On the stability of the spectrum of a chaotic quantum system

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The sensitive dependence of the positions of the discrete spectrum energy levels E_n on the parameter α of the Hamiltonian is studied, using the example of a nonlinear quantum harmonic oscillator with two degrees of freedom (the Henon-Heiles model), which in the classical case exhibits stochastic dynamics. It is shown that large susceptibilities $\chi_\alpha(n) = d^2E_n(\alpha)/d\alpha^2$ may occur both in regions of the parameter where the motion of the system is close to regular motion, and in regions where it is close to ergodic motion. Analytic estimates are obtained for the values of χ_α in these regions, estimates which agree with the results of computer experiments.

1. INTRODUCTION

The problem of quantum chaos consists in revealing specific properties of quantum systems for which the classical analogues exhibit chaotic (stochastic) unstable motions with positive maximal Lyapunov exponent σ_1 (Ref. 1.). This problem was formulated a long time ago (Refs. 2, 3), but is far from being solved and therefore has attracted increasing attention in recent years (Refs. 4, 5). A large number of papers have been devoted to establishing the criteria for quantum chaos, which correlate the differences between quantum chaotic systems (QCS) from quantum regular systems (QRS) with the degree of stochasticity of the appropriate classical motion. An example of QRS is completely integrable systems, possessing a set of N mutually commuting operators, among them the Hamiltonian (N being the number of degrees of freedom). By degree of stochasticity one usually understands the invariant measure μ of the stochastic components on the energy surface.

One of the earliest criteria for quantum chaos was proposed by Percival (Ref. 2). It consists in the assertion that the energy levels of the discrete spectrum of a QCS are more sensitive to the magnitude of perturbations to which the system is subjected, than are the energy levels of a QRS. Quantitatively, the degree of instability of the eigenvalues $E_n(\alpha)$ of a system with a Hamiltonian of the form $\hat{H} = \hat{H}_0 + \alpha\hat{V}$ can be characterized by the parameter

$$\chi_\alpha(n) = d^2E_n(\alpha)/d\alpha^2. \tag{1}$$

This parametrization was proposed in the paper of Pomphrey[6] and used for an analysis of the spectrum of the Henon-Heiles model[7]—a nonlinear oscillator with two degrees of freedom

$$H = H_0 + \alpha V = \frac{1}{2m}(p_1^2+p_2^2) + \frac{m\omega^2}{2}(q_1^2+q_2^2) + \alpha m\omega^2\left(q_1^2q_2 - \frac{1}{3}q_2^3\right), \tag{2}$$

where q_i are the Cartesian coordinates and p_i are their canonically conjugate momenta. Computer experiments have shown that in an energy region where the classical system exhibits significant stochasticity ($\mu \gtrsim 0.2$), then, together with "small" values of χ_α, which monotonically continue the dependence of the susceptibility on the energy from the region of regular motion, one also encounters values of χ_α which are larger by approximately one order (in magnitude). Their appearance can be interpreted as a confirmation of the Percival criterion. Subsequently, the existence of large susceptibilities in the parameter region corresponding to strongly stochastic classical motion has been demonstrated in computer simulations in a large number of papers (Refs. 8–13).

The quantity χ_α can be calculated perturbatively. It is expressed in terms of the matrix elements V_{mn} of the operator $\hat{V}$ between the exact eigenfunctions of $\hat{H}$, and by the eigenvalues E_n:

$$\chi_\alpha(n) = 2\sum_m{}' \frac{V_{nm}^2}{E_n - E_m}. \tag{3}$$

A detailed numerical investigation of the dependence of $E_n(\alpha)$ on α in Refs. 8, 10 has shown that large values of χ_α appear on account of and in the neighborhoods of avoided crossings as α varies. They are determined by a single anomalously large term in the right-hand side of Eq. (3): such a value will be written in the form $\chi_\alpha = 2U^2/\Delta$, where Δ is the difference between the energy levels in the denominator of the term dominating the right-hand side, and U^2 is the square of the magnitude of the corresponding matrix element.

The purpose of the present paper is to estimate the large susceptibilities of a quantum chaotic system and a clarification of the relation between their behavior and the stochastic properties of the corresponding classical system.

