An analog of the quantum Hall effect in a superfluid ^{3}He film

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The quantum Hall effect is discussed for charged and neutral superfluid Fermi liquids with an order parameter like the one of ^{3}He-A. Owing to the nontrivial structure of this order parameter a quantum Hall effect occurs in the absence of a magnetic field. Under the influence of an electric field, or its analog for the case of a neutral fluid (a gradient of the chemical potential) there appears, in addition to the longitudinal superfluid current, a transverse current with Hall conductivity σ_{xy}, which in the weak-coupling approximation takes on half-integer values (1/2)(Ne^2/h) in terms of fundamental units. The quantization of the parameter σ_{xy}, is the consequence of an integer-valued topological invariant of the Bogolyubov matrices in momentum space, invariant which in ^{3}He-A takes on the values $+1$ or -1, depending on the orientation of the orbital angular momentum vector **l** relative to the normal to the film. As the thickness of the film is varied the quantity N changes discontinuously at certain values of the thickness. At the transition point from one plateau to the next in the graph of the dependence of σ_{xy} on an external parameter, the gap in the Fermi quasiparticle spectrum vanishes. The relation of this effect to the topological Chern-Simons mass term in 2 + 1-dimensional quantum electrodynamics is discussed.

1. INTRODUCTION

In a sufficiently thin film of ^{3}He-A there must occur a phenomenon which resembles the quantization of Hall conductivity in a two-dimensional electron system in a magnetic field (Ref. 1). As the analog of an electric field (the role of which is played by a gradient of the chemical potential $\nabla\mu$) acts on the film, in addition to a longitudinal flow of particles along $\nabla\mu$, there must appear a transverse flow (here x and y are the coordinates in the plane of the film)

$$j_x=\sigma_{xy}\nabla_y\mu, \tag{1.1}$$

where in the weak-coupling approximation, when the magnitude of the gap in the fermion spectrum is small compared to the Fermi energy, the parameter σ_{xy}, takes on the quantized values (Ref. 2):

$$\sigma_{xy}=N/2h, \tag{1.2}$$

(here N is an integer, h is Planck's constant), whereas in the usual quantum Hall effect with integer quantization the Hall conductivity equals

$$\sigma_{xy}=(e^2/h)N$$

(here e is the electron charge).

This half-integer analog of the quantum Hall effect has its specific traits. In ^{3}He-A the effect exists without a magnetic field and even without its analog (an angular velocity of rotation of the vessel). Moreover, for the existence of the effect the presence of impurities is not necessary. As a result the quantum Hall effect in ^{3}He-A turns out to be much more amenable to theoretical investigation: the problem can be solved completely within the framework of the standard self-consistent field method in the theory of superfluidity and superconductivity of Fermi systems.

Nevertheless, many aspects of the quantum Hall effect are similar for ^{3}He-A in the weak-coupling approximation and for an electron system in a magnetic field, and reflect the general principles of quantization of certain physical parameters. Thus, for instance, in both systems a plateau in the dependence of σ_{xy} on any external parameter appears only in the case when dissipation is absent. For this to occur in the electron system it is necessary that there be no longitudinal current whatsoever, i.e., that the Fermi level be situated in the region of localized states (see, e.g., Ref. 3); therefore for the quantization of σ_{xy} in this system impurities are necessary. In ^{3}He-A the absence of dissipation is guaranteed by the coherence of the superfluid motion and the existence of an energy gap (in distinction from the three-dimensional case, in a ^{3}He-A film the gap in the spectrum does not vanish anywhere, owing to size quantization); as a result of this the longitudinal superfluid current is just as dissipation-free as the transverse one.

The quantization of the parameter σ_{xy} in the usual integer-valued quantum Hall effect is guaranteed by the existence of an integer-valued topological invariant, related to the first Chern characteristic class (see Refs. 4, 5, as well as 6). In ^{3}He-A the parameter σ_{xy} is also expressed, at least in the weak-coupling approximation, in terms of an integer-valued topological invariant N, which is discussed in Section 2. This invariant belongs to the second homotopy group π_2, which describes the homotopy classes of mappings of the two-dimensional momentum space (k_x, k_y) into the space of Bogolyubov matrices. Since the Bogolyubov matrix Hamiltonian resembles the Dirac Hamiltonian of two-dimensional electrons, a similar topological invariant exists also in 2 + 1-dimensional quantum electrodynamics (QED). In QED the appropriate Hall conductivity σ_{xy} has an exact, rather than approximate, expression in terms of N. In order to clarify the distinctions between the systems of the type of ^{3}He-A and those of the type of 2 + 1-dimensional QED, to which, for example multiband dielectrics may belong, we consider in Section 3 the quantization of σ_{xy} in generalized QED.

In QED the Hall conductivity is the coefficient in front of the Chern-Simons term in the effective action for the electromagnetic field A_α ($\alpha = 0, 1, 2$) (see Refs. 7 and 8):

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$$S_{CS}=\frac{1}{2}\sigma_{xy}\int d^2x\,dt\,e_{\alpha\beta\gamma}A_\alpha\nabla_\beta A_\gamma. \quad (1.3)$$

Variation of this term with respect to A_α leads to an analog of the Hall current in the electron-positron vacuum:

$$j_x=\sigma_{xy}E_y,\quad \sigma_{xy}=(e^2/h)N, \quad (1.4a)$$

where $E_i=\partial_0A_i-\partial_iA_0$ is the electric field and the topological invariant N takes on the values $+\frac{1}{2}$ or $-\frac{1}{2}$, depending on the sign of the mass m of the electron:

