Tunnelling in the vicinity of a Peierls point

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The preexponential factor for the probability of spontaneous decay of a metastable vacuum near a Peierls transition point is calculated in a $(1+1)$ dimensional scalar field theory.

1. INTRODUCTION

The problem of decay of a metastable vacuum has been addressed repeatedly in the literature in connection with applications both in statistical physics and in high-energy physics. In this article we will investigate the decay of a metastable vacuum near a Peierls transition point. This type of limiting case was investigated in Refs. 1, 2, where the basic focus of attention was the calculation of an exponential factor. Here we will evaluate the preexponential factor for the probability of such a spontaneous decay in a $(1+1)$ dimensional theory.

In the second section of this article we will describe the model and approximations; in the third section we introduce the Langer-Callan-Coleman equation in a form convenient for machine computations; in the fourth section we discuss the calculations and state the results. Some technical details are relegated to the Appendix.

2. THE MODEL

We investigate the theory of a $(1+1)$ dimensional scalar field φ described by the Euclidean action

$$S=\int d^2x[{}^1/_2(\partial_\mu\varphi)^2+U_0(\varphi)], \tag{1}$$

where the potential $U_0(\varphi)$ possesses two degenerate vacua. If we apply a stress ε, i.e., add to the potential a term

$$\Delta U=\varepsilon\varphi/a, \tag{2}$$

where a is the difference between the vacuum-averaged degenerate vacua, then one of the vacua becomes unstable. As ε increases, the maximum and minimum of the potential approach each other and disappear at a stress ε_p called the Peierls stress (see Fig. 1). For small $\varepsilon_p-\varepsilon$ it is natural to assume that the behavior of the potential far from the metastable vacuum does not affect the tunnelling process, while the potential close to the metastable vacuum can be approximated by a cubic:

$$U(\sigma)={}^1/_2\,\alpha^2\sigma^2+{}^1/_3\,\beta\sigma^3. \tag{3}$$

This theory contains only one dimensional parameter—the loop expansion constant $g^2=\beta^2/\alpha^4$—and therefore the contribution of each loop can be expressed by a certain definite combination of dimensional parameters and universal numerical functions.

Let us investigate the limits of applicability of this approximation for a theory with a potential of the form

$$U(\varphi)=\frac{\mu^2}{(2\pi b)^2}(1-\cos 2\pi b\varphi)-\varepsilon b\varphi. \tag{4}$$

For this potential the Peierls stress equals

$$\varepsilon_p=\mu^2/2\pi b^2. \tag{5}$$

It is not difficult to establish that near one of the minima the potential (4) takes the form

$$U(\sigma)=[\pi b^2\mu^2(\varepsilon_p-\varepsilon)]^{1/2}\sigma^2-{}^1/_3\,\pi b\mu^2\sigma^3+O(b^3\mu(\varepsilon_p-\varepsilon)^{1/2}\sigma^4). \tag{6}$$

For such a potential we have $g^2=\pi\mu^2/4(\varepsilon_p-\varepsilon)$. Corrections to the exponential in (7) (see below) caused by the correction terms in (6) are of order b^2/g^4 in terms of the constant g. Therefore, the validity of the near-critical-stress approximation is defined by the inequality $b\ll g^2$. At the same time, as we noted out earlier, the constant g^2 is the parameter of the loop expansion in the theory with the potential (3), and if we want to limit ourselves to a finite number of loops, then we should require that it be small. We remark that b^2 is the loop expansion parameter in the theory associated with the potential (4).

3. THE FUNCTIONAL DETERMINANT

The takeoff point for our investigation will be the Langer-Callan-Coleman equation[3–6] for the differential decay probability of the metastable vacuum, which in a theory with potential (3) after redefining the coordinates and fields $(x\to x'=x/\alpha,\ \sigma\to\sigma'=\sigma\alpha^2/\beta)$ takes the form

$$\Gamma=\left(\frac{\alpha}{g}\right)^2\frac{S}{2\pi}\left|\frac{\det'(-\Delta+1+2\sigma)}{\det(-\Delta+1)}\right|^{-1/2}\exp\left\{-\frac{S+\Delta S}{g^2}\right\}. \tag{7}$$

In Eq. (7),

$$S=\int d^2x[(\sigma_{,\mu})^2/2+\sigma^2/2+\sigma^3/3] \tag{8}$$

is a dimensionless action computed at a "bounce,"[5] i.e., for an axisymmetric continuous solution of the equation

$$\Delta\sigma=\sigma+\sigma^2 \tag{9}$$

with the boundary condition $\sigma\to0$ for $r\to\infty$. The term ΔS which stands in the exponent of Eq. (7) is the usual perturbation counterterm calculated at the bounce and used to

