Quasigas approximation in treating electron-beam bunching in a plasma and the tangential discontinuity in hydrodynamics

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We consider seven examples of a special class of problems, the equations for which can be reduced to exactly solvable equations of a negative-pressure ideal "gas."

1. INTRODUCTION

Dubrovin and Novikov[1] have considered certain general properties of one-dimensional Hamiltonian hydrodynamic-type systems describable by equations of the form

$$\dot{u}_i = M_i^j u_j', \tag{1.1}$$

where

$$u_i = u_i(t, x),\ i = 1, 2, 3, \ldots, N,\ \dot{u}_i = \partial u_i/\partial t,\ u_i' = \partial u_i/\partial x$$

and the matrix $M_i^j = M_i^j(u_1, u_2, \ldots, u_N)$ is independent of t and x. It is noted in Ref. 1 that for systems of type (1.1) there is no general integration method like the inverse-scattering-problem method applicable to a number of "soliton" equations.

The case with two dependent variables ($i = 1,2$) is considered in Ref. 2. The system (1.1) takes there the form

$$\dot{u}_1 = M_1^1 u_1' + M_1^2 u_2',\quad \dot{u}_2 = M_2^1 u_1' + M_2^2 u_2' \tag{1.2}$$

and is reduced by the hodograph transformation to a linear equation that can likewise not be solved for an arbitrary matrix $M_i^j(u_1, u_2)$, although in the case of two dependent variables one can, for example, construct a Lax operator pair.[2]

A definite class of problems such as (1.1) and (1.2), which have important physical applications, can nonetheless be completely integrated for arbitrary initial conditions, i.e., admits a solution of the general Cauchy problem. This class includes, as shown in the present paper, familiar dynamic models that describe bunching of an electron beam in a plasma[3] (see Secs. 3 and 10 below), the Buneman instability of ion and electron motion in a plasma[4–6] (Sec. 11), and also instability of tangential discontinuities in hydrodynamics[7] of various sorts for both bounded (Secs. 4 and 9) and unbounded (Sec. 2) streams.

We showed previously[8–10] that these problems can be reduced by algebraic transformation to simpler and integrable quasigas equations, confirming once more the fundamental character of the latter. Thus, for example, the bunching of a beam in a plasma and the dynamics of the instability of two bounded liquid streams are described by one and the same system of five equations of the form (see Secs. 3 and 4)

$$\dot{S}_\alpha = -(S_\alpha v_\alpha)_x',\quad \dot{v}_\alpha + v_\alpha v_{\alpha x}' = f,\quad S_1 + S_2 = S_0 = \text{const.} \tag{1.3}$$

where $\alpha = 1, 2$ is the index of the "species" of the system component. They are equivalent mathematically to the problem of "breaking" in shallow water:

$$\dot{h} + (hv)_x' = 0,\quad \dot{v} + vv_x' = gh_x', \tag{1.4}$$

which not only yields a complete solution but also introduces, in our opinion, a useful illustrative analogy with "falling drops." We present for the latter new results that supplement the previously published ones,[8–10] viz., a general solution (Sec. 7), a solution of the Cauchy problem (Sec. 8), and an elementary solution in the form of a single drop (Sec. 6). It can be shown that these results are easily generalized to include an arbitrary quasigas medium from the set cited in Ref. 10.

The integrability of the dynamic systems considered here (we discuss seven examples, but more can be cited) is important not only from the standpoint of methodology. The exact solutions we have obtained reveal new heretofore unnoticed features. For the previously considered[3] bunching problem, in particular, we demonstrate the feasibility of a dynamic potential spike and the presence of a beam-density threshold that distinguishes different bunching regimes.

2. NONLINEAR THEORY OF INSTABILITY OF A TANGENTIAL DISCONTINUITY IN AN UNBOUNDED LIQUID

Instability of tangential discontinuities (ITD) with velocity jumps in liquid flows was considered in the linear approximation by Helmholz (1868) and Kelvin (1871), and a linear ITD theory is presented, for example, in Ref. 7. This phenomenon is the cause, in particular, of flag flapping in a wind.

The nonlinear ITD equation is defined as

$$v_0^{-1} a_t' + Ha_x' = a_x' Ha_x', \tag{2.1}$$

where $a = a(t,x)$ is the perturbation of the tangential-discontinuity (TD) plane, the subscripts denote derivatives with respect to the time t and the coordinate x, and H is the Hilbert operator

$$Hf(x) = \pi^{-1} \fint (x' - x)^{-1} f(x')\,dx', \tag{2.2}$$

where the integral is taken in the sense of principal value, with $H^2 = -1$.

Since Eq. (2.1) can have also other application , we show first that it can be solved for arbitrary initial conditions, and demonstrate next how it appears in the ITD problem.

To solve (2.1), we introduce $\tau = tv_0$ and consider the two equations

$$a_\tau' + Ha_x' = a_x' Ha_x',\quad Ha_\tau' - a_x' = H(a_x' Ha_x'). \tag{2.3}$$

We introduce next two complex functions $\psi = a + iHa$ and $\psi^* = a - iHa$, for which we obtain from Eqs. (2.3) an equation with a right-hand side

$$\psi_\tau' - i\psi_x' = (1 + iH)R,\quad R = a_x' Ha_x' = i(\psi_x^{*\prime 2} - \psi_x'^2)/4, \tag{2.4}$$

and in view of the relations

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$$H\psi_x'=-i\psi_x',\quad H\psi_x'^2=-i\psi_x'^2,\quad H\psi_x^{*\prime 2}=i\psi_x^{*\prime 2},\qquad(2.5)$$

we obtain $(1+iH)R=-i\psi_x'^2/2$. Introducing now the imaginary time $T=i\tau$ and the complex "velocity $V=\psi_x'-1$, we obtain from (2.4) a simple equation with a known solution

$$V_T'+VV_x'=0,\qquad V=F(x-VT),\qquad(2.6)$$

where the function F is arbitrary, so that we have indeed the general solution of the problem.

It is more useful in practice to search for a solution in real form. We introduce to this end a new "velocity" $v=-v_0Ha_x'$ and a dimensionless "effective density" $\rho=1-a_x'$, so that $V=-\rho-iv/v_0$. Separating next in (2.6) the real and imaginary parts, we obtain the two equations

$$\rho_t'+(\rho v)_x'=0,\qquad v_t'+vv_x'=v_0^2\rho\rho_x',\qquad(2.7)$$

which describe a gas that is "ideal" but has a negative effective pressure

$$p_{\rm eff}=-\rho^\gamma v_0^2/3,\qquad \gamma=3,\qquad(2.8)$$

and it is this which leads to instability.

