Model of macroscopically inhomogeneous mixtures of a perfect conductor and an insulator near the mobility edge

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A self-similar drop model is proposed for a randomly inhomogeneous medium representing a mixture of a perfect conductor, broken up into finite clusters, and a conductor with a finite resistance. A renormalization group procedure is used to calculate the critical exponent of the conductivity in the two-dimensional case.

We shall consider the behavior of the effective conductivity of a randomly homogeneous medium near the mobility threshold. It is assumed that the medium consists of two phases, one highly conducting (metallic) with the conductivity σ_m and the other poorly conducting (insulating) with σ_i. According to percolation theory,[1,2] if the concentration of the metal p is higher than the critical value p_c, i.e., if an infinite metallic cluster exists in a medium, the effective conductivity depends in the following way on the concentration

$$\sigma^e \approx \sigma_m (p-p_c)^t, \quad p>p_c. \tag{1}$$

where t is the critical exponent depending on the dimensionality of space.

Equation (1) is derived in the zeroth approximation with respect to $h=\sigma_i/\sigma_m$ and the same approximation is used to allow for the flow of the current only along the highly conducting phase. One can use also higher approximations,[2] which are important in the description of thermoelectric effects[3] and in some other cases.[3,4]

The first model used to describe the flow of the current in a medium characterized by $p>p_c$ was that of Skal and Shklovskiĭ, proposed in Ref. 5 (see also Ref. 6). The main assumption of this model is that the current flows only along single-strand percolation channels forming the skeleton of an infinite cluster (Fig. 1). In the Skal-Shklovskiĭ model the exponent ζ governing the concentration dependence of the length of the percolation channel $\mathscr{L} \propto (p_c-p)^{-\zeta}$ is independent of the dimensionality of space and is exactly equal to unity. The critical exponent of Eq. (1) deduced on the basis of the same model is

$$t=\zeta+\nu(d-2),$$

where d is the dimensionality of space and ν is the critical exponent characterizing the behavior of the correlation length $\xi \propto (p_c-p)^{-\nu}$.

In the two-dimensional case we have $\nu \approx 1.33$ and the ratio $\mathscr{L}/\xi$ approaches zero in the limit $p \to p_c$, i.e., the distance between the sites in the network of an infinite cluster (Fig. 1b) rises faster than the length of a metallic filament and this filament "breaks." Therefore, in the two-dimensional case it is necessary to allow for the more complex components of the structure for dimensions smaller than and of the order of ξ. A drop structure of an infinite cluster was considered in Ref. 7 and this eliminated the contradiction (breaking of a filament) and made it possible to calculate the critical exponent of the conductivity t using the renormalization group approach.

In the $p<p_c$ case the conductivity rises on approach to the mobility edge:

$$\sigma^e \approx \sigma_i (p_c-p)^{-q}, \quad p<p_c, \tag{2}$$

where q is the critical exponent of the conductivity in the case when $p<p_c$. The flow of the current in the medium with $p<p_c$ differs from that in the case when $p>p_c$ because in the absence of an infinite metallic cluster the current flow along the fragments of a cluster (with characteristic dimensions of ξ) must partly take place also through the poorly conducting phase.

The simplest model of a medium near the percolation edge in the range $p<p_c$ is that proposed in Ref. 8. This model was subsequently used to describe thermogalvanomagnetic effects.[9,10] The problem of thermogalvanomagnetic effects was solved exactly for the two-dimensional case in Ref. 11. The results of Ref. 9 agreed with the conclusions of Ref. 11.

In the two-dimensional case the geometry of the model for the $p<p_c$ case follows directly from the geometry of the model for $p>p_c$: all that is necessary is to transpose the highly and poorly conducting phases (Fig. 1). The possibility of such a transposition is due to the geometrically equivalent distributions of the phases near the mobility edge of two-dimensional randomly inhomogeneous phases[12] (see also Ref. 13). It should be noted that the reciprocity relationship,[12] which is satisfied rigorously in the two-dimensional case

