Instability and collapse of solitons in media with a defocusing nonlinearity

E. A. Kuznetsov and S. K. Turitsyn

Institute of Automation and Electrometry, Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk

(Submitted 4 February 1988)
Zh. Eksp. Teor. Fiz. **94**, 119–129 (August 1988)

The nonlinear Schrödinger equation is used to show that one-dimensional solitons are unstable against transverse modulations in media with a defocusing nonlinearity. This instability is similar to the Kadomtsev–Petviashvili instability of acoustic solitons in media with a weak positive dispersion. It is shown that the nonlinear stage of the development of this instability represents collapse at low soliton amplitudes. An integral criterion of the collapse is obtained.

INTRODUCTION

The problems of soliton stability have already been investigated thoroughly using universal models based on the Korteweg–de Vries (KdV) equation and on the nonlinear Schrödinger equation with attraction, and with various generalizations of these models in plasma physics, nonlinear optics, and hydrodynamics.[1] An analysis of the stability is usually followed by an investigation of nonlinear dynamics of solitons and here the greatest progress has been made in integrable (usually one-dimensional) models which admit exact solutions describing soliton scattering[2] and the nonlinear dynamics of soliton annihilation.[3,4] The situation is more complex in nonintegrable models: the problem of the nonlinear stage of the development of an instability can frequently be understood only qualitatively. In typical situations an allowance for nonlinear effects results in collapse, i.e., creation of a field singularity, in a finite time. Self-similar solutions can be derived near a singularity in models of this kind. Among these models the best known is based on the nonlinear Schrödinger equation with attraction, which is used widely in various branches of physics. This equation demonstrates different types of nonlinear behavior, depending on the dimensionality. In the one-dimensional case the nonlinear Schrödinger equation applies to integrable models with stable solitons,[5] but when the number of dimensions is increased, solitons become unstable[6] and the nonlinear stage terminates—depending on the initial perturbations—either by complete decay or collapse.[7,8]

We shall report new results on the instability and collapse of solitons described by the nonlinear Schrödinger equation with repulsion

$$i\psi_t+{}^1/_2\Delta\psi+(1-|\psi|^2)\psi=0. \qquad (1)$$

This equation is obtained in a description of the propagation of a packet of electromagnetic waves in media with a defocusing nonlinearity. In this case the quantity ψ represents the amplitude of the electric field, while $-|\psi|^2$ in Eq. (1) is a nonlinear negative correction to the refractive index. Equation (1) was first derived by Gross and Pitaevskiĭ[9] to describe oscillations of a condensate of a slightly nonideal Bose gas [so that frequently Eq. (1) is called the Gross–Pitaevskiĭ equation]. In the latter case the symbol ψ denotes the wave function of the condensate and Eq. (1) is the Schrödinger equation with a potential $U=|\psi|^2$. If we represent the function ψ in the form $\psi=n^{1/2}e^{i\Phi}$, we can regard Eq. (1) as one of the gasdynamic models with dispersion[10]:

$$n_t+\operatorname{div} n\nabla\Phi=0,\quad \Phi_t+{}^1/_2(\nabla\Phi)^2+n-1=\Delta n^{1/2}/2n^{1/2}, \qquad (2)$$

where the pressure is $p=n^2/2$. The positive nature of the pressure or, in other words, the repulsion occurring in Eq. (1) is the reason why there are no bound states for finite distributions in Eqs. (1) and (2). The localized solutions in the form of solitons appear only for a gas, i.e., superposed on a constant density n_0 (to be specific, we shall consider the case when $n_0=1$). The simplest of these solutions are one-dimensional solitons:

$$\psi_0=\nu\,\operatorname{th}\nu(x-\varkappa t-x_0)+i\varkappa,\quad \varkappa^2+\nu^2=1. \qquad (3)$$

In the case of a Bose gas they represent wells of density $|\psi_0|^2=1-v^2/\cosh^2 v(x-\varkappa t-x_0)$ moving at a velocity $\varkappa<1$ that decreases as the amplitude ν increases. In the case of electromagnetic waves in a defocusing medium a soliton of Eq. (2) corresponds to the region of lower light intensity, i.e., to the shadow region. Observation of solitons of this type and a study of their interaction were recently reported in Ref. 11.