2. AVOIDED CROSSINGS

As an example we consider the Henon-Heiles model, Eq. (2). With minimal modifications the results of this consideration can be adapted to other nonlinear oscillators. In the sequel we shall use the oscillator unit system ($m, \omega, \hbar = 1$); this makes α into a dimensionless parameter. The stochastic properties of the classical system (for instance, μ) depend only on the magnitude of the relative energy $\varepsilon = E/D$, where E is the energy and $D = (6\alpha^2)^{-1}$ is the threshold dissociation energy of the oscillator.

We shall be interested in the quasiclassical case $\alpha \ll 1$. In this case the system (2) has a large number of quasi-stationary states with energies below the dissociation threshold D. The widths of almost all these levels are negligible (with the exception of a narrow band where $\varepsilon \gtrsim 1 - \alpha^2/12\pi$).

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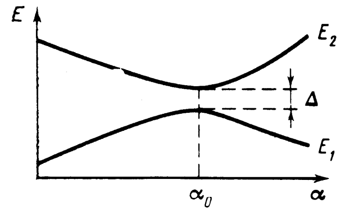

FIG. 1. The avoided crossing of the levels E_1 and E_2 as the parameter α in the Hamiltonian varies.

To an accuracy sufficient for our purposes the spectrum of the quantum system (2) can be obtained by perturbation theory for $\alpha \ll 1$ (see Ref. 11). It is convenient to use as a basis the eigenfunctions of the operators $\hat{H}_0$ and $\hat{l}_x$, the z-projection of the angular momentum; the corresponding quantum numbers will be denoted by n and l. The states of the system will henceforth be labeled by the quantum numbers n and l of the corresponding unperturbed states. To second order in α the energy levels are determined by the equation

$$E_{n,l}=n+1-\frac{\alpha^2}{12}\left(5n^2+10n-7l^2+\frac{4}{3}\right). \quad (4)$$

The expression (4) leads to the following estimate of the susceptibility of the system:

$$\chi^r={}^1/_6(5n^2+10n-7l^2+{}^4/_3). \quad (5)$$

This estimate will be called the regular susceptibility.

The range of validity of Eqs. (4) and (5) is discussed in the Appendix.

As α varies the energy levels E_n determined by Eq. (4) can cross each other. Taking into account higher order perturbative corrections converts these crossings into "avoided crossing," provided the energy levels exhibit the same symmetry, i.e., their values of l are congruent (mod 3). In the neighborhood of an avoided crossing the expression (5) loses its validity, and as explained in Sec. 1, larger susceptibilities appear. Let us determine the conditions for this to happen. It follows from Eq. (4) that the avoided crossing which has lowest energy will correspond to the levels of one symmetry labeled by the quantum numbers $|n+1, \min l\rangle$ and $n, \max l\rangle$. Setting $\min l = 0$, $\max l = n$, and $n \gg 1$, we obtain for the principal quantum number n_L of the lowest avoided crossing

$$n_L \approx ({}^{12}/_7)^{1/2}\alpha^{-1}. \quad (6)$$

Making use of the dependence $n = \varepsilon/6\alpha^2$ we find for the relative energy ε_L of the lowest avoided crossing

$$\varepsilon_L=({}^{432}/_7)^{1/2}\alpha=7.85\alpha. \quad (7)$$

For $\alpha = 0.1118$ this yields $\varepsilon_L = 0.88$ (the numerical computations in Ref. 8 yielded $\varepsilon_L = 0.96$) and for $\alpha = 0.0877$ one obtains $\varepsilon_L = 0.69$ (numerically, Ref. 6 found $\varepsilon_L = 0.75$). In the derivation of Eq. (7) we have neglected terms of the order $n^{-1} \sim \alpha \sim 0.1$, therefore the agreement should be considered satisfactory.

For the Hernon-Heiles model stochasticity becomes apparent only in the region $\varepsilon > \varepsilon_c = 0.68$, where the invariant measure μ of the stochastic component can be approximately described by the empirical formula[6,7]

$$\mu(\varepsilon) \approx {}^1/_8(25\varepsilon-17). \quad (8)$$

The quantity ε_c is usually called the threshold of stochasticity, although, strictly speaking, μ does not exactly vanish for $\varepsilon < \varepsilon_c$. From the expression (7) it can be seen that the appearance of large susceptibilities χ_α *only* in the region $\varepsilon > \varepsilon_c$ is a fortuitous consequence of the choice of parameters in Refs. 6, 8, dictated, in the final analysis, by the capabilities of current computers. In the general case the avoided crossings and the susceptibilities related to them could form at arbitrarily small energy, where the motion of the classical system is arbitrarily close to regular motion and μ and σ_1 are arbitrarily small.