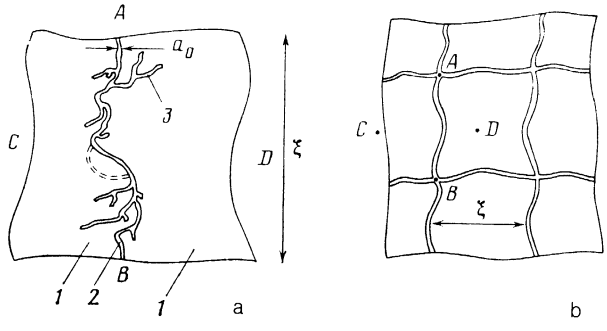

FIG. 1. Simplest model of a two-dimensional medium near the percolation edge. a) Case when $p>p_c$: 1) poorly conducting phase (insulator); 2) metallic filament of thickness a_0 forming a part of a skeleton of an infinite cluster; 3) dead ends. The current flows from A to B. The correlation length is $\xi \propto (p-p_c)^{-\nu}$ and the length of the metallic filament is $\mathscr{L} \propto (p-p_c)^{-\zeta}$. Case $p<p_c$: 1) highly conducting phase (metal); 2) insulator layer in which the main voltage drop occurs. The current flows from C to D. The correlation length is $\xi \propto (p_c-p)^{-\nu}$. The length of the insulating layer is $\mathscr{L} \propto (p_c-p)^{-\zeta}$ The dashed path is one of the doubling bonds ($p>p_c$) or layers ($p<p_c$) which are ignored in the simplest model. b) Case $p>p_c$. Network of the skeleton of an infinite cluster.

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$$\frac{\sigma^e(p<p_c)}{\sigma_i}\,\frac{\sigma^e(p>p_c)}{\sigma_m}=1, \tag{3}$$

has its approximate three-dimensional analog

$$(\sigma^e(p<p_c)/\sigma_i)^t\,(\sigma^e(p>p_c)/\sigma_m)^q\approx 1. \tag{4}$$

In the two-dimensional case we have $t=q$ and Eq. (4) reduces to the exact equality of Eq. (3).

We shall generalize the treatment of Ref. 5 (see also Ref. 1) to the case when $p<p_c$ and we shall find the length $\mathscr{L}$ of a layer in the two-dimensional case. Let us assume that p is the relative number of complete, i.e., conducting, bonds (concentration of the metal) and $1-p$ is the relative number of broken (insulating) bonds. We shall reconnect each broken bond (or, which is equivalent, we shall break down the insulator converting it into a highly conducting phase) with the probability $P=(p_c-p)/(1-p)$. Then, the fraction of the broken bonds becomes equal to $(1-p)(1-P)$ and the fraction of the unbroken bonds is

$$1-(1-p)(1-P)=p_c. \tag{5}$$

In view of the single-strand nature of the layer, its breakdown can be ensured simply by rejoining one of its broken (insulating) bonds. Therefore, we can ensure breakdown of the whole medium, i.e., that medium reaches the percolation edge, by joining a certain fraction of layers y_c:

$$\mathscr{L}P=y_c. \tag{6}$$

Hence, using the expression for P, we obtain

$$\mathscr{L}\propto(p_c-p)^{-\zeta},\quad \zeta=1. \tag{7}$$

In the three-dimensional case a similar analysis gives $\mathscr{L}\propto(p_c-p)^{-1/2}$, i.e., $\zeta=0.5$.

Knowing the dependence of $\mathscr{L}$ on the concentration of the metallic phase, we can easily estimate the critical exponent q. Equating the resistance of the medium over distances of the order of ξ (Fig. 1), given by $R=\rho_i l/S$ ($l=a_0$, $S=\mathscr{L}H$, H is the thickness of the medium), to the effective resistance $R^e=\rho^e\xi/\xi^2$, we obtain the expression $\sigma^e\propto\sigma_i\mathscr{L}/a_0\propto\tau^{-1}$, i.e., we find that $q_2\approx 1$, which (bearing in mind the simplicity of the model) is in reasonable agreement with the value $q\approx 1.3$. In the three-dimensional case a similar estimate gives $q_3\approx\nu-2\zeta\approx 1.1$, whereas numerical methods yield $q_3\approx 0.98$.

It follows from Eq. (7) that in the limit $p\to p_c$ we have $\mathscr{L}/\xi\propto(p_c-p)^{\nu-1}$ and since $\nu>1$, it follows that a single-strand model of a layer is strictly speaking invalid, because an important role is played by doubling, which occurs not only in the case of thin (of thickness a_0) oscillating layers but also in the case of bubbles of an insulator of dimensions much less than a_0 through which there is practically no flow of the current.

DROP-BUBBLE MODEL OF A MEDIUM WITH $p<p_c$

We shall assume that over distances less than the correlation length ξ the structure of a medium is self-similar and represents a set of layers, bubbles, and drops. Drops then have a complex structure and a more detailed analysis (going over to a scale with a higher resolution) shows that it consists of layers, bubbles, and drops (Fig. 2c). The length of a layer on the scale of b is selected to be b^λ, whereas its conductivity is $\sigma_p(b)\propto b^\lambda$. We shall show later that the numerical value of λ need not be specified.