$$N={}^1/_2m/|m|. \quad (1.4b)$$

In order to demonstrate that in quantum field systems the quantity σ_{xy} does not depend on the detailed structure of the Hamiltonian but is determined by its global characteristic N, we consider in Section 3 a maximally possible generalization of QED in which all symmetries are omitted with the exception of gauge symmetry, which is necessary for the existence of the quantum Hall effect in these systems. In Section 4 the parameter σ_{xy} is computed in a purely two-dimensional BCS model for σ_{xy}. It is shown that in this model, even in the case of a strongly deformed state and even if the size of the gap is not negligible compared to the Fermi energy, the parameter σ_{xy} is expressed exactly by the formula (1.2) in terms of the topological invariant N. In the two-dimensional case N takes on the values $+1$ and -1, depending on the orientation of the orbital angular momentum vector $\mathbf{l}$ relative to the normal to the film. It turns out, however, that there is an important distinction between the quantization of the parameter σ_{xy} in two-dimensional ^{3}He-A and in $2+1$-dimensional QED. In QED the paremeter σ_{xy} must not depend on the coordinates and time (and must consequently be expressible in term of the fundamental constants e and h), since a dependence of σ_{xy} on x and t would violate gauge invariance: a gauge transformation $A_\alpha\mapsto A_\alpha+\partial_\alpha\varphi$ does not change the action S_{CS} in Eq. (1.3) if and only if $\sigma_{xy}=$ const. In ^{3}He-A, even if one introduces a fictitious electric charge, the action differs from (1.3) and is constructed in such a manner that gauge invariance does not require that the parameter σ_{xy} be a fundamental constant. Therefore the equation (1.2) seems to be a consequence of an additional symmetry of the BCS model and should be violated, for example, if a Fermi-liquid interaction is switched on.

In Section 5 the finite width of the superfluid ^{3}He-A film is taken into account. It is shown that as the thickness a of the film increases the topological charge N varies discontinuously at certain values of a. The switchover from one value of N to the next occurs at the intersection with the so-called diabolic point (Refs. 6 and 9). The diabolic point in ^{3}He-A is a topologically stable point (k_x^0,k_y^0,a^0) in the three-dimensional parameter space (in this case, the space of the parameters k_x,k_y,a), where the energy of the fermionic quasiparticles vanishes (Ref. 10): $E(\mathbf{k}_0,a^0)=0$. As this point, which carries a charge of the homotopy group π_2, is crossed, the topological charge N changes by the value of the charge of the diabolic point. Since at the moment when a takes on the value a^0, the gap in the spectrum vanishes for some momentum $\mathbf{k}^0$, and the intermediate state becomes dissipative. Thus, there are different classes of two-dimensional systems: within each class which is characterized by a value of N the systems can continuously go over into each other, not passing through dissipative states. This is equivalent to the different θ-vacua in quantum field theory (see, e.g., Ref. 11). In systems of the type of QED and in the quantum Hall effect the parameter σ_{xy} is constant for a given class, thus leading to a plateau in its dependence on an external parameter. In ^{3}He-A this occurs only in the weak-coupling approximation. In the general case σ_{xy} depends on the external parameter, and as the diabolic point is crossed it will experience either a jump or a discontinuity in its derivative. As a rule, among the systems of a given class there exists one which is simplest, which is easy to compute, and still exhibits all the properties of the class. This is analogous to the fact that both the Fermi liquid and the Fermi gas belong to the same class of quantum fluids. In the case of ^{3}He-A we choose as the simplest system the so-called weak-coupling model, in which the atoms with spin up are paired independently of the atoms with spin down. For each of these projections the Hamiltonian has the form

$$H=\int d^3x\left\{\frac{1}{2m}(\nabla\psi^+,\nabla\psi)+g(\psi^+\nabla\psi^+)(\psi\nabla\psi)\right\}. \quad (1.5)$$

For low temperatures this system has a complex vectorial order parameter, namely the quasi-average

$$\Psi(\mathbf{r})=-ig\langle\psi\nabla\psi\rangle=\mathbf{\Delta}_1(\mathbf{r})+i\mathbf{\Delta}_2(\mathbf{r}) \quad (1.6)$$

(here $\mathbf{\Delta}_1$, $\mathbf{\Delta}_2$ are real vectors) which describes the pairing in a state with the Cooper pair angular momentum $L=1$ along the vector

$$\mathbf{l}=[\mathbf{\Delta}_1\mathbf{\Delta}_2]/|[\mathbf{\Delta}_1\mathbf{\Delta}_2]|. \quad (1.7)$$

In conclusion we discuss the generalization of the results obtained here to the case of systems with other values of L, which leads to the possibility of observing the fractional quantum Hall effect in superfluid films, too).

2. THE TOPOLOGY OF ^{3}He-A IN MOMENTUM SPACE

We start from purely two-dimensional ^{3}He-A postponing the discussion of the influence of the film thickness on the quantization of the parameter σ_{xy} to Section 5, and consider the simplest model for ^{3}He-A with the Hamiltonian (1.5) for each of the two spin orientations. After a standard decoupling of the fourth-order term in (1.5) and transformation to the spinor $\binom{\psi}{\psi^+}$ in the particle-hole space we obtain for this spinor the following Bogolyubov matrix Hamiltonian (see, e.g., Ref. 12):

$$H_A=\begin{pmatrix}-\frac{1}{2m}\nabla^2-\mu & \frac{\nabla}{2i}\Psi+\Psi\frac{\nabla}{2i}\\ \frac{\nabla}{2i}\Psi^*+\Psi^*\frac{\nabla}{2i} & \frac{1}{2m}\nabla^2+\mu\end{pmatrix}, \quad (2.1)$$

where μ is the chemical potential. In the spatially homogeneous case when the order parameter $\mathbf{\Delta}_1+i\mathbf{\Delta}_2$ does not depend on the coordinates, the Hamiltonian (2.1) reduces to the following matrix for the fermionic quasiparticles of momentum $\mathbf{k}$, the same for each of the spin projections

$$H_A=\tau_3\varepsilon+\tau_1\mathbf{\Delta}_1\mathbf{k}-\tau_2\mathbf{\Delta}_2\mathbf{k}, \quad (2.2)$$

where τ are the Pauli matrices acting in the particle-hole space, $\varepsilon=k^2/2m,\mu$ is the quasiparticle energy in the normal state, i.e., in the Fermi gas.