The estimate of large susceptibilities in the limiting cases of classical motion close to regular motion and close to ergodic motion, respectively, forms the subject of the two following sections.

3. LARGE SUSCEPTIBILITIES IN THE QUASIREGULAR REGION

Consider a large susceptibility at a point of avoided crossing of levels coupled by a small matrix element M. Assume that when M is neglected the pair of energy levels $E_1(\alpha)$, $E_2(\alpha)$ cross at a parameter value $\alpha = \alpha_0$ and have at that point the value $E_1(\alpha_0) = E_2(\alpha_0) = E_0$. If the coupling is taken into account in the neighborhood of the avoided crossing the position of the energy levels will be determined by the secular equation

$$(\xi_1\delta-E)(\xi_2\delta-E)-M^2=0, \quad (9)$$

where $\delta = \alpha - \alpha_0$, the energy E is measured from the crossing point E_0, and

$$\xi_1=\left.\frac{\partial E_1}{\partial\alpha}\right|_{\alpha_0}, \quad \xi_2=\left.\frac{\partial E_2}{\partial\alpha}\right|_{\alpha_0}. \quad (10)$$

We consider the upper level E_+; its position is determined by the equation

$$E_+=\delta X+(\delta^2Y^2+M^2)^{1/2}, \quad (11)$$

where

$$X={}^1/_2(\xi_1+\xi_2), \qquad Y={}^1/_2(\xi_1-\xi_2). \quad (12)$$

It follows from the expression (11) that the maximal value of the susceptibility, attained at $\alpha = \alpha_0$, equals

$$\max\chi=\left.\frac{d^2E_+}{d\alpha^2}\right|_{\alpha_0}=\frac{Y^2}{M}=2\frac{U^2}{\Delta}. \quad (13)$$

We obtain the following estimate for the matrix element U^2 of the dominant term in (12):

$$U^2=\frac{1}{4}\left(\frac{\partial E_1}{\partial\alpha}-\frac{\partial E_2}{\partial\alpha}\right)^2. \quad (14)$$

For the lower missed crossing (See Sec. 2) we have ζ_1

$= -\frac{5}{6}\alpha n^2$, $\xi_2 = \frac{1}{3}\alpha n^2$, whence $U^2 = (\frac{7}{12})^2\alpha^2 n^4$. Taking account of the relation (6), we obtain the estimate

$$U^2 \approx \alpha^{-2}. \tag{15}$$

For the conditions of Ref. 8 Eq. (15) yields $U^2 = 80$. If one uses for the calculation of ξ_i and E_0 directly the expression (4), in place of the equations corresponding to the quasiclassical limit, which conserve only the leading powers of the quantum numbers, for the avoided crossings of the levels $|13,1\rangle$ and $|12,10\rangle$ one obtains $U^2 = 119$. Computations according to trace 2 of Ref. 8 yield for these levels $U^2 = 130$.

For quasiregular classical motion the variables n and l remain close to the unperturbed values and the wave functions of the corresponding states of $\hat{H}$ in the H_0-representation are localized near the unperturbed n and l. The matrix element M, which is responsible for the splitting of the levels $|n+1, l\approx 0\rangle$ and $|n, l = n\eta\rangle$ can be estimated by means of the magnitude of the composite matrix element which couples these states. The main contribution to M will come from the shortest chains consisting of matrix elements of $\hat{V}$ (Ref. 14). Making use of Brillouin-Wigner perturbation theory, we take into account only the intermediate states with $n' = n, n\pm 1$ and estimate the energy differences in the denominator of the composite matrix element by means of the expression (4). In this approximation we obtain for M

$$M \approx \left(\frac{5}{7}\alpha^2\right)^{n\eta/6}\left[\left(n\frac{1+\eta}{2}\right)!\right]^{1/2} \times\left[\left(n\frac{1-\eta}{2}\right)!\right]^{-1/2}\left[\left(n\frac{\eta}{6}\right)!\right]^{-2}. \tag{16}$$