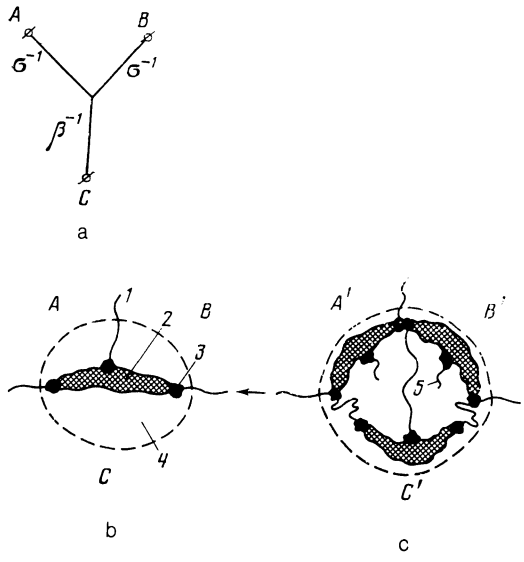

FIG. 2. Drop–bubble model of a medium near the percolation edge when $p<p_c$. a) Electrical substitution circuit for the drop (σ and β are the conductivities). b) Schematic representation of the medium: 1) layer; 2) drop; 3) insulator bubble; 4) metal. c) Fractal law of the growth of a drop. A drop of size b (Fig. 2b) consists of three drops (Fig. 2c) of dimensions $b/3^{1/2}$. Here, 5 is a "dead" layer that makes no contribution to the voltage drop.

It becomes immediately clear that a layer is no longer "drawn" through the whole correlation volume and $\mathscr{L}/\xi\propto(p_c-p)^{\nu-1}$ does not lead to a contradiction in the limit $p\to p_c$. In a real situation we must allow for structures of more than three drops. However, we shall select the simplest hierarchical structure (simpler than that selected in Ref. 7 for the case $p>p_c$) since the main task is to avoid the contradictions of the simplest hierarchical medium in the limit $p\to p_c$. The calculation of the exponent k with a sufficient accuracy represents a formidable problem but the proposed approach to the problem presents no fundamental difficulties. The subsequent calculations are analogous to those reported in Ref. 7.

Denoting the conductivity of a drop along A–B (Fig. 2b) by σ_a (b) and along A–C by σ_s(b), we find from the substitution scheme (Fig. 2a) that

$$\sigma_a^{-1}(b)=2\sigma^{-1}(b),\quad \sigma_s^{-1}(b)=\sigma^{-1}(b)+\beta^{-1}(b). \tag{8}$$

The conductivity $\sigma_a(zb)$ of a drop on a scale of zb ($z=3^{1/2}$) can be calculated by determining the value of $\sigma_a^{(i)}(zb)$, which is the conductivity of a drop in the ith configuration. All these configurations are obtained from that shown in Fig. 2b if we bear in mind that each layer (except for the dead one) may break down with a probability 0.5. Figure 3 shows different configurations: the first row corresponds to $\sigma_a(zb)$ and the second to $\sigma_s(zb)$. By way of example, we shall give an expression for the conductivity $\sigma_a^{(2)}(zb)$, representing the second configuration in the first row of Fig. 3:

$$\sigma_a^{(2)}(zb)=\sigma(b)(2+4\Psi+f+f\Psi)\times(2+2f+8\Psi+8\Psi^2+2\Psi^2 f+5f\Psi)^{-1}, \tag{9}$$

where $f(b)=\sigma(b)/\sigma_p(b)$ and $\Psi(b)=\sigma(b)/\beta(b)$.

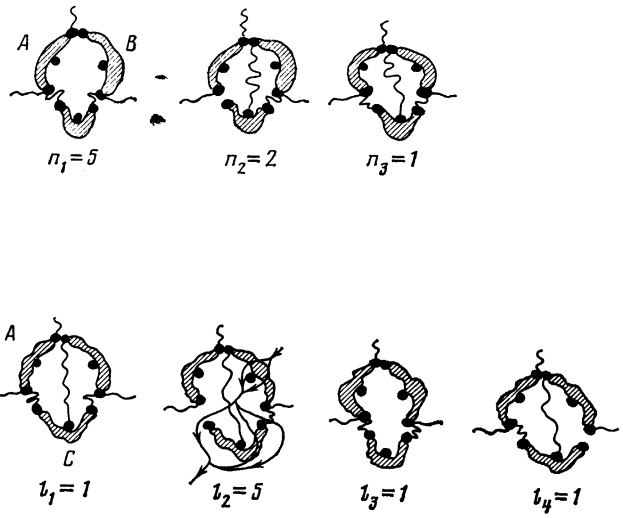

FIG. 3. Set of configurations used in the calculation of the conductivities. The first row shows the conductivities $\sigma_a(zb)$ between the points A and B and the second between A and C. The configurations, which have the values of the conductivity coinciding with one of those mentioned in the text, are not given. The number of configurations with the same conductivity is shown below for each case. Their total number is 16. In the second configuration in the second row we are showing the paths along which the current flows from the point A to B.