In the equilibrium state of ^{3}He-A the vectors $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ are orthogonal to each other and have the same magnitude, and their vector product (1.7) determines the orientation of the angular momentum vector $\mathbf{l}$ of the Cooper pair. In the ^{3}He-A film the vector $\mathbf{l}$ is normal to the film (for a discussion of the order parameters in films of superfluid ^{3}He, see Refs. 13 and 14), thus the vectors $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$ lie in the plane of the film, and therefore, as required, the matrix H_A depends only on the 2-momentum $\mathbf{k} = (k_x, k_y)$. The energy of the fermionic quasiparticles, obtained by squaring the matrix (2.2):

$$E^2(\mathbf{k}) = (k^2/2m - \mu)^2 + (\mathbf{\Delta}_1\mathbf{k})^2 + (\mathbf{\Delta}_2\mathbf{k})^2, \qquad (2.3)$$

does not vanish anywhere, since the vector $\mathbf{k}$ is in the plane of the film. This constitutes the important difference from the three-dimensional case, where the energy vanishes for two values of the momentum $\mathbf{k} = \pm (2m\mu)^{1/2}\,\mathbf{l}$, perpendicular to $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$.

The Hamiltonian H_A reminds one of the Hamiltonian of Dirac electrons in 2 + 1-dimensional spacetime

$$H_{\rm QED} = \tau_3 mc^2 + \tau_1 ck_x + \tau_2 ck_y, \qquad (2.4)$$

where c is the speed of light. Indeed, Eq. (2.2) becomes Eq. (2.4) if one sets $\mathbf{\Delta}_1 = c\hat{\mathbf{x}}$, $\mathbf{\Delta}_2 = -c\hat{\mathbf{y}}, \varepsilon = mc^2$. In the case of Eq. (2.4) the fermion energy $E^2 = m^2c^4 + k^2c^2$ does not vanish anywhere if $m \neq 0$. Therefore it is natural to expect a similarity between the properties of these systems, including the quantization of the parameter σ_{xy}. However, the matrices (2.2) and (2.4) differ in the magnitude of the topological invariant N, which, as will be shown in the following sections, determines the quantization of the parameter σ_{xy}.

In order to determine this invariant we write a two-dimensional nonsingular traceless matrix in the following generic form, introducing the three-dimensional vector $\mathbf{m} = (m_1, m_2, m_3)$, which depends on $\mathbf{k}$:

$$H = \boldsymbol{\tau}\mathbf{m}(\mathbf{k}), \qquad (2.5)$$

where the vector $\mathbf{m}$ does not vanish for any value of $\mathbf{k}$. One can then define an invariant which describes the mapping of the two-dimensional plane (k_x, k_y) into the space over which the vector $\mathbf{m}$ varies:

$$N = \frac{1}{4\pi}\int d^2k\,|\mathbf{m}|^{-3}\mathbf{m}\left[\frac{\partial \mathbf{m}}{\partial k_x}\frac{\partial \mathbf{m}}{\partial k_y}\right]. \qquad (2.6)$$

This invariant does not change under continuous deformations of the field $\mathbf{m}(\mathbf{k})$ which do not change the direction of the vector $\mathbf{m}$ at infinity.

For the Hamiltonian (2.2) this invariant takes on the values $+1$ or -1, depending on the orientation of the vector $\mathbf{l}$ relative to the normal $\hat{\mathbf{z}}$ to the film, for arbitrary noncollinear vectors $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$. For Dirac fermions $N_{\rm QED}$ takes the half-integer values $+\frac{1}{2}$ or $-\frac{1}{2}$ depending on the sign of the mass m:

$$N_A = \mathbf{l}\hat{\mathbf{z}}, \quad N_{\rm QED} = {}^1/_2 m/|m|. \qquad (2.7)$$

The difference between the two cases is related to the following circumstance. In ^{3}He-A for $k \to \infty$ the direction $\boldsymbol{\nu}(\mathbf{k}) = \mathbf{m}(\mathbf{k})/|\mathbf{m}(\mathbf{k})|$ of the vector $m(k)$ converges to a unique value $\boldsymbol{\nu}(\infty) = (0, 0, 1)$, independently of the direction of the vector $\mathbf{k}$. Therefore N_A is the degree of the mapping $S^2 \to S^2$ of a sphere onto a sphere, namely the two-dimensional $\mathbf{k}$-space which is equivalent to a sphere since infinity has been compactified to a single point, is mapped onto the unit sphere on which the vector $\boldsymbol{\nu}$ is situated. The degree of such a mapping can only take on integer values. For Dirac fermions $\boldsymbol{\nu}(\mathbf{k} \to \infty) \to \mathbf{k}/|\mathbf{k}|$, therefore the $\mathbf{k}$-space is mapped onto a hemisphere, and consequently the invariant $N_{\rm QED}$ takes on half-integer values.

The difference between ^{3}He-A and QED resides not only in the fact that the fermions in these two field theories may have different values of the invariant N, but also in the symmetry of the interaction of the fermions with bosonic fields (the electromagnetic field in QED, and in ^{3}He-A the field of the order parameter + the electromagnetic field, if a phase of the type of ^{3}He-A is realized in a superconductor). Therefore the quantization of the parameter σ_{xy} exhibits different traits in ^{3}He-A and in QED. We first consider the quantization of σ_{xy} in QED.

3. QUANTIZATION OF THE HALL CONDUCTIVITY AND THE TOPOLOGICAL INVARIANT IN QUANTUM ELECTRODYNAMICS

In order to clarify the topological character of the Hall conductivity σ_{xy} in the action (1.3) we consider the general case of the Hamiltonian (2.5), i.e., give up the restrictions imposed by Lorentz invariance and of necessity leading to the Hamiltonian (2.4), but we retain gauge invariance. Then the Hamiltonian of charged fermions in an electromagnetic field has the form (2.5) in which the momentum $\mathbf{k}$ is to be replaced by the operator $\hat{\mathbf{p}} = -i\nabla - e\mathbf{A}$:

$$H_{\rm QED} = \boldsymbol{\tau}\mathbf{m}(\hat{\mathbf{p}}). \qquad (3.1)$$

In order to find the vacuum Hall current we shall calculate the action S for the gauge field A_μ obtained after taking the expectation value over the vacuum of fermions situated in the classical field A_μ. This action has the standard form