We note also the identity $2H(fHf)=(Hf)^2-f^2$ that permits Eqs. (2.7) to be deduced directly from (2.3), bypassing the stage (2.6) which can, however, be useful since it yields a general solution.

We show now how Eq. (2.1) arises in the problem of the nonlinear evolution of a jump with a tangential discontinuity.

We assume that the TD is located in the $z=0$ plane, and the unperturbed velocities of the (identical) liquids are respectively $v_{1x}=v_0>0$ for $z>0$ and $v_{2x}=-v_0<0$ for $z<0$. If $z=a(t,x)$ is the perturbation of the TD plane then, recognizing that the two flows are incompressible ($j=1,2$, div $\mathbf{v}_j=0$) and irrotational, we have approximately the boundary conditions

$$v_{jz}|_{z=a}=v_{jz}|_{z=0}-a(v_{jx})_x'|_{z=0}=a_t'+a_x'v_{jx}|_{z=0},\qquad(2.9)$$

and hence

$$v_{1x}|_0=v_0-Hv_{1z}|_0,\qquad v_{2x}|_0=-v_0+Hv_{2z}|_0.$$

Introducing the convenient notation

$$\alpha(t,x)=(v_{1z}+v_{2z})_0/2,\qquad \beta(t,x)=(v_{1z}-v_{2z})_0/2,\qquad(2.10)$$

we obtain approximately from (2.9), taking quadratic corrections into account,

$$\alpha=v_0[a_\tau'-(aHa_x')_x'],\qquad \beta=v_0[a_x'-(aHa_\tau')_x'].\qquad(2.11)$$

At the boundary $z=a$ we have the condition that the pressures be equal, $p_1=p_2$, meaning that, since the flows are irrotational,

$$(\varphi_{1t}'-\varphi_{2t}')_{z=a}=(\varphi_{1t}'-\varphi_{2t}')_0+2v_0^2aa_x'=(\mathbf{v}_2^2-\mathbf{v}_1^2)_{z=a}/2$$
$$=2v_0^2[H(a-aHa_\tau')_x'-aa_{x\tau}''-a_x'a_\tau'-(Ha_x')(Ha_\tau')]\qquad(2.12)$$

[relations (2.11) are taken into account here]. Differentiating (2.12) with respect to x and allowing for (2.11), we obtain an equation that contains only the functions $a(t,x)$:

$$a_{\tau\tau}''+a_{xx}''=\{(aHa_x')_\tau'+(aHa_\tau')_x'$$
$$-H[2aa_{x\tau}''+a_x'a_\tau'+(Ha_x')(Ha_\tau')]\}_x'.\qquad(2.13)$$

This equation, however, can be simplified by using in the right-hand side the linear relations $a_\tau'=-Ha_x'$, $a_x'=Ha_\tau'$ that are valid only for the perturbation branch that grows like $\exp(\gamma t)$, where $\gamma=|k|v_0$ is the linear growth rate. The right-hand side of (2.13) is then equal to

$$R=(aa_x')_{xx}''-(aa_x')_{\tau\tau}''-2H(aa_{x\tau}'')_x',\qquad(2.14)$$

and replacing a temporarily by $b=a+aa_x'$ we get the equation

$$b_{\tau\tau}''+b_{xx}''=2[(bb_x')_x'-H(bb_{x\tau}'')]_x',\qquad(2.15)$$

the left-hand side of which can be approximated by

$$b_{\tau\tau}''+b_{xx}''=-2H(b_\tau'+Hb_x')_x'.\qquad(2.16)$$

Equation (2.15) can then be reduced to the form

$$(b-bb_x')_\tau'+H(b-bb_x')_x'=b_x'Hb_x'.\qquad(2.17)$$

Returning now to the function $a=b-bb_x'$ we obtain our fundamental nonlinear ITD equation (2.1).

The quasigas system (2.7) is encountered not only in the ITD problem but also, as shown in Ref. 10, in problems involving nonlinear long-wave solition perturbations of the nonlinear Schrödinger equation (NSE) and two-dimensional ("rational") Kadomtsev–Petviashvili (KP) solitions, which can consequently also be described by the nonlinear ITD equation. Although ITD in unbounded streams is not considered in Ref. 10, for the KP solition, it does contain a derivation of the parametric equations

$$\gamma t=-\xi/\rho,\qquad \rho=\operatorname{sh}\xi/(\operatorname{ch}\xi+\cos\eta),$$
$$\gamma x/v_0=\eta-\xi\sin\eta/\operatorname{sh}\xi,\qquad(2.18)$$

integration of which yields the simplest perturbation of the tangential-discontinuity-plane which is periodic in x and is a solution of (2.1):

$$a(\xi,\eta)=\frac{2v_0}{\gamma}\left[\operatorname{arctg}\left(\frac{\sin\eta}{e^\xi+\cos\eta}\right)+\frac{\rho-1}{2\operatorname{sh}\xi}\xi\sin\eta\right].\qquad(2.19)$$

In the limit $\xi\gg1$ it yields

$$a(t,x)=\frac{2v_0}{\gamma}e^{\gamma t}\sin\left(\frac{\gamma x}{v_0}\right),$$

which coincides with the linear ITD theory.[7]

It is curious that the nonlinear maxima of the $a(t,x)$ profile are shifted not downstream but upstream relative to the maxima of the linear theory (see Fig. 1), and that the solution (2.18), (2.19) acquires in this case a singularity, so that the equations are not valid for $T>-2$. It can be assumed that vortex generation on the interface and a vortex street of the Kelvin-Taylor "cat's-eye" type[11] set in at the instant $T=-2$.