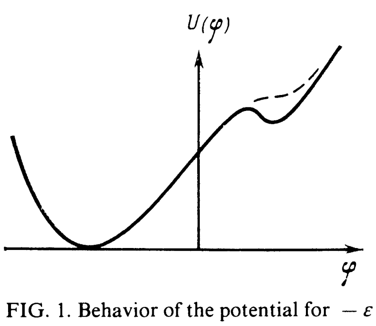

FIG. 1. Behavior of the potential for $-\varepsilon\to0$.

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cancel the ultraviolet divergences which will arise from the determinant.

Let us now turn to the determinant. The symmetry of the bounce allows us to rewrite the determinant in the form of partial wave expansion:

$$\det(-\Delta+1+2\sigma)=\prod_{l=0}^{\infty}[\det(-\Delta_r+d_l/r^2+1+2\sigma)]^{C_l}, \quad (10)$$

where $d_1 = l^2$ is the centrifugal potential and C_1 is the degree of degeneracy, which equals 1 for $l = 0$ and 2 for $l \geqslant 1$. For the one-dimensional determinants appearing in (10), we will use the well-known relation[7]:

$$\det(-\Delta_r+d_l/r^2+1+2\sigma) \propto F_\sigma(l), \quad (11)$$

where $F_\sigma(l)$ is the Jost function of the operator whose determinant is calculated. We recall that the Jost function is defined in terms of the solution to the equation

$$(-\Delta_r+d_l/r^2+1+2\sigma)\psi_l(r)=0, \quad (12)$$

which is normalized at the point $r = 0$ as the coefficient of the part of the solution which grows exponentially as $r\to\infty$. Relations (10) and (11) in principle reduce the problem of calculating the determinant to more or less standard problems: finding the asymptotic solutions of a linear differential equation and calculating an infinite product. We need only be careful to exclude the null mode from the determinant with $l = 1$ and take care to regularize an ultraviolet divergence which manifests itself in the divergence of the product with l as $l\to\infty$ in Eq. (10).

Let us first investigate the null modes. It is well-known that the wave function for the null mode of the operator $(-\Delta r + d_1/r^2 + 1 + 2\sigma)$ is proportional to $d\sigma/dr$ (Ref. 6), where σ is the bounce. Let us determine the Jost function $F_\sigma(1,\varkappa)$ for the operator $(-\Delta r + d_1/r^2 + \varkappa^2 + 2\sigma)$. Using (11), we have

$$\det{}'\left(-\Delta_r+\frac{d_1}{r^2}+1+2\sigma\right)$$

$$=\lim_{\varkappa\to1}\frac{\det(-\Delta_r+d_1/r^2+\varkappa^2+2\sigma)}{\varkappa^2-1}=\frac{1}{2}\frac{\partial F_\sigma(1,\varkappa)}{\partial\varkappa}\bigg|_{\varkappa=1}. \quad (13)$$

From scattering theory it is well known that the product of Jost functions like that appearing in (13) is related to the normalization of the wave function, in our case to the normalization of the null-mode wave function. Following this procedure, it is not difficult to obtain the relation (see Appendix A)

$$\frac{\partial F_\sigma(1,\varkappa)}{\partial\varkappa}\bigg|_{\varkappa=1}=\frac{S}{\pi a_1 a_2}, \quad (14)$$

where

$$a_1=\lim_{r\to0} r^{-1}\sigma'(r), \qquad a_2=\lim_{r\to\infty} r^{1/2}e^r\sigma'(r).$$

Taking into account the fact that for the assumed normalization the Jost function of the operator $(-\Delta r + d_1/r^2 + 1)$ equals $(2\pi)^{-1/2}$, from (13) and (14) we obtain

$$\frac{\det'(-\Delta_r+d_1/r^2+1+2\sigma)}{\det(-\Delta_r+d_1/r^2+1)}=\frac{S}{(2\pi)^{1/2}a_1a_2}. \quad (15)$$

Now that we have explained our method of including the null mode, let us turn to an investigation of the regularization of the determinant. We will use dimensional regularization, i.e., perform all calculations not with $\nu = 2$ (ν is the space-time dimensionality), but with $\nu = 2-\delta$. For this case, Eq. (10) is easily modified: first of all, the centrifugal potential will depend on δ; secondly, the degree of degeneracy C_l will depend on δ. For the contribution of terms with high l to the product (10) we can use the asymptotic form of the Jost function for large l. This contribution can be explicitly evaluated; the result is singular as $\nu\to2$. This singularity sould be exactly cancelled by the counterterm ΔS in Eq. (7). Since the remainder of the product (10) is finite for finite values of l, dimensional regularization is not even necessary; this remainder can be expressed in terms of a finite product of Jost functions and the asymptotics of the bounce with the help of Eqs. (11), (15).