We shall show that solitons described by Eq. (2) are unstable in the presence of transverse perturbations. This instability is analogous to the instability of acoustic solitons in media with a weak positive dispersion, discovered by Kadomtsev and Petviashvili (KP).[12] In the final analysis this instability is entirely due to the positive nature of the dispersion of the spectrum of small oscillations. It follows from this that, firstly, one-dimensional solitons represent moving density wells and, secondly, that the velocity of these wells is less than the velocity of sound and that it decreases as the soliton amplitude increases. Therefore, for a soliton modulated weakly in the transverse direction, regions with small amplitude overtake those with large amplitude. This gives rise to an instability of the self-focusing type (Ref. 13).[1)]

We shall show that the analogy with the KP instability is in fact deeper: it applies not only to qualitative aspects, but also to the equations. It is known that for Eq. (1) the spectrum of small oscillations superposed on a constant density $n_0=1$,

$$\omega=k(1+k^2/4)^{1/2}, \qquad (4)$$

is characterized by positive dispersion and becomes acoustic at long wavelengths. It therefore follows directly that the three-dimensional KP equation can be used to describe small-amplitude solitons ($\nu\ll 1$) and their stability. This embedding of the KP equation into the nonlinear Schrödinger equation is important not only for linear stability, but also in studies of nonlinear dynamics. One should add that, as shown in Refs. 15 and 16 and in contrast to the two-dimen-

 0038-5646/88/081583-06$04.00

sional case, the three-dimensional KP equation can describe the phenomenon of collapse which can be regarded as a nonlinear stage of the development of an instability of one-dimensional[12,14] or two-dimensional[17] solitons.

The present paper is organized as follows. We shall first (Sec. 1) consider general properties of the linear stability problem for solitons with an arbitrary amplitude ν in the presence of transverse perturbations. We shall use the results of this analysis to determine the limit of the bound state spectrum in which the frequency of small oscillations vanishes. In the second section we shall use perturbation theory to find the instability growth rate in the long-wavelength limit. The third section deals with an analysis of the stability of the particular equation (2) in the form of a "domain wall" $\psi_0 = \tanh x$, which describes the behavior of a condensate near a solid boundary, where $\psi_0 = 0$. In the concluding section we shall use the Whitham method to study the nonlinear stage of the instability of one-dimensional solitons in the presence of long-wavelength perturbations. We shall also establish the relationship with the collapse of sound in media with a positive dispersion.

1. STABILITY PROBLEM

We shall consider the problem of stability of solitons described by Eq. (3) in the presence of small perturbations in more than one dimension. Adopting a coordinate system moving at a velocity $\varkappa$ and describing a perturbation by

$$\delta\psi=\varphi_1 \exp(-i\Omega t+iky), \quad \delta\psi^*=\varphi_2 \exp(-i\Omega t+iky),$$

we can find the spectrum $\Omega(k)$ by solving the following problem:

$$\Omega\sigma_3 u-{}^1/_2 k^2 u+Lu=0, \tag{5}$$

where

$$L=-i\varkappa\sigma_3\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial x^2}-\begin{pmatrix} 2|\psi_0|^2-1 & \psi_0^2 \\ \psi_0^{*2} & 2|\psi_0|^2-1 \end{pmatrix},$$
$$\sigma_3=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad u=\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

In addition to the representation by Eq. (5), we shall describe this spectral problem also by another representation which is obtained by rotating the eigenfunctions $\varphi_{1,2} = f_1 \pm if_2$:

$$\Omega\sigma_1\tilde{u}-\frac{1}{2}k^2\tilde{u}+\tilde{L}\tilde{u}=0, \tag{6}$$

where

$$\tilde{L}=-i\varkappa\sigma_1\frac{\partial}{\partial x}+\frac{1}{2}\frac{\partial^2}{\partial x^2}+\begin{pmatrix} \dfrac{3\nu^2}{\mathrm{ch}^2\,\nu x}-2\nu^2 & -2\varkappa\nu\,\mathrm{th}\,\nu x \\ -2\varkappa\nu\,\mathrm{th}\,\nu x & \dfrac{\nu^2}{\mathrm{ch}^2\,\nu x}-2\varkappa^2 \end{pmatrix},$$
$$\sigma_1=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \tilde{u}=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

The last form is remarkable because for $\varkappa = 0$, i.e., in the case of a domain wall ($\psi_0 = \tanh x$), the operator $\tilde{L}$ becomes diagonal. The spectral problem of Eq. (6) then simplifies greatly:

$$-i\Omega f_1=(L_0+k^2/2)f_2, \quad +i\Omega f_2=(L_1+k^2/2)f_1, \tag{7}$$

where

$$L_0=-\frac{1}{2}\frac{\partial^2}{\partial x^2}-\frac{1}{\mathrm{ch}^2\,x}, \quad L_1=-\frac{1}{2}\frac{\partial^2}{\partial x^2}-\frac{3}{\mathrm{ch}^2\,x}+2.$$