We conclude this section by pointing out that surface equations with Hilbert operators are encountered also in a number of other problems, such as the Benjamin-Ono equation. We have also obtained an equation of the form

$$a_t'+\varepsilon vHa_x'\to{}^1/_2v(a_x')^2,\qquad \varepsilon=(\rho_1-\rho_2)/(\rho_1+\rho_2),\qquad(2.20)$$

which describes, when systematic account is taken of all the

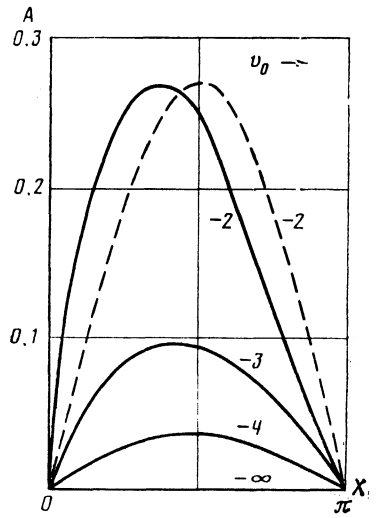

FIG. 1. Profile, described by Eq. (2.19), of the boundary of the perturbed tangential discontinuity $a(t,x)$. The numbers on the curve indicate the times $T<0$. The dashed curve is the sinusoid of the linear theory. A vortex street $A=\gamma a/v_0$ sets in at $T=-2$.

quadratic corrections, the instability of a flame front during slow combustion of a gas moving with front velocity v. In the linear approximation this problem was first considered by Landau, and also by Dardier, and a qualitative picture of the nonlinear flame-front stabilization was proposed by Shchelkin and independently by Zel'dovich. We were unable, however, to find a general method of solving Eq. (2.20), in contrast to the solution of the ITD equation (2.1).

3. ELECTRON-BEAM BUNCHING IN A PLASMA

As the next example we consider the problem of bunching of an electron beam passing through a plasma. This problem was studied earlier in Ref. 3, where certain particular solutions were indicated, but the general Cauchy problem was not solved. If the ions are regarded as immobile and we assume zero temperature, the equations for the electron beam (b) and for the electrons of the plasma proper (e) are

$$\dot n_b+(n_bv_b)_x'=0,\quad \dot v_b+v_b(v_b)_x'=-|e|E/m,$$
$$\dot n_e+(n_ev_e)_x'=0,\quad \dot v_e+v_e(v_e)_x'=-|e|E/m,\qquad (3.1)$$
$$\operatorname{div}\mathbf{E}=E'_x=4\pi|e|(N-n_e-n_b).$$

Here $N=n_i=\text{const}$ is the density of the immobile ions. If n_b^0 and v_e^0 are the initial beam density and velocity, and the quasineutrality condition is satisfied initially, the system (3.1) yields in the linear approximation the dispersion relation

$$1-(1-\varepsilon)\omega_0^2/\omega^2-\varepsilon\omega_0^2/(kv_b^0-\omega)^2=0,\qquad (3.2)$$

where $\varepsilon=n_b^0/N=\text{const}$, $\omega_0^2=4\pi Ne^2/m$. In the region

$$kv_b^0/\omega_0<k_{max}v_b^0/\omega_0=[(1-\varepsilon)^{1/3}+\varepsilon^{1/3}]^{3/2}\ (\sim 1\ \text{for}\ \varepsilon\ll 1)\qquad (3.3)$$

this dispersion relation has two complex roots $\omega=\omega(k)$. If $k\ll k_{\max}$ we can neglect the 1 in (3.2). This corresponds to satisfaction of the quasineutrality condition $n_e+n_b=N$ and yields roots

$$\omega\approx kv_b^0\{1-\varepsilon\pm i[\varepsilon(1-\varepsilon)]^{1/2}\}\qquad (3.4)$$

in a region far from the growth-rate maximum.

Thus, only when the initiating perturbations have long length ($\lambda\gg v_b^0/\omega_0$) can their subsequent development be described by using the quasineutrality condition $n_b+n_e=N=\text{const}$ in lieu of the last Poisson equation $\operatorname{div}\mathbf{E}=4\pi\rho$ in the system (3.1).

Assuming the quasineutrality condition, we rewrite the set of equations for the long-wave perturbations in the form

$$\sum_\alpha n_\alpha=N=\text{const},\quad \dot n_\alpha=-(n_\alpha v_\alpha)_x',$$

$$\dot v_\alpha+v_\alpha(v_\alpha)_x'=|e|\varphi_x'/m,\qquad (3.5)$$

where $\alpha=b$ and $e=1,2$, with the function φ the same for both particle "species": $\alpha=b=1$, $\alpha=e=2$. This is in fact that system used as the starting point for the problem[3] of electron-beam bunching in a plasma. We shall show below however, that the very same equations describe a few other interesting problems involving tangential discontinuities of the velocity of bounded plasma streams.

4. LONG-WAVE MODELS OF THE EVOLUTION OF A TANGENTIAL DISCONTINUITY IN A BOUNDED STREAM

Consider a rigid-wall pipe of radius R, filled with a liquid of density $\rho_0=\text{const}$. Part of this liquid, in the form of a cylindrical jet, moves in the region $0<r<a_0$ with initial velocity v_0, while in the region $a_0<r<A$ the liquid is initially (at $t=-\infty$) at rest, but instabilities of the tangential velocity discontinuity give rise on the jet boundary $r=a_0$ to perturbations that increase with time. If $r=a(t,x)$ is the perturbed boundary of the jet, it must be subject to the condition $(v_\perp)_{r=a}=\dot a+a_x'(v_\parallel)_{r=a}$, where $v_{\perp,\parallel}$ are velocities determined from the continuity equation

$$\operatorname{div}\mathbf{v}=\frac{\partial}{r\,\partial r}rv_r+\frac{\partial}{\partial x}v_x=0.\qquad (4.1)$$

Assuming potential flow, $\mathbf{v}=\nabla\psi$, $\Delta\psi=0$, and using the condition

$$\psi(t,r,x)=\psi_0(t,x)+r\psi_1(t,x)+r^2\psi_2(t,x)+\dots\qquad (4.2)$$

and the condition $v_\perp=v_r=\partial\psi/\partial r=0$ on the rigid wall at $r=R$, it can be shown that long perturbation waves ($\lambda\gg R$) are subject to the conservation laws

$$\dot S_1=-(S_1v_1)_x',\quad \dot S_2=-(S_2v_2)_x',\qquad (4.3)$$

where $S_1=\pi a^2$ is the cross section of the inner jet, $S_2=\pi(R^2-a^2)$ is the cross section of the layer outside the jet, and the velocities $v_{1,2}$ are equal to

$$v_1=v_{int}(t,x)=(\psi_{0x}')_{int},\quad v_2=v_{ext}(t,x)=(\psi_{0x}')_{ext}.$$

Finally, in the long-wave approximation the pressure $p(t,r,x)$ can be regarded as independent of the radius r and the same in the jet ($0<r<a$) and in the layer ($a<r<R$) outside the jet, and the problem of the "constricted" cylindrical jet and the broader tube is described by the equations

$$\dot S_\alpha=-(S_\alpha v_\alpha)_x',\quad \dot v_\alpha+v_\alpha(v_\alpha)_x'=-\frac{1}{\rho_0}p_x',$$

$$\sum_\alpha S_\alpha=\pi R^2=\text{const},\qquad (4.4)$$

where $\alpha = 1,2$. This system is obviously completely analogous to Eqs. (3.5) for bunching.