Considering the arguments of the factorials large and making use of the Stirling approximation, we represent Eq. (16) in the form (with the dependence on $\hbar$ made explicit):

$$M \approx \exp[-S(\varepsilon, \eta)/\hbar]. \tag{17}$$

The magnitude of the action S which depends only on the classical dimensionless variables ε and $\eta = l/n$ is determined by the expression

$$S(\varepsilon, \eta) = \frac{\varepsilon\eta}{36\alpha^2}\left[\ln\frac{\eta^2}{\varepsilon} + K - \frac{3}{4}f(\eta)\right], \tag{18}$$

where the constant is $K = \ln(180/7) - 1 = 2.247$, and $f(\eta)$ is given by the expression

$$f(\eta) = \frac{1}{\eta}\ln\left(\frac{1+\eta}{1-\eta}\right) + \ln\left(\frac{1-\eta^2}{4}\right). \tag{19}$$

Eq. (18) is valid if the argument of the logarithm is large compared to one, i.e., for $l \gg \alpha n^{3/2}$.

The existence of exponentially small splittings $\Delta E \sim \exp(-S/\hbar)$ between the levels of multidimensional nonlinear oscillators, related to dynamic tunneling between tori localized in different regions of phase space was first pointed out by Berry[15] and was discussed in Refs. 16–18. The tunneling splitting of eigenfrequencies for interacting symmetric classical oscillations with nonintersecting oscillation zones (in the two-dimensional billiard model) has been estimated in Ref. 19.

One may assume the existence of a relation between the exponentially small thickness of the stochastic layers of the classical system in the region $\varepsilon < \varepsilon_c$ and the exponentially small level splitting of the corresponding quantum system in that region. Such a relation was first proposed in Ref. 20.

From the expressions given above we obtain for the susceptibility corresponding to the lower avoided crossing the expression

$$\chi \approx \frac{1}{\alpha^2}\exp\left(\frac{1}{(21)^{1/4}\alpha}\ln\frac{1.20}{\alpha}\right). \tag{20}$$

Although for the conditions of this paper S the assumption on localization of the wave function in the space of the quantum numbers n, l, made in the derivation of Eq. (16), can no longer be explained by the regularity of the classical motion, Eq. (20) still yields a reasonable estimate $\chi_L \approx 1.1\times 10^4$, whereas the result of the numerical simulation (Ref. 8) is $\chi_L \approx 5.3\times 10^3$. At approximately the same value of α as the lower avoided crossing, avoided crossings of the levels $|n+1, l_1 = \eta_1 n\rangle$ and $|n, l_2 = \eta_2 n\rangle$ will also occur, for which the condition $(n/n_0)^2 \equiv \nu^2 = (\eta_1^2 - \eta_2^2)^{-1}$. The matrix elements for such avoided crossings will be given, to exponential accuracy, by the formulas

$$\ln M = -\frac{1}{\hbar}|S(\varepsilon, \eta_1) \pm S(\varepsilon, \eta_2)| \tag{21}$$

where the sign is chosen to agree with that of $\eta_1\eta_2$. Thus, as ε increases, one encounters avoided crossings which are both narrower or wider than the lowest one, with matrix elements from the region

$$S(\varepsilon, 1) + S(\varepsilon, (1-1/\nu)^{1/2}) > -\hbar\ln M > S(\varepsilon, 1) - S(\varepsilon, (1-1/\nu)^{1/2}). \tag{22}$$

Owing to the fact that n and l take on only discrete values, the crossings under consideration will occur in a band of width $\Delta \approx \alpha/n$. We also note that at present there are no numerical results for the susceptibilities related to avoided crossings in the regular region for nonlinear oscillators

4. LARGE SUSCEPTIBILITIES IN THE QUASIERGODIC REGION

According to current thinking, based on the analysis of a large mass of numerical simulations, in the region where the classical motion is close to ergodic ($\mu \approx 1$), the coefficients α_{mn} in the expansion of the eigenfunctions Ψ_n of the Hamiltonian $\hat{H}$ with respect to the basis $\{\Phi\}$ of eigenfunctions of $\hat{H}_0$:

$$\Psi_n = \sum_m a_{nm}\Phi_m, \tag{23}$$

are Gaussian random variables (Refs. 21–23). Together with them the matrix elements V_{mn} also acquire a stochastic character. The structure of the spectrum in the quasiergodic region becomes complex for $\alpha \to 0$ (owing to the confluence of avoided crossings), and it becomes impossible to separate the values of χ_α into large susceptibilities, related to the avoided level crossings, and into small ones, related to the curvature of the dependence of $E_n(\alpha)$ for isolated energy levels. We retain the designation "large" for the susceptibilities which are determined by one dominant term in the right-hand side of Eq. (3):