It is clear from Eqs. (5) and (6) that the main information on the spectrum $\Omega(k)$ is provided by the properties of the operator $\tilde{L}$. Since the operator $\tilde{L}$ is Hermitian, its spectrum is purely real:

$$\tilde{L}\varphi=-{}^1/_2 p^2\varphi, \quad p=p^*. \tag{8}$$

To find the eigenfunctions of $\tilde{L}$ we start by making the following substitutions in Eq. (8):

$$\nu x\to x, \quad \Omega/\nu^2\to\Omega, \quad \varkappa/\nu\to\varkappa, \quad k/\nu\to k, \quad p/\nu\to p; \tag{9}$$

we then seek the solution in the form

$$f_1=e^{iqx}g_1(x), \quad f_2=e^{iqx}g_2(x), \tag{10}$$

where at this stage $g_1(x)$ and $g_2(x)$ are unknown functions, which are finite in the limit $|x|\to\infty$. Considering the asymptotic behavior of Eq. (8) in the limit $|x|\to\infty$, we can find the relationship between p^2 and q^2 making the substitutions given by Eq. (9):

$$p_\pm^2=2(\varkappa^2+1)+q^2\pm2[(\varkappa^2+1)^2+q^2\varkappa^2]^{1/2}. \tag{11}$$

Since Eq. (6) contains only the powers of tanh x, we have to find g_1 and g_2 in the form of a series in powers of tanh x. After substitution it is quite readily found that such series terminate at the second and third terms. We thus obtain

$$f_1=-e^{iqx}[q^2+1+3iq\,\mathrm{th}\,x-3\,\mathrm{th}^2\,x],$$
$$f_2=\frac{(4+p^2+q^2)}{2\varkappa}e^{iqx}[iq-\mathrm{th}\,x]. \tag{12}$$

It is clear from Eqs. (11) and (12) that the continuous spectrum of the operator $\tilde{L}$ consists of two branches: $4(1+\varkappa^2)<p_+^2<\infty$, $0<p_-^2<\infty$. The corresponding eigenfunctions are given by the expressions in the system (12). In addition to a continuous spectrum, the operator $\tilde{L}$ contains also a discrete spectrum:

$$q=i, \quad p_\pm^2=2\varkappa^2+1\pm2(\varkappa^4+\varkappa^2+1)^{1/2}, \tag{13}$$
$$f_1=\frac{3\,\mathrm{th}\,x}{\mathrm{ch}\,x}, \quad f_2=\frac{3+p^2}{2\varkappa}\frac{1}{\mathrm{ch}\,x};$$
$$q=2i, \quad p_\pm^2=0, \quad f_1=3/\mathrm{ch}^2\,x, \quad f_2=0. \tag{14}$$

The second eigenfunction of the discrete spectrum has a simple meaning: in Eq. (5) or (6) it corresponds to a mode characterized by a neutral stability ($\Omega = k = 0$) and a shift of the soliton as a whole:

$$u_0=\frac{\partial}{\partial x}\begin{pmatrix} \psi_0 \\ \psi_0^* \end{pmatrix}, \quad \tilde{u}_0=\begin{pmatrix} 1/\mathrm{ch}^2\,x \\ 0 \end{pmatrix}. \tag{15}$$

For the first eigenfunction in the discrete spectrum of the operator $\tilde{L}$, we find that in the spectral problem of Eq. (5) it corresponds to a second bound state with $\Omega = 0$:

$$k_{cr}^2=-(2\varkappa^2+1)+2(\varkappa^4+\varkappa^2+1)^{1/2},$$

or in units of ν [see Eq. (9)]

$$k_{cr}^2=\nu^2-2+2(\nu^4-\nu^2+1)^{1/2}. \tag{16}$$

The points $\Omega = 0$, $k = 0$ and $\Omega = 0$, $k = k_{cr}$ determine, as shown below, the limit of the spectrum of bound states. From the point of view of stability only the bound states are of interest. This is because an unstable mode must be localized. It cannot belong to functions of the continuous spectrum, which have exactly the same asymptotes at infinity as the condensate and are therefore stable.

2. SOLITON INSTABILITY

We shall now calculate directly the spectrum $\Omega(k)$ of bound states. We shall consider the long-wavelength limit $k \to 0$ and we shall seek the solution of Eq. (5) in the form of a series in powers of k:

$$u = u_0 + u_1 + u_2 + \ldots, \quad \Omega = \Omega_1 + \Omega_2 + \ldots$$

We shall assume that u_0 is a neutrally stable function of Eq. (15), for which we have

$$Lu_0 = 0.$$

In the next order of perturbation theory, we find that

$$\Omega \sigma_3 u_0 + L u_1 = 0. \tag{17}$$

To find u_1 we consider the stationary form of the equation for a soliton, Eq. (8):

$$-i\varkappa \frac{\partial \psi_0}{\partial x} + \frac{1}{2} \frac{\partial^2 \psi_0}{\partial x^2} + (1 - |\psi_0|^2)\psi_0 = 0. \tag{18}$$

We can readily see that differentiation in Eq. (18) and of the complex conjugate of this equation with respect to $\varkappa$ gives

$$-i\sigma_3 u_0 + L \frac{\partial}{\partial \varkappa}\begin{pmatrix} \psi_0 \\ \psi_0^* \end{pmatrix} = 0,$$

which differs from Eq. (17) only by the factor $i\Omega$. Hence, we obtain

$$u_1 = i\Omega \frac{\partial}{\partial \varkappa}\begin{pmatrix} \psi_0 \\ \psi_0^* \end{pmatrix}. \tag{19}$$