It is easy to verify that were we to consider not a cylindrical but a planar "constricted" jet in a region $|z| < a(t,x)$, flowing in a broader planar gap $-L < z < L$, and consider only perturbations symmetric in z, we would obtain in the long-wave approximation $\lambda \gg L$ precisely the same system (4.4).

The next variant of the tangential-discontinuity problem is obtained by introducing in the preceding problem an immobile plane $z = 0$ and considering only the half $z > 0$ of the "constricted" jet described above. This problem is also obviously described by the system (4.4).

One more variant is the problem of two shallow parallel brooks with free surfaces located in a gravitational field $\mathbf{g} = -|g|e_z$ and flowing with different velocities in a trough with rigid vertical walls (or symmetry lines for a periodic system) $y = 0$ and $y = L$ along the x axis above a bottom $z = 0$, with the friction between the latter and the water neglected. It can be shown that the relations that hold in this case are

$$\dot{v}_1 + v_1 v_{1x}' = -g h_x' = \dot{v}_2 + v_2 v_{2x}', \qquad \dot{S}_1 + (S_1 v_1)_x' = 0 = \dot{S}_2 + (S_2 v_2)_x', \tag{4.5}$$

where $S_1 = (L - \xi)h$, $S_2 = \xi h$ are the cross sections of the streams, and $h \approx h(t,x)$ is the depth of the brooks. The instability here has a threshold and sets in if

$$|\Delta v| = |v_1^0 - v_2^0| < V_{max} = 2(gh_0)^{1/2},$$

which differs only by a factor $2^{1/2}$ from the familiar result for the short-wave limit.[13] Equations (4.5) can be simplified near the threshold, when $|\Delta v| \approx V_{\max}$, or under hard-excitation conditions, $|\Delta v| \ll V_{\max}$. In the latter case $h \approx h_0 = \text{const}$ and Eqs. (4.5) reduces to (4.4).

Thus, several different problems are described in the long-wave perturbation approximation by one and the same system of equations (3.5) [or (4.4)], which is partly similar to the system (1.2). We shall therefore consider below a method of solving the system (3.5), which permits us, in particular, to solve the general Cauchy problem also.

5. REDUCTION TO THE PROBLEM OF A "FALLING CEILING"

To be specific, we consider the system (3.6), which we rewrite in the form

$$\begin{gathered} S_1 + S_2 = S_0 = \text{const}, \qquad S_1 v_1 + S_2 v_2 = Q_0 = \text{const}, \\ \dot{S}_1 + (S_1 v_1)_x' = 0, \qquad \dot{S}_2 + (S_2 v_2)_x' = 0, \\ \dot{v}_1 + v_1 v_{1x}' = \dot{v}_2 + v_2 v_{2x}'. \end{gathered} \tag{5.1}$$

The symmetry with respect to the subscripts 1 and 2 suggests that it is advantageous here to introduce the differences

$$s = S_1 - S_2, \qquad u = v_1 - v_2, \tag{5.2}$$

in terms of which the remaining quantities are expressed:

$$S_{1,2} = {}^1/_2 (S_0 \pm s), \qquad v_{1,2} = \overline{V}_0 \pm {}^1/_2 u (1 \mp s/S_0), \tag{5.3}$$

where $\overline{V}_0 = Q_0/S_0$ is the average velocity of the initial flow. The derivatives of the differences (5.2) are respectively

$$\dot{s} = -\overline{V}_0 s_x' - S_0 u_x'/2 + (us^2)_x'/2S_0, \quad \dot{u} = -\overline{V}_0 u_x' + (su^2)_x'/2S_0 \tag{5.4}$$

and are obviously related to the general equation type (1.2).

We assume that for unperturbed motion the values of s and u are respectively s_0 and u_0, and introduce in place of s and u the new variables

$$\rho = \frac{S_0^2 - s^2}{S_0^2 - s_0^2} \left(\frac{u}{u_0} \right)^2, \qquad w = \frac{s_0 u_0 - su}{S_0}. \tag{5.5}$$

It is then easy to verify that Eqs. (5.4) take the form

$$\dot{\rho} + [(w + V_{dr})\rho]_x' = 0, \quad \dot{w} + (w + V_{dr}) w_x' = c_0^2 \rho_x'; \tag{5.6}$$

they contain as the initial unperturbed parameters the combinations

$$\begin{gathered} c_0^2 = {}^1/_4 u_0^2 (1 - s_0^2/S_0^2) = u_0^2 S_{10} S_{20}/S_0^2, \\ V_{dr} = \overline{V}_0 - s_0 u_0/S_0 = (S_{10} v_{20} + S_{20} v_{10})/S_0. \end{gathered} \tag{5.7}$$

The characteristic velocity c_0 determines the growth rate of the perturbations during the linear stage, and $V_{dr} \neq 0$ specifies the systematic drift.

The initial variables S_α, and v_α ($\alpha = 1,2$) with the new ρ and w are related by

$$\begin{gathered} S_\alpha = S_0 (1 \mp \eta)/2, \qquad v_\alpha = \overline{V}_0 + \tilde{w}(\eta \pm 1)/2\eta, \\ s/S_0 = \eta = \tilde{w}/u, \qquad u = (4c_0^2 \rho + \tilde{w}^2)^{1/2}, \end{gathered} \tag{5.8}$$

where we put for brevity $\tilde{w} = w + V_{dr} - \overline{V}_0$ and assume, to be specific, that $u_0 > 0$. It is particularly easy to determine the pressure in the liquid

$$(p - p_0)/\rho_0 = c_0^2 (1 - \rho) \tag{5.9}$$

in the case of a tangential discontinuity, or the electric-field potential

$$|e|\varphi/m = c_0^2 (\rho - 1) \tag{5.10}$$

in the case of beam bunching in a plasma.