For the asymptotics of the ratio of determinants we obtain in Appendix B

$$\frac{\det(-\Delta_r+d_l/r^2+1+2\sigma)}{\det(-\Delta_r+d_l/r^2+1)}=\exp\left\{\frac{1}{l}\int_0^\infty r\sigma\,dr+O\left(\frac{1}{l^5}\right)\right\}. \quad (16)$$

Using (10), (16), and the fact that

$$C_l(\nu)=\frac{(2l+\nu-2)\Gamma(l+\nu-2)}{l!\Gamma(\nu-1)}=\frac{2}{\Gamma(1-\delta)l^\delta}+O\left(\frac{\delta}{l}\right), \quad (17)$$

we obtain

$$\frac{\det(-\Delta+1+2\sigma)}{\det(-\Delta+1)}=\left\{\prod_{l=0}^{\Lambda}\left[\frac{\det(-\Delta_r+d_l(2)/r^2+1+2\sigma)}{\det(-\Delta_r+d_l(2)/r^2+1)}\right]^{C_l(2)}\right\}$$

$$\cdot\exp\left\{\prod_{l=\Lambda+1}^{\infty}\frac{2}{l^{1+\delta}}\int_0^\infty r\sigma\,dr+O\left(\frac{1}{\Lambda^4}\right)\right\}=\left\{\prod_{l=0}^{\Lambda}[\dots]^{C_l(2)}\right\}$$

$$\cdot\exp\left\{-\sum_{l=1}^{\Lambda}\frac{2}{l}\int_0^\infty r\sigma\,dr+O\left(\frac{1}{\Lambda^4}\right)+\frac{2}{\delta}\int_0^\infty r\sigma\,dr\right\}, \quad (18)$$

where Λ is a certain whole number which generally speaking is much larger than unity. As it should be, the pole terms in the exponent (18) are exactly cancelled by the counterterm ΔS, which is determined by the "bulb-shaped" diagrams (see Fig. 2); in the minimal subtraction scheme[8] this term equals

$$\Delta S=-\frac{g^2}{2\pi\delta}\int d^2x\,\sigma(x). \quad (19)$$

Substituting (11), (15), (18), and (19) into (7), we obtain

$$\Gamma=\left(\frac{\alpha}{g}\right)^2\frac{a_1a_2}{4(2\pi)^{1/2}}\left(\frac{F_0(l=0)}{F_\sigma(l=0)}\right)^{1/2}\prod_{l=2}^{\Lambda}\frac{F_0(l)}{F_\sigma(l)}$$

$$\cdot\exp\left\{\sum_{l=1}^{\Lambda}\frac{1}{l}\int_0^\infty r\sigma\,dr+O\left(\frac{1}{\Lambda^4}\right)-\frac{S}{g^2}\right\}. \quad (20)$$

FIG. 2. The diagram which determines the counterterm ΔS in the theory with the potential (3).

4. COMPUTATIONS

It is not possible to push the calculation of Eq. (20) any further analytically; however, (20) is a convenient starting point for machine calculations. These calculations first require that we solve equation (9), which, taking into account the symmetry of the bounce, can be written

$$\sigma''+\sigma'/r=\sigma+\sigma^2. \tag{21}$$

In solving this equation it is useful to invoke the standard mechanical analogy: Eq. (21) describes the motion of a particle (σ—coordinate, r—time) in a potential $\tilde{V}=(-\sigma^2/2-\sigma^3/3)$ and with friction which is inversely proportional to time. In this case the boundary condition requires that after infinite time the particle arrives at the point $\sigma=0$, while continuity requires that the particle start out with zero velocity. The following algorithm naturally suggests itself: to begin with we find the point σ_0 by iteration (at each step we solve a Cauchy problem), where σ_0 is determined by the condition that the particle passes into the vicinity of $\sigma=0$ after a sufficiently long time; then for this known σ_0 the bounce is also found by solving a Cauchy problem. For this known bounce, it is not difficult to find the constants a_1 and a_2, which are defined following Eq. (14). So as to find the ratio of Jost functions entering into (20), it is convenient to solve the equation for the ratio χ_l between Jost solutions for the bounce background and for the vacuum background:

$$-\chi_l''-(2I_l'/I_l+1/r)\chi_l'-2\sigma\chi_l=0 \tag{22}$$

with the boundary condition $\chi_l(0)=1$ (here I_l is a modified Bessel function). Then $\chi_l(\infty)=F_\sigma(l)/F_0(l)$.