It should be pointed out that this solution is, firstly, the particular solution of Eq. (17) determined to within zeroth eigenfunctions of the operator $L - u_0$ and to within the function of the continuous spectrum of Eq. (12) characterized[2)] by $q = 0$ and $p = 0$; secondly, and this is most important, u_1 tends to a constant in the limit $|x| \to \infty$ and, therefore, it can be formally regarded as one of the functions in the continuous spectrum which, as pointed out already, describe stable modes. These two paradoxes can be resolved because the function u_i described by Eq. (19) is simply an intermediate asymptote of the exact solution in the range $v^{-1} \lesssim x \ll k^{-1}$. If $kx \gtrsim 1$, then the eigenfunction exhibits weak damping with are exponent $\sim k$. We can introduce this damping by supplementing the operator L of Eq. (17) with the term $-k^2/2$:

$$(L - k^2/2) u_1 = -\Omega \sigma_3 u_0. \tag{20}$$

If $k^2 < k_{cr}^2$, it follows from Eqs. (11)–(14) that the operator $L - k^2/2$ is invertible. The existence of $(L - k^2/2)^{-1}$ lifts directly the indeterminacy of the solution and ensures, as is easily confirmed, that the eigenfunction damps proportional to $\exp(-k|x|/v)$. We can demonstrate this by matching the solutions obtained in the regions $v^{-1} \ll |x| \ll v/k$. If x is positive, then in this region we can simplify Eq. (20) by dropping the right-hand side so that the left-hand side is converted into a system with constant coefficients:

$$\begin{aligned} -i\varkappa \varphi_{1x} + {}^1/_2 \varphi_{1xx} - \varphi_1 - {}^1/_2 k^2 \varphi_1 &= (\varkappa + i\nu)^2 \varphi_1, \\ +i\varkappa \varphi_{2x} + {}^1/_2 \varphi_{2xx} - \varphi_2 - {}^1/_2 k^2 \varphi_2 &= (\varkappa - i\nu)^2 \varphi_2. \end{aligned} \tag{21}$$

The solutions of this system are proportional to $e^{-\beta x}$, where in accordance with Eq. (11), the exponent β is described in the limit $k \to 0$ by

$$\beta_{1,2} = \pm k/\nu, \quad \beta_{3,4} = \pm 2\nu.$$

It is obvious that among the required solutions we need to retain only one with the exponent β_1 and all the others can be dropped because they are growing or have already been damped out too.

The relationship between the components φ_1 and φ_2 is easiest to find if we make a number of substitutions:

$$\varphi_1 = (\varkappa + i\nu)\chi_1, \quad \varphi_2 = (\varkappa - i\nu)\chi_2,$$
$$w = \chi_1 - \chi_2, \quad v = \chi_1 + \chi_2.$$

Then, the functions w and v are accurately described by

$$w_{xx} = \frac{k^2}{\nu^2} w, \quad v = -\frac{i\varkappa}{2} w_x.$$

Hence, it follows that $w = w_0 \exp(-kx/\nu)$, $|v| \ll |w|$, or $\chi_1 \approx -\chi_2$. If we now consider the asymptote of Eq. (19) in the range $\nu x \gg 1$, we find that it matches the solution of the system (21) found earlier:

$$\begin{aligned} \varphi_1 &= \frac{\Omega}{\nu}(-\nu - i\varkappa)\exp\left(-\frac{k}{\nu}x\right), \\ \varphi_2 &= \frac{\Omega}{\nu}(\nu - i\varkappa)\exp\left(-\frac{k}{\nu}x\right). \end{aligned} \tag{22}$$

The matching at negative values of x is carried out similarly.

The second order of perturbation theory for the range $x\nu \lesssim 1$ gives

$$\Omega \sigma_3 u_1 - {}^1/_2 k^2 u_0 = -L u_2. \tag{23}$$

The solubility condition on Eq. (22) is the orthogonality of its left-hand side to the zeroth eigenfunction u_0 of the operator L. Since the function u_0 decays rapidly over distances $x \sim \nu^{-1}$, all the scalar products are finite and terminate at the same distance. When we allow for Eq. (19), we find that the instability is described by

$$i\Omega \langle u_0 | \sigma_3 | u_1 \rangle = {}^1/_2 k^2 \langle u_0 | u_0 \rangle$$

or

$$\Omega^2 = -k^2 \nu^2 / 3 < 0. \tag{24}$$

As explained in the Introduction, the reason for this instability is a reduction in soliton velocity described by Eq. (3) on increase of its amplitude.

We shall show that the instability of Eq. (24) represents a continuation of the KP instability to the case of large amplitudes. We can easily show that in the case of low amplitudes the KP equation is contained in the nonlinear Schrödinger equation (1).