It is remarkable that, apart from the drift that can be easily eliminated formally by the substitution $x \to \tilde{x} + V_{dr} t$, the resultant system (5.6) is equivalent to the system describing "falling ceiling" that is artifically free of surface tension,[8] or describing nondiffractive self-focusing of light.[14] In the case of "a falling ceiling drop" the effective density ρ is the reduced layer thickness h/h_0:

$$\dot{\rho} + (w\rho)_x' = 0, \quad \dot{w} + w w_x' = c_0^2 \rho_x', \quad c_0^2 = g h_0. \tag{5.11}$$

Here g is the gravity acceleration, h_0 the initial unperturbed liquid-layer thickness, and w the longitudinal velocity. This analogy is quite useful and illustratively represents the evolution of the instability in terms of a "falling ceiling," since the shape of a dripping water drop (more accurately, of a flat clot separated from a homogeneous layer) corresponds directly to the profile (5.9) of the pressure or (5.10) of the potential.

6. EXACT SOLUTION IN THE FORM OF A SPONTANEOUS DROP

The system (5.11) is a particular, but quite representative in applications, subclass of a number of unstable media described by the dynamics equations for a polytropic liquid with negative compressibility.[10] We have obtained[9,10] for such quasi-Chaplygin media the most typical spontaneously growing perturbations that are either lengthwise periodic or local, in the form of a "pit," "hump," or a "doublet"

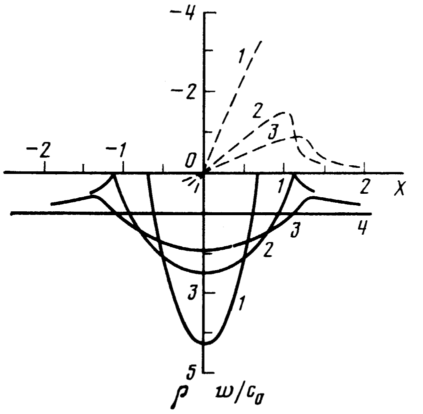

FIG. 2. "Elementary" "falling ceiling." Solid lines—relative thickness of water layer, dashed—longitudinal-velocity profiles. In all the figures, the parameter of the curves is the time $\Delta T = T - T_{cr}$ measured from the instant of drop separation: 1—$\Delta T = +0.2$; 2—0; 3— -0.2; 4— $-\infty$.

("pit" + "hump"), and can now be used as examples of exact solutions for bunching or tangential-discontinuity problems.

These exact solutions are expressed in terms of elliptic integrals, but we have recently been able to identify a number of explicit solutions containing only elementary functions. For example, separation of one drop or clot (see Fig. 2) from a homogeneous layer is described by the parametric equation

$$\rho = s^2 + (s-1)^{1/2}/T, \quad w = 2c_0(X/T)(s-1)/[1+3(s-1)], \quad X = \gamma x/c_0 = \pm(3s-2)\left[|T|\left(s^{-1}(s-1)^{-1/2}+T\right)\right]^{1/2}, \tag{6.1}$$

where $T = \gamma t < 0$, $\gamma > 0$ is a free parameter (growth rate) that specifies the characteristic time or the perturbation length and assumed given by $1 < s < s_{max}$, $s_{max}(1 - s_{max})^{1/2} = |T|^{-1}$. During the linear stage, when $|T| \to \infty$, we obtain from (6.1)

$$\rho \approx 1 + (T^2 - X^2)(T^2 + X^2)^{-2}, \quad w = 2c_0 XT(T^2 + X^2)^{-2}, \tag{6.2}$$

corresponding to two "pits" separated by a "hump." The amplitude of the "pits" increases next and the drop is separated from the layer at an instant $T = T_{cr} = -3^{3/2}/16$ at the points $X = \pm 9/8$ near which

$$\rho \approx \frac{1}{3}\left(\frac{48}{5}\left[1 - \left(\frac{8x}{9}\right)^2\right]\right)^{2/3}, \quad w \approx \mp\frac{2c_0}{3^{1/2}}. \tag{6.3}$$

The drop changes next into a downward-trickling and contracting "streamline" that as $t \to 0$ collapses to the point $X = 0$. The solution (6.1) is close here to the self-similar one determined in Ref. 9:

$$\rho \approx |T|^{-2/3}(1 - X^2/9|T|^{4/3}), \quad w \approx 2c_0 X/3T. \tag{6.4}$$

The particular solution (6.1)–(6.4) provides a quite simple illustrative picture of the drop (clot) development. It is easily transformed with the aid of relations (5.8)–(5.10) into the picture of the evolution of tangential discontinuities in the cases listed above (Sec. 4), or of electron-beam bunching, which will be discussed below for specific examples. We report beforehand, however, some general results for the "falling ceiling" problem. It is worth recalling that the subject here is a model problem with "long-wave" drops which moreover have no surface tension.

7. HODOGRAPH TRANSFORMATION AND GENERAL SOLUTION OF THE "FALLING CEILING" PROBLEM

The solutions noted above and those similar to them can be found by using the hodograph transformation. For the inverse functions $t = t(\rho, w)$, $x = x(\rho, w)$ we obtain the equations

$$x_w' = wt_w' - \rho t_\rho', \quad x_\rho' = wt_\rho' + c_0^2 t_w', \tag{7.1}$$

which are compatible under the condition[8]

$$t_{rr}'' + 3r^{-1}t_r' + t_{zz}'' = 0, \tag{7.2}$$

where we have introduced the convenient variables $r = \rho^{1/2}$, $z = w/2c_0$. Using known analogs (see, e.g., Ref. 15), one can indicate for this equation a formal general solution

$$t = \int dz' R^{-3}[\tau_1(z') + \tau_2(z')R^4 r^{-2}], \quad R^2 = r^2 + (z - z')^2, \tag{7.3}$$

where $\tau_{1,2}(z')$ are arbitrary functions. Using next, for example, the second relation of (7.1) we obtain also the coordinate

$$x = 2zc_0 t + c_0 \int dz' \left\{ \frac{\tau_1'(z')}{R} - \tau_2'(z')\left[R + (z - z')\,\mathrm{arcsh}\frac{z - z'}{r}\right]\right\}. \tag{7.4}$$

Equations (7.3) and (7.4) yield the solution of the Cauchy problem for initial conditions $r = r_0(x)$, $z = z_0(x)$ specified at $t = t_0$. It is much more convenient, however to reduce (7.2) to a simple Laplace equation,[9] as will be done below, and solve the Cauchy problem by electrostatics methods.