$$S = i\,\mathrm{Tr}\ln\mathcal{G} = i\,\mathrm{Tr}\int_0^1 du\,\mathcal{G}\partial_u\mathcal{G}^{-1}, \qquad (3.2)$$

where Tr denotes the trace over all states and $\mathcal{G}$ is the Green's function:

$$\mathcal{G}^{-1} = i\partial_0 - A_0 - H_{\rm QED}. \qquad (3.3)$$

The extra variable u has been introduced here in order to get rid of the logarithm, with the assumption that the dependence of the external field $A_\mu(x_\nu, u)$ on u is such that

$$A_\mu(x_\nu, 0) = A_\mu(x_\nu), \quad A_\mu(x_\nu, 1) = 0. \qquad (3.4)$$

Considering the classical field $A_\mu = (A_i, A_0)$ to be slowly varying in space and time, so that in the zeroth approximation with respect to the gradients of the field, the Green's function has a classical dependence on both the coordinates and the momenta:

$$\mathcal{G}_{(0)}^{-1}(\mathbf{k}, \mathbf{r}, \omega, t, u) = \omega - eA_0(\mathbf{r}, t, u) - \boldsymbol{\tau}\mathbf{m}(\mathbf{k} - e\mathbf{A}(\mathbf{r}, t, u)), \qquad (3.5)$$

we carry out a gradient expansion of the action (3.2), expressing it in terms of the Green's function $\mathcal{G}_{(0)}$ and its derivatives (see, e.g., Refs. 15 and 8). In order to obtain an action of the form (1.3) it suffices to retain terms which are linear in the gradients of the field A_μ, and consequently in the gradients of the function $\mathcal{G}_{(0)}$. In this linear approximation we have

$$S=\frac{1}{2}\int_0^1 du\int d^2x\,dt\int\frac{d^2k\,d\omega}{(2\pi)^3}\,\mathrm{tr}\{\Lambda_u(\Lambda_k\Lambda_\tau-\Lambda_\tau\Lambda_k)-\Lambda_u(\Lambda_\omega\Lambda_t-\Lambda_t\Lambda_\omega)\},\quad(3.6)$$

where tr denotes the trace over the Pauli matrices and the matrices Λ denote various derivatives of $\mathscr{G}_{(0)}$

$$\Lambda_u=\mathscr{G}_{(0)}\partial_u\mathscr{G}_{(0)}^{-1},\quad \Lambda_t=\mathscr{G}_{(0)}\partial_t\mathscr{G}_{(0)}^{-1},\ldots.\quad(3.7)$$

Substituting (3.5) into (3.7), taking into account that

$$\Lambda_{x_\mu}=-e\Lambda_{k_j}\partial_\mu A_j-e\Lambda_\omega\partial_\mu A_0,$$
$$\Lambda_u=-e\Lambda_{k_j}\partial_u A_j-e\Lambda_\omega\partial_u A_0,\quad(3.8)$$

we obtain for the action (3.6)

$$S=\frac{e^2}{2}\int_0^1 du\int d^2x\,dt\,e_{\alpha\beta\gamma}\partial_u A_\alpha\partial_\beta A_\gamma\times\int\frac{d\omega\,d^2k}{(2\pi)^3}\,\mathrm{tr}\{\Lambda_\omega(\Lambda_{k_x}\Lambda_{k_y}-\Lambda_{k_y}\Lambda_{k_x})\}.\quad(3.9)$$

After taking the trace with respect to the spin variables and integrating with respect to the frequency, which is done along the imaginary axis, one gets the topological invariant (2.6):

$$\int\frac{d\omega\,d^2k}{(2\pi)^3}\,\mathrm{tr}\{\Lambda_\omega(\Lambda_{k_x}\Lambda_{k_y}-\Lambda_{k_y}\Lambda_{k_x})\}$$
$$=2i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}(\omega^2+m^2)^{-2}\int\frac{d^2k}{(2\pi)^2}\,\mathrm{tr}\left\{(\boldsymbol{\tau}\mathbf{m})\left(\boldsymbol{\tau}\frac{\partial\mathbf{m}}{\partial k_x}\right)\left(\boldsymbol{\tau}\frac{\partial\mathbf{m}}{\partial k_y}\right)\right\}$$
$$=\int\frac{d^2k}{(2\pi)^2}\frac{1}{|\mathbf{m}|^3}\mathbf{m}\left[\frac{\partial\mathbf{m}}{\partial k_x}\frac{\partial\mathbf{m}}{\partial k_y}\right]=\frac{1}{\pi}N.\quad(3.10)$$

Since this invariant does not depend on the gauge fields A_μ, the action (3.9) turns out to be quadratic in A_μ:

$$S=\frac{e^2}{h}N\int_0^1 du\int d^2x\,dt\,e_{\alpha\beta\gamma}\partial_u A_\alpha\partial_\beta A_\gamma.\quad(3.11)$$

Integrating over u taking (3.4) into account, we obtain the resulting Chern-Simons action (1.3)

$$S=\frac{e^2}{2h}N\int d^2x\,dt\,e_{\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma.\quad(3.12)$$

This action leads to the vacuum current

$$j_\mu=\delta S/\delta A_\mu=(e^2/2h)Ne_{\mu\alpha\beta}F_{\alpha\beta}.\quad(3.13)$$

Thus, for fermionic systems with a Hamiltonian of the form (3.1) which exhibits gauge invariance, the Hall conductivity

$$\sigma_{xy}=(e^2/h)N\quad(3.14)$$

does not depend on the details of the fermionic spectrum, but is determined by its global characteristic—the topological invariant N. Therefore, for deformations of the system which are not too large, i.e., deformations which don't change its global characteristics, the parameter σ_{xy} remains a constant quantity expressed in terms of fundamental constants (e and h). To this class of systems belong, e.g., two-zone dielectrics (cf. Ref. 16) with the Hamiltonian

$$H=\begin{pmatrix}\varepsilon_1(\mathbf{k}) & \mathbf{Pk}\\ \mathbf{P}^*\mathbf{k} & \varepsilon_2(\mathbf{k})\end{pmatrix},\quad(3.15)$$

where $\mathbf{P}$ is the interband matrix element. This Hamiltonian may have a nonvanishing topological invariant if $\mathbf{P}\times\mathbf{P}^*\neq0$. The latter means that there exists orbital ferromagnetism. Thus, multiband insulators with spontaneous orbital magnetic moment may exhibit an integer-valued quantum Hall effect in the bsence of an external magnetic field.