Following Ref. 18, we shall introduce slow coordinates and a slow time:

$$t'=\varepsilon^3 t, \quad x'=\varepsilon(x-t), \quad y'=\varepsilon^2 y, \quad z'=\varepsilon^2 z,$$

and we shall seek then the solution of the system (2) in the form of a series in powers of a small parameter ε:

$$n=1+\sum_{k=1}^{\infty}\varepsilon^{2k}n_k(x',y',z',t').$$

This is equivalent to going over to small-amplitude waves propagating within a narrow cone of angles. In third order ($\sim\varepsilon^3$) we obtain the KP equation with a positive dispersion:

$$\frac{\partial}{\partial x}\left(n_{1t}+\frac{3}{2}n_1n_{1x}-\frac{1}{8}n_{1xxx}\right)=-\frac{1}{2}\Delta_\perp n_1, \tag{25}$$

where $\Delta_\perp=\partial^2/\partial y^2+\partial^2/\partial z^2$ and the primes are omitted. Then the one-dimensional (1-D) soliton Eq. (3) in the nonlinear Schrödinger equation of Eq. (1) transforms into a (1-D) soliton of Eq. (25):

$$n_1=-\frac{\nu^2}{\operatorname{ch}^2\nu(x+\nu^2t/2-x_0)}. \tag{26}$$

Its velocity in a coordinate system moving at the velocity of sound is negative, i.e., it decreases with increasing amplitude. As is well known, the solution (26) is unstable in the presence of transverse perturbations with a growth rate[16]

$$\Gamma=\frac{k}{3^{1/2}}\left(\nu^2-\frac{2k}{3^{1/2}}\right)^{1/2}. \tag{27}$$

At low values of k, Eq. (27) for $\Gamma=k\nu/3^{1/2}$ is the same as Eq. (24). Then the instability limit $k_*=(3^{1/2}/2)\nu^2$ is identical with k_{cr} of Eq. (16) in the limit of small values of ν.

We shall conclude this section by noting that the exact solution of the problem obtained by Zakharov[16] for the KP equation, like that for the nonlinear Schrödinger equation, demonstrates slow exponential decay $\sim\exp(-kx/\nu)$. Therefore, the instability of Eq. (24) is a direct continuation of the KP instability.

Figure 1 shows computer-generated traces of the growth rate γ plotted as a function of k for different values of ν, including $\nu=1$.

3. STABILITY OF A "DOMAIN WALL"

We shall now consider the stability of the solution of Eq. (1) in the form of $\psi_0=\tanh x$, describing the behavior of the wave function of the condensate near a solid wall at $x=0$. In this case the stability problem is described by Eqs. (7), which can be reduced to one equation for f_1 (or f_2):

$$\Omega^2 f_1=(L_0+k^2/2)(L_1+k^2/2)f_1 \tag{28}$$

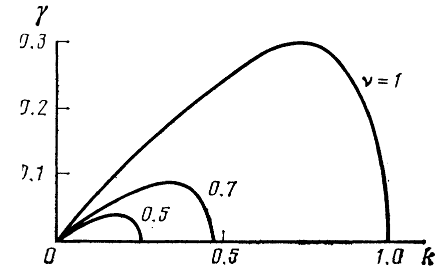

FIG. 1.

subject to an additional boundary condition that the wave function vanishes at the wall:

$$f_1|_{x=0}=0. \tag{29}$$

The operators L_0 and L_1 in Eq. (28) are the Schrödinger operators for one- and two-soliton potentials, respectively. The operator L_0 has only one discrete level with $E_0=-1/2$, which corresponds to the eigenfunction $g_0=1/\cosh x$:

$$L_0g_0=-{}^1/{}_2g_0. \tag{30}$$

The operator L_1 has two bound states:

$$E_1=0, \quad g_1=1/\operatorname{ch}^2 x, \tag{31}$$

$$E_2={}^3/{}_2, \quad g_2=\operatorname{th} x/\operatorname{ch} x. \tag{32}$$

By virtue of the boundary conditions and since the operators L_0 and L_1 exhibit even behavior when x goes into $-x$, all the eigenfunctions of the spectral problem of Eq. (28) are odd. For this class of functions the operator $L_0+k^2/2$ is reversible and positive for all x. In order to ensure stability we need thus to show simply that Ω^2, defined as the lower limit of the functional

$$\Omega^2=\min\frac{\langle f|L_1+k^2/2|f\rangle}{\langle f|(L_0+k^2/2)^{-1}|f\rangle}, \tag{33}$$

is positive. The proof of this is obvious: the operator $L_1+k^2/2$ is positive definite in the class of odd functions and this follows from Eqs. (31) and (32).