As is clear from Eq. (20), the calculation error falls off very rapidly as Λ grows, and to attain an accuracy of $\sim 1\%$ it is sufficient to solve Eq. (22) for three values of l. Calculations using the algorithm described above give for the decay width the expression

$$\Gamma=0.18(\alpha/g)^2\exp\{-7.80/g^2\}, \tag{23}$$

which completes our investigation.

We note that the method used here to calculate the functional determinant is easily generalized to the case of a higher number of dimensions.

APPENDIX A

In order to obtain relation (14), we will investigate the radial Schroedinger equation for $l=1$:

$$\left[-\frac{d^2}{dr^2}+\frac{3/4}{r^2}+\varkappa^2+2\sigma\right]\psi(\varkappa,r)=0 \tag{A1}$$

with the boundary condition

$$\lim_{r\to 0} r^{(\nu+1)/2}\psi(r)=0.$$

Then the Jost function $F_\sigma(1,\varkappa)$ is defined as

$$F_\sigma(1,\varkappa)=\lim_{r\to\infty} e^{-\varkappa r}\psi. \tag{A2}$$

Differentiating (A1) with respect to $\varkappa$ and setting $\varkappa=1$, we obtain

$$\left[-\frac{d^2}{dr^2}+\frac{3/4}{r^2}+1+2\sigma\right]\frac{\partial\psi(\varkappa,r)}{\partial\varkappa}\bigg|_{\varkappa=1}=-2\psi(1,r). \tag{A3}$$

Multiplying (A3) by $\psi(1,r)$ and integrating with respect to r gives

$$\frac{\partial F}{\partial\varkappa}\bigg|_{\varkappa=1}=\frac{1}{b}\int_0^\infty\psi^2(1,r)\,dr, \tag{A4}$$

where

$$b=\lim_{r\to\infty} r\psi(1,r).$$

Introducing the proportionality coefficient a_1 such that

$$\psi(1,r)=a_1^{-1}r^{1/2}d\sigma/dr$$

(where σ is the bounce), and taking into account the fact that $b=a_2/a_1$, we obtain Eq. (14).

APPENDIX B

In order to obtain (16), we investigate the Schroedinger equation

$$\left[-\frac{d^2}{dr^2}+\frac{\lambda^2-1/4}{r^2}+U\right]\psi_l(r)=0 \tag{B1}$$

where in a ν-dimensional space time $\lambda=1+\nu/2-1$. We cast $\psi_\lambda(r)$ in the form

$$\psi_\lambda(r)=r^{\lambda+1/2}\exp\left\{\int_0^r\chi_\lambda\,dr\right\}.$$

Then for $\chi_\lambda(r)$ we will have the equation

$$\chi_\lambda'+\chi_\lambda^2+r^{-1}(2\lambda+1)\chi_\lambda=U. \tag{B2}$$

This solution for large λ admits the asymptotic expansion

$$\chi_\lambda(r)=\sum_{k=1}^{\infty}\frac{\chi_k(r)}{\lambda^k}. \tag{B3}$$

Substituting (B3) into (B2), we obtain the recursion relation

$$\chi_{k+1}(r)=-\frac{1}{2}\chi_k(r)-\frac{r}{2}\chi_k'(r)-\frac{r}{2}\sum_{l=1}^{k-1}\chi_l(r)\chi_{k-l}(r),$$

$$\chi_1(r)=1/2\,rU(r). \tag{B4}$$

Now, in order to derive (16) there remains only to use the equation

$$\frac{\det(-d^2/dr^2+(\lambda^2-1/4)/r^2+U^{(1)})}{\det(-d^2/dr^2+(\lambda^2-1/4)/r^2+U^{(2)})}=\exp\left\{\int_0^\infty\chi_\lambda^{(1)}dr-\int_0^\infty\chi_\lambda^{(2)}dr\right\}, \tag{B5}$$

which is obvious if we take (11) into account.

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