4. THE TOPOLOGICAL INVARIANT AND HALL CONDUCTIVITY IN ^{3}He-A

The Hall conductivity (1.2) in ^{3}He-A was calculated in Ref. 2. However, in Ref. 2 it was not made clear to what degree of accuracy the relation (1.2) is valid, i.e., to what degree it is sensitive to external perturbations, e.g., to deformations of the order parameter, which take the film out of the A-phase. Here we shall show that in the two-dimensional model (1.5) for ^{3}He-A, just as in QED, the parameter σ_{xy} is determined by a global characteristic of the fermionic spectrum and therefore does not depend on deformations of the order parameter.

In order to determine the parameter σ_{xy} we introduce for the atom of ^{3}He-A a fictitious electric charge e. On the one hand this makes the computations easier, since one can use gauge invariance. In the final result one may set $e=0$. On the other hand, this allows one to extend the result to superconductors with heavy fermions, where a Cooper pairing with nontrivial symmetry is also possible (see the reviews 17 and 18). In charged ^{3}He-A an electromagnetic field A_μ acts on the fermions and the order parameter field is a complex vector field $\boldsymbol{\Psi}=\boldsymbol{\Delta}_1+i\boldsymbol{\Delta}_2$; the fermionic Hamiltonian has the following expression in terms of these fields [compare with Eq. (2.1)]:

$$H_A=\begin{pmatrix}\frac{1}{2m}(-i\nabla-e\mathbf{A})^2-\mu-eA_0 & \frac{\nabla}{2i}\boldsymbol{\Psi}+\boldsymbol{\Psi}\frac{\nabla}{2i}\\ \frac{\nabla}{2i}\boldsymbol{\Psi}^*+\boldsymbol{\Psi}^*\frac{\nabla}{2i} & -\frac{1}{2m}(-i\nabla+e\mathbf{A})^2+\mu+eA_0\end{pmatrix}.\quad(4.1)$$

The distinction from the form (3.1) for QED is related to the fact that the matrices τ for ^{3}He-A act in the particle hole space, and the particles and holes carry opposite charges. The gauge symmetry group for ^{3}He-A is the following: the Green's function $\mathscr{G}$ ($\mathscr{G}^{-1}=i\partial_0-H_A$) does not change under the transformations

$$\boldsymbol{\Delta}_1+i\boldsymbol{\Delta}_2\to\exp(2i\varphi)(\boldsymbol{\Delta}_1+i\boldsymbol{\Delta}_2),\quad eA_\mu\to eA_\mu+\partial_\mu\varphi,$$
$$\mathscr{G}^{-1}\to\exp(-i\tau_3\varphi)\mathscr{G}^{-1}\exp(i\tau_3\varphi).\quad(4.2)$$

We must determine the effective bosonic action S of the type of (1.3), which, however, in addition to the gauge field A_μ may contain the order parameter Ψ. We first find the action that depends on the gauge field A_μ and then, making use of the gauge group (4.2) we complement it including the dependence on the order parameter. In the approximation linear in the gradients of A_μ we have Eqs. (3.6) and (3.7), with the classical approximation to the Green's function, Eq. (3.5), has the following form:

$$\mathcal{G}^{-1}=\omega-e\mathbf{k}\mathbf{A}(\mathbf{r}, u, t)-e\tau_3A_0(\mathbf{r}, u, t)+\tau_1\mathbf{k}\Delta_1-\tau_2\mathbf{k}\Delta_2+\tau_3(k^2/2m-\mu). \quad (4.3)$$

In the sequel, in order to verify that the parameter σ_{xy} is invariant to different perturbations, we carry out a maximal generalization of the Hamiltonian (4.1), retaining the gauge and Galilean invariances. The latter requirements are quite stringent for a Hamiltonian of the form (4.1), since they allow only deformations of the order parameter Ψ, but do not allow for an arbitrary momentum dependence, as was the case for Eq. (2.5). Just as in (4.3) the vectors Δ_1 and Δ_2 are arbitrary and the vector $\mathbf{l}$ is defined by Eq. (1.7). The only restrictions which we impose on Δ_1 and Δ_2 is that they should not be collinear, otherwise the fermion spectrum (2.3) vanishes for some momentum and the vector $\mathbf{l}$ stops being determined.

Substituting (4.3) into (3.7):

$$\Lambda_{x_\mu}=-e\Lambda_\omega(k_i\partial_\mu A_i+\tau_3\partial_\mu A_0), \quad (4.4)$$

$$\Lambda_u=-e\Lambda_\omega(k_i\partial_u A_i+\tau_3\partial_u A_0)$$

and introducing the notations

$$\Lambda_{k_j}^3=\mathcal{G}\partial_{k_j}(\varepsilon\tau_3)=\Lambda_\omega\tau_3k_j, \qquad \Lambda_{\mathbf{k}}^{1,2}=\mathcal{G}\partial_{\mathbf{k}}(\tau_{1,2}\mathbf{k}\Delta_{1,2}), \quad (4.5)$$

we obtain for the action (3.6)

$$S_A=\frac{e^2}{2}\int_0^1 du\int d^2x\,dt\,(\partial_u A_0\partial_i A_j-\partial_i A_0\partial_u A_j)\times\int\frac{d\omega\,d^2k}{(2\pi)^3}\operatorname{tr}\Lambda_\omega[\Lambda_{k_j}^3,\Lambda_{k_i}]; \quad (4.6)$$

here the square brackets denote the commutator of the matrices.