8. CAUCHY PROBLEM FOR "FALLING CEILING"

We introduce in addition to the variables r and z a fictitious azimuthal angle and regard r, φ, and z as cylindrical coordinates. Next, putting

$$\psi = rt(r, z)\cos\varphi, \tag{8.1}$$

we obtain from (7.2)

$$\Delta\psi(r, \varphi, z) = 0. \tag{8.2}$$

Now, to solve the Cauchy problem for the evolutional initial conditions $\rho(t_0, x) = \rho_0(x)$, $w(t_0, x) = w_0(x)$ specified at $t = t_0 < 0$, it suffices to use the known relation from electrostatics

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\oint [R^{-1}\nabla'\psi(\mathbf{r}') - \psi(\mathbf{r}')\nabla' R^{-1}]\,dS_0', \tag{8.3}$$

where $\mathbf{R} = \mathbf{r}' - \mathbf{r}$ and $\mathbf{r}$ is a point outside the azimuth-symmetric toroid of the initial values $r = r_0(x)$ and $z = z_0(x)$. Specified on its surface S_0 are a potential

$$\psi = \psi_0 = t_0 r_0(x)\cos\varphi \tag{8.4}$$

and the components of its gradient, which can be readily shown to be

$$\psi_{r0}' = (t_0 + J_0 r_0 z_0'(x))\cos\varphi, \quad \psi_{z0}' = -r_0 J_0 r_0'(x)\cos\varphi, \tag{8.5}$$

where $J_0 = -1/c_0 r_0(r_0'^2 + z_0'^2)$ is the Jacobian of the hodograph transformation. It is assumed that there are no "charges" outside the toroid.

9. COUNTERSTREAMING FLOWS WITH A TRANSITION LAYER

As an example with "constricted" TD we consider the simplest particular case of initially symmetric flow for which $v_{10} = -v_{20} \equiv v_0$, $S_{10} = S_{20} \equiv S_0/2$. The quasigas variables ρ and w of such counter streaming flows are given by

$$\rho = -v_1 v_2 / v_0^2, \quad w = v_1 + v_2, \tag{9.1}$$

with $c_0^2 = v_0^2$ and, furthermore, $\overline{V}_0 = V_{\rm dr} = 0$, so that there is no drift of the perturbations, and Eqs. (5.6) for ρ and w have exactly the same form as Eqs. (5.4) that describe a "falling ceiling." The inverse-transition equations are here

$$v_{1,2} = \frac{1}{2} w \pm v_0 \zeta, \quad \xi = L \frac{s}{S_0} = \frac{Lw}{2v_0 \zeta},$$

$$\zeta = \left[\rho + \left(\frac{w}{2v_0} \right)^2 \right]^{1/2}, \tag{9.2}$$

where $\xi = \xi(t,x)$ is the surface separating the streams and $2L$ is the width of the planar channel.

As shown above, in the quasigas approximation it is also possible to exactly solve the more complicated problem of instability of two oppositely flowing streams separated by a transition zone of finite but small thickness, say $2l$. The simplification that permits solution of this problem is the neglect of the influence of this zone, in view of its smallness, on the bending of the layer as a whole, so that the layer is described as before by Eqs. (5.1). The presence of an inflection point leads to modulation of the transition-zone thickness, and it can be shown that we arrive at a dynamic system of the form

$$\dot\rho + (\rho w)_x' = 0, \quad \dot w + w w_x' = c_0^2 \rho_x', \tag{9.3}$$

$$\dot l + (lV)_x' = 0, \quad \dot V + V V_x' = -\Omega^2 l l_x' + c_0^2 \rho_x', \tag{9.4}$$

where $\Omega = {\rm curl}_y \mathbf{v} \equiv v_0 / l_0$ is the velocity curl in the layer and is assumed constant, l and V are the thickness and velocity of the flow (meaning approximately the average velocity at the center of the layer in the case of a small bend) in the transition zone. The first two equations describe the nonlinear bending of the layer as a whole, and the last two the flow induced by the bend inside the transition zone.

It is curious that the last equations are no longer of the quasigas type, but simply gas equations and have the same form as the dynamic equations of ordinary stable flow of a one-dimensional gas with an adiabatic exponent $\gamma = 3$. Its flow, however, is excited by an external force (here, by the bending) which is determined by the nonlinear system (9.3). It is therefore somewhat unexpected that the motion induced in a layer by a nonlinear bend can be determined accurately and is described by the simple equations

$$V = w/2, \quad l/l_0 = [\rho + (w/2v_0)^2]^{1/2}. \tag{9.5}$$

Thus, knowing how the "falling ceiling" grows, we can draw a fairly complete "long-wave" picture of the instability dynamics of counterstreaming flows of a nonviscous incompressible liquid, with a transition layer of finite thickness besides. For example, the local drop (6.1) generates the flow shown in Fig. 3. The initially horizontal layer tends next to break up and partition off the channel. Its thickness at the center is thereby catostrophically increased and the flow is choked.

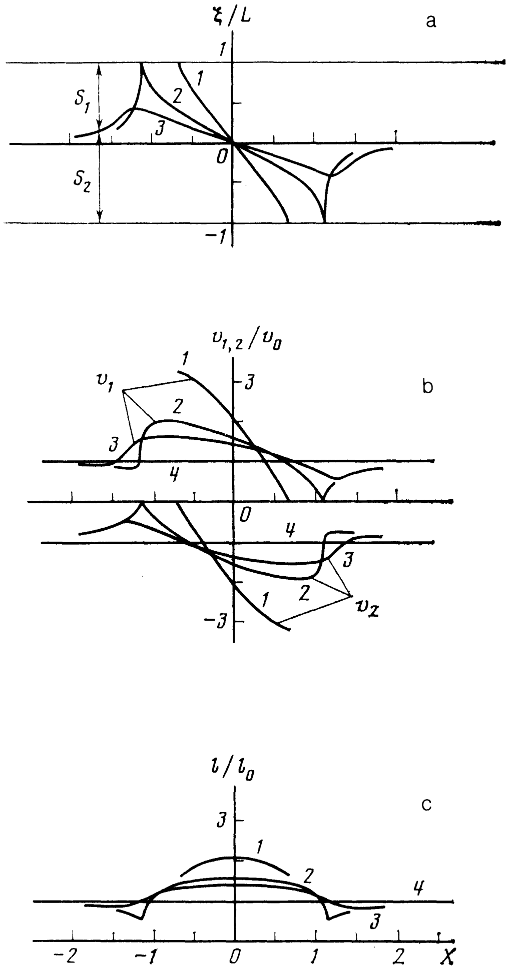

FIG. 3. Dynamics generated by a drop [Eq. (6.1)], of the instability of counter streaming liquid streams; a—nonlinear bending of the stream boundary, b—longitudinal-velocity profiles, c—evolution of transition-zone thickness.

