D. E. Khmel'nitskiĭ for valuable remarks in connection with gauge invariance.

[10]The operator $\hat{R}$ takes the form (22) only for action on states of the form (20) and (21), in which the fields $\partial_+ Q$ and $\partial_- Q$ alternate. We do not present here the more complicated equations for the action of the operator $\hat{R}$ on states with arbitrary alternations of $\partial_+ Q$ and $\partial_- Q$, since they are not needed to determine the largest eigenvalue .

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Translated by J. G. Adashko