$$\chi_a = 2U^2/\Delta. \tag{24}$$

Utilizing the random nature of Ψ_n one can estimate the squared matrix element U^2 in Eq. (24) by the typical value $U^2 = V_s^2/C$, where C is the complexity of the state Ψ_n (the number of basis functions which contribute significantly to Ψ_n) and V_s^2 is the sum of the squares of the matrix elements between the state Ψ_n and all the other states:

$$V_s^2 = \sum_m V_{nm}^2 = (\hat{V}^2)_{nn}. \tag{25}$$

In our model one can use this approach in the neighborhood of $\varepsilon = 1$; we adopt it in the region $\mu > 0.5$ ($\varepsilon > 0.84$). The magnitude of V_s^2 can be estimated by making use of the definition

$$V_s^2 = \int \Psi_n(\mathbf{q}) V^2(\mathbf{q}) \Psi_n(\mathbf{q}) d\mathbf{q}, \tag{26}$$

where $\Psi_n(\mathbf{q})$ is the wave function of the state with energy E in the coordinate representation. An approximate calculation of V_s^2 is possible in the quasiclassical case, when all the quantum numbers are large and the wave function $\Psi_n(\mathbf{q})$ has a large number of nodes in any direction. Averaging with respect to the oscillations of the wave functions one can rewrite the expression (26) in the form

$$V_s^2 = \int V^2(\mathbf{q}) W(\mathbf{q}) d\mathbf{q}, \tag{27}$$

where $W(\mathbf{q})$ is the classical probability density in coordinate space, which does not depend on the initial conditions on account of the assumed almost complete ergodicity of the motion. For a particle moving in a two-dimensional potential the density $W(\mathbf{q})$ is constant inside the classically accessible region (Ref. 24), and we obtain:

$$V_s^2 \approx \frac{1}{A(\Omega)} \int_\Omega \frac{r^6}{9} \sin^2 3\theta r\, dr\, d\theta, \tag{28}$$

where A is the area of the classically accessible region Ω. For $\varepsilon \to 0$ the expression (28) is elementary to calculate, as an integral over a disk of radius $R = (2n)^{1/2}$; the result is

$$V_{s0}^2 \approx \frac{n^3}{9} \approx \frac{1}{1944} \varepsilon^3 \alpha^{-6}. \tag{29}$$

This expression coincides with the value of V_s^2 calculated for the unperturbed problem,

$$V_s^2 = \frac{n}{36}(5n^2 - 3l^2), \tag{30}$$

and averaged over the value of the angular momentum. In the region $\varepsilon \approx 1$ Eq. (29) is valid only in order of magnitude, since the boundaries of the classically accessible region deviate strongly from a circle. Setting $V_s^2(\varepsilon) = V_{s0}^2(\varepsilon) F(\varepsilon)$, the correction factor $F(\varepsilon)$ can be determined numerically; its order of magnitude is several times unity. In order to estimate the complexity of the wave functions Ψ in terms of the basis formed by eigenfunctions of the Hamiltonian H_0 we make use of the following considerations. The Hamiltonian of the Henon-Heiles model has the following form in terms of the variables n, l:

$$H = n + \frac{\alpha}{3 \cdot 8^{1/2}} [s^{3/2} \cos(3\varphi + 3\theta) + 3sr^{1/2} \cos(\varphi + 3\theta) + 3s^{1/2} r \cos(\varphi - 3\theta) + r^{3/2} \cos(3\varphi - 3\theta)], \tag{31}$$

where $s = n + l$, $r = n - l$, and θ and φ are the angles canonically conjugate to the action variables n and l. In the quasiergodic case one may assume that the trajectories of the variables n and l will fill the whole classically accessible region. A count of the number of states in this region yields

$$C(\varepsilon) = \frac{1}{216} \alpha^{-4} \varepsilon^2 G(\varepsilon), \tag{32}$$

where

$$G(\varepsilon) = \int_{-1}^{+1} [z_+(\varepsilon, \eta) - z_-(\varepsilon, \eta)] d\eta, \tag{33}$$

and the functions $z_\pm(\varepsilon, \eta)$ are defined by

$$z_\pm = 1 \pm (\varepsilon/1728)^{1/2} z_\pm^{3/2} [(1+\eta)^{1/2} + (1-\eta)^{1/2}]^3. \tag{34}$$

For $\varepsilon \to 0$ we have approximately $G \approx 1.30\varepsilon^{1/2}$; in the region of interest $\varepsilon \approx 1$ the function $G(\varepsilon)$ can be determined numerically: it is on the order of several times unity.