10. DYNAMIC POTENTIAL SPIKE IN A BUNCHING BEAM

The analogy between the problem of beam bunching in a plasma and the problem of light self-focusing (mathematically identical, if diffraction is neglected, with that of the "falling ceiling") was first established by Bulanov and Sasorov, although only for a low-density beam, $n_b \ll N$. It is clear from the foregoing that the analogy is complete in the sense that beam bunching in a plasma reduces exactly, without any additional assumptions whatever, to the equations for the "falling ceiling" with arbitrary initial parameters.

We write down the final equations of type (5.7) and (5.8) as applied to the bunching problem for the particular case $\overline{V}_0 = 0$, when the beam current is neutralized. It is convenient to denote

$$v_b^0 = v_0, \quad \varepsilon = n_b^0 / N < 1/2. \tag{10.1}$$

In this notation, we obtain for the characteristic velocities (5.7)

$$c_0^2 = \varepsilon v_0^2 / (1-\varepsilon), \quad V_{\rm dr} = v_0 (1-2\varepsilon)/(1-\varepsilon), \tag{10.2}$$

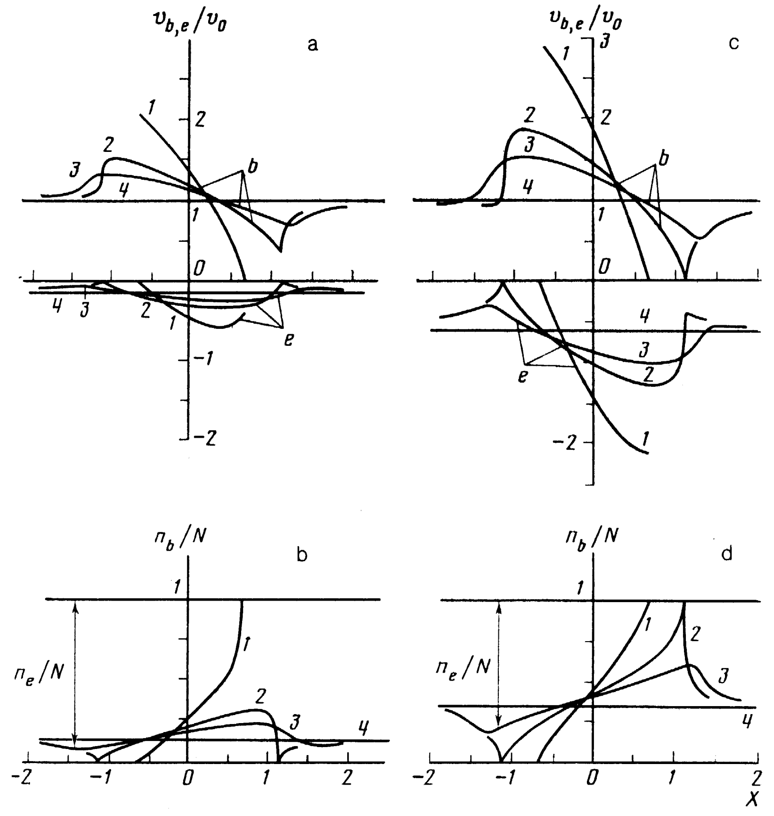

FIG. 4. Bunching, described by the drop (6.1), of a weak (a, b; $\varepsilon < 1/4$) and intense (c, d; $\varepsilon > 1/4$) electron beam in a plasma; a, c—particle-velocity profiles, b, d—density profiles. It is assumed that $\varepsilon^{a,b} = 1/8$ and $\varepsilon^{c,d} = 3/8$.

and the parameters of the plasma–beam system are connected with the parameters ρ and w for the drops by the relations

$$n_{b,e}=N(1\mp\eta)/2,\quad v_{b,e}=(w+V_{dr})(\eta\pm1)/2\eta,\qquad \eta=(w+V_{dr})/[4c_0^2\rho+(w+V_{dr})^2]^{1/2}. \tag{10.3}$$

It is assumed that $\rho^0 = 1$ and $w^0 = 0$ in the unperturbed state.

We illustrate the instability dynamics by using the exact solution of (6.1) for a single dropbunch, assuming that the drift has already been substrated, i.e., a coordinate change $x \to x - V_{dr}t$.

With increase of the perturbation, the initially homogeneous beam breaks up into clusters or bunches (Fig. 4). The instant of bunching corresponds exactly to the time when the drop separates from the layer of water, when $\rho \to 0$. It is curious that the character of the bunching for an initial beam density $\varepsilon < 1/4$ (Fig. 4, a and b) differs substantially from that for $\varepsilon > 1/4$ (Fig. 4, c and d). The beam remains continuous at $\varepsilon > 1/4$. If $\varepsilon < 1/4$ a discontinuity takes place and a bunch breaks away from the beam.

During the concluding phase of the instability the bunch electrons crowd out the plasma electrons completely, the bunch tends to take the form of a step, and a catastrophic potential spike is produced on its trailing edge:

$$\frac{|e|}{m}\varphi|_{x=0}\approx c_0^2\rho\approx c_0^2|T|^{-2/3}\to+\infty. \tag{10.4}$$

It must be recognized, of course, that we are restricted to a long-wave approximation governed mainly by the assumption that the plasma is quasi-neutral (see Sec. 3). The long-wave picture is valid only under the condition

$$\left|1-\frac{n_e+n_b}{N}\right|=\left|\frac{\delta n}{N}\right|=\left|\frac{\varphi_{xx}''}{4\pi eN}\right|=\frac{c_0^2}{\omega_0^2}|\rho_{xx}''|\ll1, \tag{10.5}$$

where $\omega_0 = (4\pi Ne^2/m)^{1/2}$ and relation (5.10) is used. Let us estimate now to what extent (10.5) is violated at the single points.