We can thus see that the imposition of the boundary condition of Eq. (29) ensures the stability of the solution near a solid wall in spite of the fact that the "domain wall" is essentially unstable. It should be pointed out that the instability of a "domain wall" follows also from analysis of the variational problem (33), as first demonstrated in Ref. 19. The expression for the growth rate was found in the long-wavelength limit in Ref. 22.

4. NONLINEAR STAGE OF THE INSTABILITY OF ONE-DIMENSIONAL SOLITONS

In this section we shall consider the role of nonlinear effects in the development of the instability of one-dimensional solitons. We shall employ an adiabatic approach frequently called the Whitham method, which is based on averaging of the solutions over the fast motion. When applied to the problem in hand, this method allows us to describe the nonlinear stage of the instability at the long-wavelength limit. It should be pointed out that the application of this method to such problems was discussed in detail recently in a review of Trubnikov and Zhdanov.[20] We shall therefore omit many details and give only the principal features of this approach.

We shall begin with an expression for the action in Eq. (1):

$$S=\int dt\,dx\,d\mathbf{r}_\perp\left\{\frac{i}{2}(\psi\psi_t^*-\psi^*\psi_t)+\frac{1}{2}|\nabla_\perp\psi|^2+\frac{1}{2}|\psi_x|^2+\frac{1}{2}(|\psi|^2-1)^2\right\} \tag{34}$$

and average over x the soliton-type solutions

$$\psi_0 = \nu \,\mathrm{th}\, \nu(x-x_0) + i\varkappa, \qquad \varkappa^2+\nu^2=1$$

with the parameters $\varkappa$ and x_0 dependent only on y, z, and t. The adiabaticity of changes in these parameters means that bends in the soliton front are slow and long-wavelength, namely

$$|\nabla_\perp \nu| \ll |\nabla_\perp x_0|, \qquad |\nu_t| \ll |x_{0t}|. \tag{35}$$

After integration with respect to x and subject to Eq. (35), we find that S is given by

$$S = \int dt\, d\mathbf{r}_\perp \left\{ 2\left[\mathrm{arctg}\frac{\nu}{\varkappa} - \nu\varkappa \right] x_{0t} - \frac{4}{3}\nu^3 - \frac{2}{3}\nu^3(\nabla_\perp x_0)^2 \right\}.$$

Variation of this expression with respect to ν and x_0 gives

$$\nu\varkappa_t + {}^1/_3\, \mathrm{div}\, \nu^3 \nabla_\perp x_0 = 0, \qquad x_{0t} - {}^1/_2\varkappa(\nabla_\perp x_0)^2 - \varkappa = 0. \tag{36}$$

We can easily show that the system of equations (36) describes the instability of Eq. (24) in the long-wavelength limit. It is sufficient to consider linearized equations of the system (36) superposed on the exact solution $\varkappa = \varkappa_0$, $x_0 = \varkappa_0 t$. For low values of ν, when the KP approximation is valid, the system (36) is converted into gasdynamic equations with a negative pressure (compare with Refs. 18 and 20)

$$\rho_t + \mathrm{div}\, \rho\nabla\varphi = 0, \qquad \varphi_t + {}^1/_2(\nabla\varphi)^2 - {}^1/_2\rho^{2/3} = 0,$$

where $\rho = \nu^3$, $\varphi = t - x_0$. These equations have self-similar solutions describing strong collapse of a wave in which a finite energy falls into a singularity[7,21]:

$$\rho = \frac{\lambda^3}{a^2}\left(1 - \frac{\mathbf{r}_\perp^2}{a^2}\right)^{3/2}, \qquad \varphi = \frac{1}{2}\int_0^t \frac{\lambda}{a^{4/3}}\,dt + \frac{a_t}{2a}\mathbf{r}_\perp^2,$$

where λ^2 is an arbitrary positive parameter and $a(t)$ obeys Newton's law

$$a_{tt} = -\frac{\partial}{\partial a}U(a) \tag{37}$$

for the impact of a particle on the center in the potential $U(a) = -\frac{3}{4}\lambda^2/a^{4/3}$. As we approach the singularity, $t \to t_0$, we find that $a \propto (t_0 - t)^{3/5}$ (Ref. 20) and the longitudinal size of a soliton obeys $\nu^{-1} \propto (t_0 - t)^{2/5}$, so that the ratio of the longitudinal to the transverse size ν^{-1}/a increases as $(t_0 - t)^{-1/5}$. This means that the criteria of adiabaticity given by Eq. (35) are quite rapidly disobeyed and in the subsequent analysis we have to turn either directly to Eq. (1) or to the system of equations (36), depending on the initial data.

We shall now show that within the framework of Eq. (1) a density well can not disappear as a result of soliton instability.