Thus, the characteristic magnitude of the squared matrix element U^2 of the perturbation, taken between two states with neighboring energies is determined by the formula.

$$U^2 = \frac{1}{\alpha^2} L(\varepsilon), \quad L(\varepsilon) = \frac{\varepsilon F(\varepsilon)}{9G(\varepsilon)}. \tag{35}$$

Since $L \sim 1$ the matrix elements which determine the large susceptibilities in the quasiergodic region turn out to be of the same order as those for the avoided crossings in the regular region; cf. Eq. (15).

We now estimate the large χ, considering the distribution of the interlevel spacings to be random. For the description of the structure of the energy spectrum it is customary (Refs. 3, 5) to make use of the distance between neighboring levels

$$t_n = \rho(E_n)(E_{n+1} - E_n), \tag{36}$$

where $\rho(E_n)$ is the density of levels of a given symmetry. The equations (24), (35) and the expression

$$\rho(E) \approx \varepsilon/18\alpha^2 \tag{37}$$

yield the estimate

$$\chi \approx \alpha^{-4} \left(\frac{\varepsilon L(\varepsilon)}{9} \right) \frac{1}{t}. \tag{38}$$

The expression of χ corresponding to typical values of the susceptibility is obtained from Eq. (38) for $t \sim 1$: $\chi \approx (30\alpha^4)^{-1}$. This quantity is of the same order as the maximum values of the regular susceptibility determined according to Eq. (5): $\max \chi^r \approx (43\alpha^4)^{-1}$. This agreement is plausible, but hardly obvious, since the premises used in the derivations of the expressions (5) and (38) are totally different.

The extremal χ_a which have a fluctuational character are related to anomalously small t. One can estimate the probability of occurrence of the latter, making use of the known results on the statistics of the energy spectrum of QCS. The majority of numerical simulations of the structure of the energy spectra agree with the assumption that the form of the distribution function $P(t)$ is Poisson: $P(t) = \exp(-t)$, and in the quasiperiodic case, for $\mu \approx 1$ a Wigner distribution $P(t) = (\pi t/2)\exp(-\pi t^2/4)$ applies. For an interpolation between these two limiting forms one can utilize the Brody distribution[25]

$$P_B(\beta, t) = At^\beta \exp(-Bt^{1+\beta}), \tag{39}$$

where

$$A=(1+\beta)B, \quad B=\left[\Gamma\left(\frac{2+\beta}{1+\beta}\right)\right]^{1+\beta}, \tag{40}$$

and $\Gamma(x)$ is the Euler gamma function. This parametrization, proposed in Ref. 21, satisfactorily describes the structure of the spectrum of nonlinear oscillators (Refs. 26, 27). Within the limits of accuracy attained in numerical calculations the function $\beta(\mu)$ can be approximated by the expression

$$\beta=1.8\mu(1+0.8\mu^{2,25})^{-1}.$$

We define $t(\nu)$ so that with probability ν the inequality $t \leqslant t(\nu)$ should be satisfied. For $\nu \ll 1$ we obtain from Eq. (39)

$$t(\nu) \approx (\nu/B)^{1/(1+\beta)}. \tag{41}$$

The equations (38) and (41) determine the character of the decrease of the distribution function $W(\chi)$ for large χ: it is a slow power-law decay. The value of ν in Eq. (41) can practically not be made arbitrarily small: the maximal χ in the quasiergodic region corresponds to a value $\nu \sim 1/\mathcal{N}$, where $\mathcal{N}$ is the number of levels in the quasiergodic region. This yields the estimate

$$\max \chi^e \sim (15\alpha^2)^{-1} \max \chi^r \approx (650\alpha^6)^{-1}. \tag{42}$$

For the conditions of Ref. 6 we obtain $\max \chi^e \sim 3\times 10^3$, which is compatible with the numerical results, according to which in the region where $\varepsilon > 0.84$ we have $\max \chi = 4\times 10^3$.