The trace of the matrices Λ will be rewritten in the following form:

$$\operatorname{tr}\Lambda_\omega[\Lambda_{k_j},\Lambda_{k_i}]$$

$$={}^1/_2\operatorname{tr}\Lambda_\omega[\Lambda_{k_j},\Lambda_{k_i}]-{}^1/_2\operatorname{tr}\Lambda_\omega([\Lambda_{k_j},\Lambda_{k_i}]-[\Lambda_{k_i},\Lambda_{k_j}]). \quad (4.7)$$

One can show that the momentum integral in the second term in Eq. (4.7) vanishes owing to the particle-hole symmetry. Indeed,

$$\int\frac{d\omega\,d^2k}{(2\pi)^3}\operatorname{tr}\Lambda_\omega[\Lambda_{k_j}^1,\Lambda_{k_i}^2]=\Delta_{1j}\Delta_{2i}\int\frac{d\omega\,d^2k}{(2\pi)^3}\operatorname{tr}(\Lambda_\omega\tau_1\Lambda_\omega\tau_2\Lambda_\omega-\Lambda_\omega\tau_2\Lambda_\omega\tau_1\Lambda_\omega)=\Delta_{1j}\Delta_{2i}\int\frac{d^2k}{(2\pi)^2}\frac{\varepsilon}{E^3} \quad (4.8)$$

and the last integral vanishes since it is antisymmetric with respect to the transformation $\mathbf{k}\mapsto\mathbf{k}k_F^2/k^2$ which corresponds to a transition from particles to holes, and under these transformations

$$\varepsilon\to-\varepsilon k_F^2/k^2, \quad E\to Ek_F^2/k^2,$$
$$d^2k\to d^2k(k_F^2/k^2)^2, \quad k_F^2=2m\mu, \quad (4.9)$$

as a result of which the integral changes sign.

The contribution to the action from the first term in Eq. (4.7), just as was the case for QED, [see Eq. (3.10)], is expressed in terms of the topological invariant N. Finally, integrating (4.6) with respect to u, we obtain a term of the type of (1.3) in the action:

$$S_A=\frac{e^2}{4h}N\int d^2x\,dt\,e_{ij}(A_0\partial_iA_j+A_i\partial_jA_0), \qquad e_{ij}=e_{ijk}\hat{z}_k. \quad (4.10)$$

Thus, the Hall conductivity in the two-dimensional charged superfluid system (4.1), is determined in the absence of an external magnetic field by the topological invariant of this system:

$$\sigma_{xy}=(e^2/2h)N. \quad (4.11)$$

For charged ^{3}He-A in a purely two-dimensional situation we have, according to Eq. (2.7),

$$\sigma_{xy}^{(A)}=(e^2/2h)\mathbf{l}\hat{\mathbf{z}} \quad (4.12)$$

—a half-integer quantum Hall effect. For a two-dimensional electronic system in a magnetic field half-integer quantization has been discussed in the recent paper, Ref. 19. For two-dimensional ^{3}He-A_1 in which only atoms with one spin projection are paired, the effect should be half as large:

$$\sigma_{xy}^{(A_1)}=(e^2/4h)\mathbf{l}\hat{\mathbf{z}}. \quad (4.13)$$

In connection with this we note one technical detail. To the action (3.2) in the case of superfluid Fermi fluids one must add a factor $\frac{1}{2}$ since in passing from particles to the Bogolyubov representation in terms of particles and holes the number of states is artificially doubled. We have not taken this factor into account in ^{3}He-A since it was compensated by the summation over the two spin projections. In ^{3}He-A_1, where superfluidity appears only for one spin component, there is no such compensation, and this leads to the factor $\frac{1}{4}$ in the Hall effect.

We now discuss the differences between the Hall effect in ^{3}He-A and in QED. First, the formulas (3.14) in QED and (4.11) in ^{3}He-A have different dependences on the topological invariant N. Second, in distinction from the action (3.12) for QED, the action (4.10) is not gauge invariant, since it does not contain the term $e_{ij}A_j\partial_0A_i$. This is not surprising, since the gauge group (4.2) in the Bogolyubov Hamiltonian for ^{3}He-A also contains a transformation of the order parameter. Therefore, in order to reestablish the gauge invariance, we must supplement the action (4.10) by adding to it a dependence on the order parameter. Neglecting spin rotations which are not essential for the Hall effect, in ^{3}He-A in a film the only degree of freedom of the order parameter is related to the phase Φ of the Bose condensate:

$$\Psi(\mathbf{r}, t)=\text{const}\,(\hat{x}+i\hat{y})\exp[i\Phi(\mathbf{r}, t)]. \quad (4.14)$$

However, such a dependence of the order parameter on coordinate and time in Eq. (4.1) can be removed by means of a transformation (4.2), as a result of which the term $-(\frac{1}{2}e)\partial_\mu\Phi$ is added to the gauge field A_μ. Therefore Eq. (4.10) is transformed into the following gauge-invariant expression

$$S_A(A_\mu,\Phi)=\frac{N}{4h}\int d^2x\,dt\,e_{ij}\Big[\Big(\frac{1}{2}\dot{\Phi}-eA_0\Big)\partial_i(mv_{sj}-eA_j)+(mv_{si}-eA_i)\,\partial_j\Big(\frac{1}{2}\dot{\Phi}-eA_0\Big)\Big],\qquad(4.15)$$

where

$$\mathbf{v}_s=(\hbar/2m)\,\nabla\Phi\qquad(4.16)$$

is the superfluid velocity.