We recall first that Eq. (1), like the whole system (2), is of the Hamiltonian type:

$$i\psi_t = \delta H/\delta\psi^* \tag{38}$$

or

$$\frac{\partial n}{\partial t} = \frac{\delta H}{\delta \Phi}, \qquad \frac{\partial \Phi}{\partial t} = -\frac{\delta H}{\delta n}, \tag{39}$$

where

$$H = \int\left[\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}(|\psi|^2-1)^2\right]d\mathbf{r}$$
$$= \int\left[\frac{1}{2}n(\nabla\Phi)^2 + \frac{1}{2}(\nabla n^{1/2})^2 + \frac{1}{2}(n-1)^2\right]d\mathbf{r}. \tag{40}$$

In addition to H, Eqs. (1) and (2) conserve the momentum $\mathbf{p} = \int (n-1)\nabla\Phi d\mathbf{r}$ and the number of particles $N = \int (n-1) d\mathbf{r}$.

We shall consider the integral in the Hamiltonian of Eq. (40):

$$H_{int} = \frac{1}{2}\int n(\nabla\Phi)^2\, d\mathbf{r}.$$

Applying the mean-value theorem, we find that

$$H_{int} \geqslant \min_{\mathbf{r}} n \int \frac{(\nabla\Phi)^2}{2}\, d\mathbf{r}.$$

Substitution of the above inequality in Eq. (40) gives

$$H \geqslant \int\left[\frac{1}{2}(n-1)^2 + \min_{\mathbf{r}} n \frac{1}{2}(\nabla\Phi)^2 + \frac{1}{2}(\nabla n^{1/2})^2\right]d\mathbf{r}. \tag{41}$$

The first two terms on the right-hand side of Eq. (41) can be estimated in terms of the momentum p:

$$\frac{1}{2}\int [n-1)^2\, d\mathbf{r} + \min_{\mathbf{r}} n \int \frac{(\nabla\Phi)^2}{2}\, d\mathbf{r} \geqslant (\min_{\mathbf{r}} n)^{1/2}|\mathbf{p}|.$$

Combining this with Eq. (41), we find that the minimum value of the density is bounded above by the conserved quantity

$$\min_{\mathbf{r}} n \leqslant H^2/|\mathbf{p}|^2. \tag{42}$$

Therefore, a density well can exist $(n<1)$ if we satisfy the condition $H^2 - p^2 < 0$. This quantity is kept negative only by the nonlinear interaction, which is described by the term $H_{\rm int}$ in the Hamiltonian. In the case of small-amplitude waves, when we can ignore $H_{\rm int}$, this criterion is always positive: $H^2 - p^2 > 0$. If we assume that H and p in Eq. (42) are their values for the soliton solution of Eq. (3), we find that the ratio

$$\left(\frac{H}{p_x}\right)^2 = \frac{4}{9}\nu^6\left(\nu\varkappa - \mathrm{arctg}\frac{\nu}{\varkappa}\right)^{-2} \tag{43}$$

represents the upper permissible limit for $\min_{\mathbf{r}} n$ for the development of the instability of Eq. (24). The ratio (43) varies smoothly from 1 for $\nu = 0$ to $(16/9)\pi^2$ for $\nu = 1$. Therefore, the initial density well cannot disappear and its depth given by Eq. (43) is always less than the average level $n_0 = 1$.

In the case of a narrow distribution of small-amplitude waves, when the KP equation is valid, the criterion of Eq. (42) can be rewritten in the form[15]

$$\min_{\mathbf{r}} n_1 \leqslant H_{\rm KP}/p_{\rm KP}, \tag{44}$$

where

$$H_{\rm KP} = \int \frac{1}{2}\left(\frac{1}{4}n_{1x}^2 + n_1^3 + (\nabla\Phi_1)^2\right)d\mathbf{r}$$

is the Hamiltonian for the KP equation [Eq. (25)], and

$$\Phi_{1x} = n_1, \qquad p_{\rm KP} = \frac{1}{2}\int n_1^2\, d\mathbf{r}$$

is the corresponding projection of the momentum along the x axis.

The relationship (44), like Eq. (42), is valid for an arbitrary region. It is the relationship (44) that is the central criterion of collapse in the three-dimensional KP equation.[15]

We now consider briefly the reasons for collapse. We assume first of all that they are related to the infinite value of $H_{\rm KP}$ when $p_{\rm KP}$ is conserved as a result of the nonlinear interaction. Consequently, collapse should be regarded as the process of impact of a particle in an unbounded potential [compare with Eq. (37)], where the role of the friction is played by the emission of small-amplitude waves. Such emission of waves results, at first sight quite naturally, in acceleration of the collapse itself. If we consider a certain region with a lower density characterized by $H_{\rm KP} < 0$, then obviously we find that small-amplitude waves emitted from this region carry away positive portions of $H_{\rm KP}$ and $p_{\rm KP}$. The ratio $H_{\rm KP}/p_{\rm KP}$, taken in the region $H_{\rm KP} < 0$, becomes even more negative and eventually tends to $-\infty$. It follows from Eq. (44) that this reduces also $\min_r n_i$. It is known as a weak collapse regime.[7,21] In this process the singularity formally receives, in contrast to a strong collapse, zero energy. Numerical experiments[14,15] demonstrate that the KP equation leads to a weak collapse with maximum emission corresponding to the self-similar solution. In the process of collapse a density well decreases and reaches a value of order unity with a considerable reduction in the longitudinal and transverse dimensions. It should be pointed out that this stage of the collapse is valid in a fairly wide range, because the KP approximation applies for $v^2 \ll 1$, i.e., that a satisfactory description by means of the KP equation is available even for solitons with $v \sim 1/2$. Since the physical reasons for the instability are valid for arbitrary soliton amplitudes and the characteristic growth rates grow, it follows that during the next stage the tendency to collapse will be retained. This process represents cavitation. At present we do not know whether a density wave reaches a minimum value $n = 0$ or how its compression proceeds subsequently. In our opinion, such information can be obtained from numerical experiments.