5. CONCLUSIONS

The methods of estimating χ_α for the Henon-Heiles model given in this paper can easily be adapted to other systems. The structure of the expressions remains the same and the changes have only a quantitative character. This allows one to formulate the following theses.

1. For strongly quasiclassical systems ($\hbar \to 0$) of the nonlinear oscillator type the avoided level crossings and the large susceptibilities related to them can be observed in a region where the classical motion is arbitrarily close to regular motion. Large values of the χ_α appear in this case on account of tunneling between tori (which is essentially related to the nonintegrability of the system), and are exponentially large: $\ln \chi \sim S/\hbar$, where the action under the barrier, S, depends only on the classical variables. Practically, such "tunneling" values of χ_α may exceed the typical (regular) values of χ_α by many orders of magnitude, but manifest themselves only in narrow intervals of variation of $\hbar$ and are therefore hard to observe.

2. In the region where the classical motion is close to ergodic large susceptibilities appear on account of the fluctuational getting closer of neighboring levels and have a slowly decreasing (power-law) distribution function $W(\chi)$. Practically such "fluctuational" susceptibilities may exceed the typical (regular) values by only 1–2 orders of magnitude, but do not require a special choice of the parameters in the Hamiltonian in order to be observed.

These conclusions are possibly true also for a wider class of systems. In a recent paper (Ref. 28) it was established by means of numerical simulations for a two-dimensional model with homogeneous potential (for such systems μ does not depend on the energy) that the mean susceptibility of the energy levels with respect to variations of the parameters of the Hamiltonian is largest for conditions in which the motion of the system is close to regular, which agrees with what was said before.

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APPENDIX

We discuss the question of applicability conditions for Eqs. (4) and (5). We consider corrections of the next order. Making use of a Brillouin-Wigner perturbation expansion to fourth order in α we find for the energy of the state $|i\rangle = |n,l\rangle$ the equation

$$E_i = E_i^{(0)} + \sum_j \frac{V_{ij}^2}{E_i - E_j^{(0)}} + \sum_j \sum_k \sum_m \frac{V_{ij}V_{jk}V_{km}V_{mi}}{(E_i - E_j^{(0)})(E_i - E_k^{(0)})(E_i - E_m^{(0)})}. \tag{A1}$$

Solving Eq. (A1) iteratively, we obtain for the energy correction of order α^4 the expression

$$E_i^{(4)} = -\alpha^4\left(\sum_j \frac{V_{ij}^2}{\Delta_{ij}}\right)\left(\sum_j \frac{V_{ij}^2}{\Delta_{ij}^2}\right) + \alpha^4 \sum_j {\sum_k}' \sum_m \frac{V_{ij}V_{jk}V_{km}V_{mi}}{\Delta_{ij}\Delta_{ik}\Delta_{im}}, \tag{A2}$$

where $\Delta_{ij} = E_i^{(0)} - E_j^{(0)}$, and the sum over k omits the terms with $|k\rangle = |n, l+6\rangle$ and $|k\rangle = |n, l-6\rangle$: for our system these terms only contribute in order α^6. Although each term of the triple sum is a polynomial of the 6th degree in n and l, in the summation in Eq. (A2) the term of the *three* highest orders cancel, and as a result we obtain

$$E_{n,l}^{(4)} = -\frac{\alpha^4}{432}[67n^3+21nl^2+201n^2+21l^2+178n+44]. \tag{A3}$$

By comparing Eqs. (4) and (A3) it can be seen that the correction to (4) and the corresponding correction to (5) are small if $\alpha^2 n \ll 1$; this condition is satisfied for all levels since $\max n\alpha^2 \approx 1/6$. Thus, for $\alpha \ll 1$ the equation (5) is valid if for the calculation of the regular susceptibility practically without any restrictions. An exception is formed only by a narrow band of quantum numbers where $5n^2 \approx 7l^2$ and the susceptibility χ^r determined by Eq. (5) is *small*: here the correction $\chi^4 = d^2E^{(4)}/d\alpha^2$ may become noticeable. In this paper we have not considered this case specially.

One establishes similarly that Eq. (4) allows one to calculate the positions of the lowest avoided crossings (6) and the magnitude of the matrix elements (15) with an error which scales as $\alpha^2 n$.

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