Averaging over all the configurations, we obtain the following functional equations for $\sigma_s(zb)$ and $\sigma_a(zb)$:

$$\sigma_a(zb)=\Big[\prod_{k=1}^{3}(\sigma_a^{(k)}(zb))^{n_k}\Big]^{1/P}, \tag{10}$$

$$\sigma_s(zb)=\Big[\prod_{k=1}^{4}(\sigma_s^{(k)}(zb))^{l_k}\Big]^{1/L} \tag{11}$$

$$\Big(\sum_{k=1}^{3}n_k=P,\ \sum_{k=1}^{4}l_k=L\Big).$$

Using the notation $f(b)$ and $\Psi(b)$, and bearing in mind that $\sigma_p(b)\propto b^{\lambda}$, $\sigma_a(b)=\sigma(b)/2$, and $\sigma_s=\sigma(b)/[1+\Psi(b)]$ (Fig. 2a), we find from specific expressions for $\sigma_a^{(k)}(zb)$ and $\sigma_s^{(k)}(zb)$ that Eqs. (10) and (11) can be rewritten in the form

$$\begin{aligned} f(zb)&=2z^{-\lambda}f(b)G(f,\Psi,b),\\ [1+\Psi(zb)]^{-1}&=L(f,\Psi,b)G^{-1}(f,\Psi,b),\end{aligned} \tag{12}$$

where

$$G(f,\Psi)=[(3+f)(2+4\Psi+f+f\Psi)^2(3+3f+6\Psi+2f\Psi)^{-1}\times(1+2\Psi)^{-5}(2+2f+8\Psi+8\Psi^2+2\Psi^2f+5\Psi f)^{-2}]^{1/6}, \tag{13}$$

$$L(f,\Psi)=[(1+4f+6\Psi+2f\Psi+f^2)(1+2\Psi+f)(2+4\Psi+f\Psi+f)\times(1+2\Psi)^{-7}(3f+f\Psi+4\Psi+2)^{-1}(3+12\Psi+18f\Psi+8f+12\Psi^2+6\Psi^2f+3f^2+2\Psi f^2)^{-1}(1+2\Psi+2f)^{-1}]^{1/6}.$$

Assuming that the procedure of going over to increasing scales ($z^2, z^3, \ldots$ times greater) results in convergence and linearizing Eq. (12) near the stable points f^* and Ψ^* of the system (12), which are equal $f^*=\Psi^*=0$, we obtain

$$f(zb)=2z^{-\lambda}f(b)G(0,0). \tag{14}$$

The self-similar solution of the functional equation (14) is

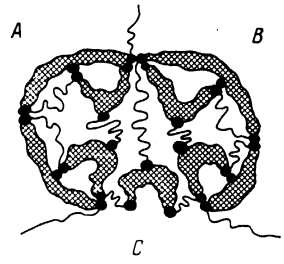

FIG. 4. General form of a specific configuration with nine drops.

$$f(b)\propto b^{\mu},\quad \mu=\frac{\ln 2}{\ln z}-\lambda. \tag{15}$$

We consider the last over the correlation distance (we recall that $f=\sigma/\sigma_p$), and we obtain $\sigma(\xi)\propto\xi^{\ln 4/\ln 3}$ and hence allowing for $\xi\propto(p_c-p)^{-\nu}$, we find that

$$\sigma^e(p_c-p)\infty(p_c-p)^{-\nu\ln 4/\ln 3}. \tag{16}$$

It follows that the critical exponent q considered within the framework of the proposed model is

$$q=\nu\frac{\ln 4}{\ln 3}=\frac{4}{3}\frac{\ln 4}{\ln 3}\approx 1.68, \tag{17}$$

which allowing for the adopted approximation (when each drop consists of three) is in satisfactory agreement with the familiar value found by numerical methods: $q\approx 1.3$.

If we select a more complex structure of the medium, for example that shown in Fig. 4, we find that the critical exponent decreases from 1.68 to 1.39.

It should be noted that if $p>p_c$, then the geometric structure of the medium is the same for the two- and three-dimensional cases: the percolation channel is a metallic filament, whereas in the case when $p<p_c$ the transition to the three-dimensional case converts an insulating layer (Fig. 1a) into a surface of thickness a_0.

In conclusion it should be pointed out that there is one further approach to the description of the transport effects in macroscopically inhomogeneous media near the mobility edge, which is related to an explicit introduction of fractal percolation structures both for $p>p_c$ (Ref. 14) and for $p<p_c$ (Refs. 15–17).

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