The authors are grateful to E. G. Shapiro for calculating the instability spectrum numerically.

[1)] The exact solution[14] shows that such a mobility picture, which is based on simple kinematic considerations, describes fairly long-wavelength transverse perturbations but not the most dangerous short-wavelength perturbations or the instability limit.

[2)] This function is a linear combination of u_1 and a function which appears because of the gradient invariance of Eq. (1): $\psi \to \psi e^{i\alpha}$.

[1] E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, Phys. Rep. **142**, 103 (1986).

[2] S. P. Novikov, S. V. Manakov, L. P. Pitaevskiĭ, and V. E. Zakharov, *Theory of Solitons*, Consultants Bureau, New York (1984).

[3] G. E. Fal'kovich, M. D. Spector, and S. K. Turitsyn, Phys. Lett. A **99**, 271 (1983); A. Ya. Orlov, Preprint No. 221, Institute of Atomic Energy, Moscow (1983).

[4] V. E. Zakharov and S. V. Manakov, Zh. Eksp. Teor. Fiz. **69**, 1654 (1975) [Sov. Phys. JETP **42**, 842 (1975)].

[5] V. E. Zakharov and A. B. Shabat, Zh. Eksp. Teor. Fiz. **64**, 1627 (1973) [Sov. Phys. JETP **37**, 823 (1973)].

[6] N. G. Vakhitov and A. A. Kolokolov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **16**, 1020 (1973).

[7] V. E. Zakharov and E. A. Kuznetsov, Zh. Eksp. Teor. Fiz. **91**, 1310 (1986) [Sov. Phys. JETP **64**, 773 (1986)].

[8] G. M. Fraĭman, Zh. Eksp. Teor. Fiz. **88**, 390 (1985) [Sov. Phys. JETP **61**, 228 (1985)].

[9] E. P. Gross, Phys. Rev. **106**, 161 (1957); L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **40**, 646 (1961) [Sov. Phys. JETP **13**, 451 (1961)].

[10] A. V. Gurevich and A. L. Krylov, Zh. Eksp. Teor. Fiz. **92**, 1684 (1987) [Sov. Phys. JETP **65**, 944 (1987)].

[11] P. Emplit, J. P. Hamaide, F. Reynaud, C. Froehly, and A. Barthelemy, Opt. Commun. **62**, 374 (1987).

[12] B. B. Kadomtsev and V. I. Petviashvili, Dokl. Akad. Nauk SSSR **192**, 753 (1970) [Sov. Phys. Dokl. **15**, 539 (1970)].

[13] B. B. Kadomtsev, *Collective Effects in Plasma* [in Russian], Nauka, Moscow (1976).

[14] V. E. Zakharov, Pis'ma Zh. Eksp. Teor. Fiz. **22**, 79 (1975) [JETP Lett. **22**, 172 (1975)].

[15] E. A. Kuznetsov, S. L. Musher, and A. V. Shafarenko, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 204 (1983) [JETP Lett. **37**, 241 (1983)].

[16] E. A. Kuznetsov and S. L. Musher, Zh. Eksp. Teor. Fiz. **91**, 1605 (1986) [Sov. Phys. JETP **64**, 947 (1986)].

[17] E. A. Kuznetsov and S. K. Turitsyn, Zh. Eksp. Teor. Fiz. **82**, 1457 (1982) [Sov. Phys. JETP **55**, 844 (1982)].

[18] V. E. Zakharov and E. A. Kuznetsov, Physica D (Utrecht) **18**, 455 (1986).

[19] E. W. Laedke and K. H. Spatschek, J. Math. Phys. (N. Y.) **20**, 1838 (1979).

[20] B. A. Trubnikov and S. K. Zhdanov, Phys. Rep. **155**, 137 (1987).

[21] V. E. Zakharov, E. A. Kuznetsov, and S. L. Musher, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 125 (1985) [JETP Lett. **41**, 154 (1985)].

[22] E. Infeld and G. Rowlands, Z. Phys. B **37**, 277 (1980).

Translated by A. Tybulewicz