At time $t = t_{cr}$ when the beam breaks we have, according to (6.3), $\rho \sim |\delta X|^{2/3} \to 0$ and the condition (10.5) is certainly violated near the singularity, but over a length

$$\omega_0|\delta x|/c_0\sim(\gamma/\omega_0)^{1/2}\ll1, \tag{10.6}$$

much shorter than the Debye radius ($\sim v_0/\omega_0$), since it is assumed that $\gamma \ll \omega_0$. Here γ is a parameter in the solution (6.1) and determines the characteristic "bare" length $l_{char} = c_0/\gamma$ of the perturbation. At the limit of applicability we have $\rho \sim \gamma/\omega_0 \ll 1$ and the discontinuity should be distinctly traceable.

In the field-collapse phase, using the self-similar asymptotic relation (6.4), we get

$$c_0^2|\rho_{xx}''|/\omega_0^2\approx2\gamma^2/9\omega_0^2T^2. \tag{10.7}$$

Quasineutrality is violated only when at a time such that $|\gamma t| = |T| \sim \gamma/\omega_0 \ll 1$ holds and the length of the collapse region, which decreases like $\delta X \approx 6|T|^{2/3}$, already is shorter than the "bare" length

$$|\delta x/l_{char}|\sim(\gamma/\omega_0)^{2/3}\ll1, \tag{10.8}$$

but is still long compared with the Debye radius

$$\omega_0|\delta x|/c_0\sim(\omega_0/\gamma)^{1/3}\gg1. \tag{10.9}$$

One can therefore hope that the quasigas approximation yields, in general outline, a qualitatively correct picture, but must be supplemented by allowance for dispersion effects in the vicinity of the singular points, a task outside the scope of the present article.

11. BUNEMAN INSTABILITY OF A PLASMA

A situation similar to the dynamic spike of the potential takes place during the nonlinear stage of a Buneman instability.[4-6] For long perturbation waves, this instability is described by the system of equations (Z_i is the plasma-ion charge)

$$Z_i^{-1}m_i(\dot{v}_i+v_i(v_i)_x')=-m_e(\dot{v}_e+v_e(v_e)_x')=-|e|\varphi_x',$$
$$\dot{n}_e+(n_ev_e)_x'=0=\dot{n}_i+(n_iv_i)_x', \quad n_e=n_iZ_i\equiv n, \tag{11.1}$$

which is similar to the bunching system (3.5) above. It reduces, however, not to the "falling ceiling," but to the simpler dynamics of a Chaplygin "gas" (Refs. 8–10), with account taken of the finite mass ratio and for arbitrary Z_i. This is one more example of exact reduction to a simpler integrable system, and we therefore present the derivation. This reduction is achieved by a transformation similar to a change to center-of-mass coordinates, and we therefore introduce

$$u=v_e-v_i, \quad w=(m_iv_i+Z_im_ev_e)/(m_i+Z_im_e). \tag{11.2}$$

The system (11.1) yields the current conservation law $un = u_0n_0 =$ const, and for w we obtain by direct calculation the equation

$$\dot{w}=-\frac{1}{2}\left[w^2+\frac{\mu}{M}u^2\right]_x', \quad \mu=\frac{Z_im_em_i}{M}, \quad M=Z_im_e+m_i. \tag{11.3}$$

We note next that the continuity equation leads to the relation

$$\dot{n}=-(nw)_x'. \tag{11.4}$$

Taken together, this yields in fact the equations for the Chaplygin "gas":

$$\dot{n}=-(nw)_x', \quad Mn(\dot{w}+ww_x')=-p_x', \quad p=\mu u_0^2n_0^2/n. \tag{11.5}$$

In contrast to Refs. 8–10, we have here $w_0\neq 0$, but this is not of fundamental significance. The weak drift of the perturbations ($\sim\mu/M$) due to the finite inertia of the electrons was noted earlier.[6]

A more important fact, not noted in those papers, is that the exact equation for the potential

$$\varphi_x'(x,t)=\frac{\mu}{Z_i|e|}\left[\frac{m_i-Z_im_e}{2M}(u^2)_x'+2uw_x'\right] \tag{11.6}$$

contains a correction $\sim(\mu/M)^{1/2}$ that leads to a potential jump in the perturbation region. For example, taking for the Chaplygin "gas" an exact solution[10] in the form of a "hump" or a "pit" ($|\theta|<\pi/2$)

$$n_0/n=u/u_0=1\pm T^{-1}\cos^2\theta,$$
$$w=w_0-(c_0/2T)\sin 2\theta, \quad X=\gamma x/c_0=\theta\pm T\,\mathrm{tg}\,\theta, \tag{11.7}$$

where $c_0 = u_0(\mu/M)^{1/2}$, $T=\gamma t<0$ [$T<-1$ for the (+) mode], and γ is a parameter, we find that in both cases the potential jump increases like

$$\delta\varphi=\int_{-\infty}^{+\infty}dx\,\varphi_x'=\pi\mu u_0^2\left(\frac{\mu}{M}\right)^{1/2}\Big/2Z_i|e|T^2. \tag{11.8}$$

For an arbitrary local structure (such that $n\to n_0$ as $|x|\to\infty$) we have according to (11.6), in the variables $r=n_0/n$, $z=w/c_0$, which for a Chaplygin "gas" are the most natural ones,[1)]

$$\delta\varphi=2(\mu u_0^2/|e|Z_i)(\mu/M)^{1/2}\int_{t=\mathrm{const}}r\,dz. \tag{11.9}$$

By the same token, the potential jump is proportional to the area of the intersection of the surface $t=t(r,z)$ and the plane $t=$ const.

An estimate of the degree to which plasma quasineutrality is violated for a Buneman instability is given in Ref. 6.

We have thus shown in the present paper that, for certain particular cases, systems of the form (1.1) are completely integrable under arbitrary initial conditions.

[1)]In terms of these variables, the problem of determining the inverse functions $x=x(r,z)$ and $t=t(r,z)$ reduces to the "planar" Laplace equation[8-10] $t_{rr}''+t_{zz}''=0$.

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