We call attention to the fact that the action (4.15) is even "more" gauge invariant than the action (1.3) in QED. In charged ^{3}He-A the integrand is gauge invariant, whereas in QED only the integral as a whole is gauge invariant, and the noninvariance of the integrand in QED dictates the quantization of the parameter σ_{xy}: this parameter must be expressible in terms of the fundamental constants and must be independent of external perturbations; otherwise, if the external perturbation were inhomogeneous in space or time, the coordinate-time dependence of the parameter σ_{xy} would lead to the action not being gauge invariant. In ^{3}He-A the gauge invariance is valid locally, therefore there are no symmetry reasons for the quantization of σ_{xy}. In the transition to a coordinate system which moves with a constant velocity $\mathbf{u}$ the superfluid velocity transforms according to the law $\mathbf{v}_s\mapsto\mathbf{v}_s+\mathbf{u}$, therefore one must add to the action (4.15) the term

$$\frac{1}{2}\int d^2x\,dt\,\sigma_{xy}\mathbf{u}\Big[\nabla\Big(\frac{1}{2}\dot{\Phi}-eA_0\Big),\mathbf{l}\Big],\qquad(4.17a)$$

which vanishes only if the parameter σ_{xy} is independent of the coordinates. However, for the general case, other than the model (1.5) this reasoning does not apply. Consideration of subsequent terms in the gradient expansion leads to the result that the integrand in (4.17a) is reproduced up to a total derivative of the particle density ρ:

$$\mathbf{u}\,[\nabla(\sigma_{xy}(^1/_2\dot{\Phi}-eA_0)),\mathbf{l}]={}^1/_2\mathbf{u}[\nabla\rho,\mathbf{l}],\qquad(4.17b)$$

even in the case when σ_{xy} depends on the coordinates. Therefore the quantization rule (4.11) is a property of the model rather than of ^{3}He-A.

We now analyze the effect of the action (4.15) on the dynamics of neutral ^{3}He-A. In a neutral Fermi liquid one can set the charge $e=0$ in the action (4.15); the gauge fields then vanish, but there remains a response of the particle density and current density to changes in the order parameter:

$$\rho_{(F)}=\frac{\delta S}{\delta(eA_0)}=\frac{N}{2h}e_{ij}\partial_i v_{sj}=\frac{1}{2h}\mathbf{l}\,\mathrm{rot}\,\mathbf{v}_s,\qquad(4.18a)$$

$$\mathbf{j}_{(F)}=\frac{\delta S}{\delta(e\mathbf{A})}=\frac{1}{2h}[\mathbf{l},{}^1/_2\nabla\dot{\Phi}]\qquad(4.18b)$$

[in Eq. (4.18a) we have retained the expression curl $\mathbf{v}_s$, in spite of the fact that in ^{3}He-A it vanishes on account of Eq. (4.16): in ^{3}He-A_1 the superfluid flow may be nonpotential]. Such terms in the density do not influence the equation of motion, since, on the one hand, such anomalous contributions satisfy the conservation law

$$\partial_t\rho_{(F)}+\nabla\mathbf{j}_{(F)}=0,\qquad(4.19)$$

and, on the other hand, the current $\mathbf{j}_{(F)}$ is an exact spatial derivative and therefore does not change the equations for the momentum.

The hydrodynamical action functional for a ^{3}He-A film at $T=0$, in the quadratic approximation in Φ has the form:

$$S=\int d^2x\,dt\Big\{\frac{1}{8}\rho\Big((\nabla\Phi)^2-\frac{1}{c^2}\dot{\Phi}^2\Big)-\frac{1}{4}\sigma_{xy}e_{ij}\nabla_i\Phi\nabla_j\dot{\Phi}\Big\},\qquad(4.20)$$

where c is the speed of sound. The variation of the second term with respect to Φ yields zero. Therefore the Hall term has no influence whatsoever on the dynamical equation for Φ, and the equation has the same wave-equation form as in the usual superfluid liquid:

$$\ddot{\Phi}=c^2\nabla^2\Phi.\qquad(4.21)$$

Nevertheless, if one excites oscillations of the superfluid velocity $\mathbf{v}_s$ along a direction, then in a transverse direction there must appear a particle flow, which according to Eq. (4.18b) has the current density

$$\mathbf{j}=\sigma_{xy}[\mathbf{l}\dot{\mathbf{v}}_s].\qquad(4.22)$$

5. DIMENSIONAL QUANTIZATION OF THE HALL CONDUCTIVITY IN A ^{3}He-A FILM AND THE DIABOLIC POINTS

The results of the preceding section refer to purely two-dimensional ^{3}He-A. On the other hand, for three-dimensional ^{3}He-A a current similar to (1.1), or what amounts to the same, (4.22) was derived by Mermin and Muzikar (Ref. 20):

$$\mathbf{j}={}^1/_4[\mathbf{l}\nabla\rho].\qquad(5.1)$$

For a thick film of thickness a much larger than the interatomic distances, this current, integrated over the thickness of the film corresponds to the Hall current (1.1) with parameter

$$\sigma_{xy}={}^1/_4a\partial\rho/\partial\mu\approx k_Fa/2\pi h\gg 1/h,\qquad(5.2)$$

which substantially exceeds the parameter α_{xy} for purely two-dimensional ^{3}He-A. Therefore, as the thickness a of the film increases, α_{xy} must increase, i.e., the topological invariant N in Eq. (1.2) varies. But N cannot vary continuously. Consequently discontinuities must appear.

We consider such a discontinuous variation of the physical parameter on the simplest model, since the details of the model will only influence the values of the thickness of the film at which jumps occur in N, but not the topological character of the process, nor magnitude of the jump. As a model we consider a three-dimensional Bogolyubov Hamiltonian (2.1) with transverse (i.e., along the z axis) dimensional quantization. Then for each level n and transverse motion with energy α_n there exists a distinct Bogolyubov matrix

$$H_{nn}=\boldsymbol{\tau}\mathbf{m}_n(\mathbf{k})=\varepsilon_n(k)\tau_3+\Delta_{1n}\mathbf{k}\tau_1-\Delta_{2n}\mathbf{k}\tau_2,\qquad(5.3)$$

where the spectrum of the two-dimensional Fermi gas at the level n has the form

$$\varepsilon_n(k)=k^2/2m+\alpha_n-\mu.\qquad(5.4)$$

In the simplest case of a mirror reflection, when the normal derivative of a Bogolyubov spinor vanishes at both surfaces of the film, $\alpha_n=\frac{1}{2}(\pi n/a)^2$. The orbital angular momentum vector

$$\mathbf{l}=[\mathbf{\Delta}_{1n}\mathbf{\Delta}_{2n}]/|[\mathbf{\Delta}_{1n}\mathbf{\Delta}_{2n}]| \tag{5.5}$$

will be chosen to be the same for all levels, which must be true owing to the combined rotation and gauge symmetry of the A-phase. However, in magnitude the order parameters $\mathbf{\Delta}_{1n} + i\mathbf{\Delta}_{2n}$ must not be the same for all the levels; in particular, they may vanish for those levels for which the energy is above the Fermi surface, and which therefore do not contain any particles.

Since in such a model fermions on different levels do not interact with each other, one can calculate α_{xy} for each level separately. The contribution to α_{xy} from each level consists of two parts: the contribution of the topological invariant N_n of the level, which upon summation over the levels yields

$$\sigma_{xy}^{(N)}=\frac{1}{2h}N=\frac{1}{2h}\sum_n N_n, \tag{5.6}$$

and the contribution from Eq. (4.8). The topological invariant N_n has the form

$$N_n=\mathbf{l}\hat{\mathbf{z}}\theta(\mu-\alpha_n) \tag{5.7}$$

(here θ is the Heaviside step function), i.e., contributions to (5.6) come only from levels for which the energy α_n of transverse motion is below the Fermi surface. The contribution from Eq. (4.8), on the other hand, differs from zero only for $\alpha_n > \mu$, since for $\alpha_n < \mu$ it vanishes according to the symmetry of (4.9), where μ should be replaced by $\mu - \alpha_n$. This contribution is, however, proportional to the order parameter on levels on which there are no particles, consequently it vanishes if the levels do not interact. Therefore, in the limit of noninteracting levels the quantization (1.2) holds, where N coincides with the number of levels below the Fermi surface. In the limit of large thickness of the film the number of such levels converges to $k_F a/2\pi$, yielding the value of the volume effect (5.1) and (5.2).

The discontinuity in N occurs when one of the levels passes through the Fermi surface. At this instant the fermion energy on this level

$$E_n^2(\mathbf{k})=(k^2/2m+\alpha_n-\mu)^2+(\mathbf{k}\mathbf{\Delta}_1)^2+(\mathbf{k}\mathbf{\Delta}_2)^2 \tag{5.8}$$

vanishes for $k = 0$ (see the figure, which shows the dependence of the minimal energy of the fermionic quasiparticles on the film thickness; the energy can vanish only at the discontinuity), thus the transition between two values of N is realized via a dissipative state.

We consider this transition in the three-dimensional space of the parameters $\mathbf{q} = (k_x, k_y, \alpha_n)$. The energy (5.8) vanishes at the point $\mathbf{q}_0 = (0, 0, \mu)$ in this three-dimensional space. The zero in the energy is topologically nonremovable: small movements can only change the position $\mathbf{q}_0$ of this point where the energy vanishes, but cannot destroy this zero. This is a diabolic point (Refs. 6, 9 and 10) where two branches of the spectrum of the Bogolyubov Hamiltonian touch each other: the positive spectrum of the particles and the negative spectrum of the holes. The diabolic point has a topological invariant which guarantees its topological stability. This invariant is the degree of the mapping of a sphere σ centered at the point $\mathbf{q}_0$ in three-dimensional $\mathbf{q}$-space onto the sphere of the unit vector $\mathbf{v}_n(\mathbf{q})$:

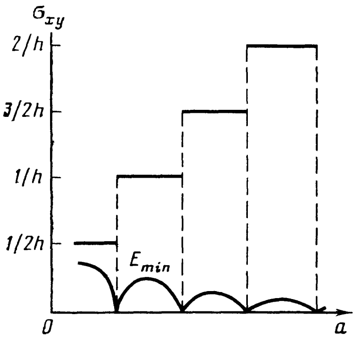

FIG. 1. The dependence of the Hall conductivity σ_{xy} in a film of superfluid ^{3}He-A on the thickness a of the film for $T = 0$, calculated in the BCS model in the limit when the order parameter is negligible compared to the Fermi energy. The lower curve shows the character of the dependence of the gap in the spectrum of fermionic quasiparticles on the thickness of the film. A discontinuity in σ_{xy} occurs for those values of a for which the gap in the spectrum vanishes: for those values of a one of the levels of the transverse dimensional quantization in the film crosses the Fermi surface.

$$N=\frac{1}{8\pi}\int_\sigma dS^i\, e^{ijk}|\mathbf{m}_n(\mathbf{q})|^{-3}\mathbf{m}_n\left[\frac{\partial \mathbf{m}_n}{\partial q^i}\frac{\partial \mathbf{m}_n}{\partial q^j}\right]. \tag{5.9}$$

For the Hamiltonian (5.3) $N = 1$. Now it is understandable why a discontinuity of N occurs as a level passes through the Fermi surface: the integral (2.6) over the two-dimensional plane (k_x, k_y) for $\mu < \alpha_n$ differs from that for $\mu > \alpha_n$ exactly by the topological charge of the diabolic point, and a transition through the diabolic point automatically means a compulsory transition through a dissipative state. A similar switchover of the topological charge in the spectrum of magnetic Bloch wave-functions was described by Novikov in Ref. 6.

The same situation occurs for a deformation which changes the direction of the vector $\mathbf{l}$. According to Eq. (1.7), when the sign of $\mathbf{l}$ changes, the order parameter passes through a state with collinear $\mathbf{\Delta}_1$ and $\mathbf{\Delta}_2$. According to Eq. (2.3) in this state the energy also vanishes for some momentum. In this case the momentum is not zero, but is perpendicular to the unique real order-parameter vector in this intermediate state, which corresponds to the polar phase. Such a zero of the energy is also a diabolic point. As a result of passing through this point the parameter σ_{xy}, Eq. (4.12) changes its sign discontinuously.

Let us analyze how the results of this section would change were one to include an interaction between the levels of transverse motion, as a result of which there appears an off-diagonal order parameter $\mathbf{\Psi}_{nm}$, corresponding to pairing of particles from different zones. In this case in place of the diagonal matrix H_{nn} in Eq. (5.3) there appears the nondiagonal matrix

$$H_{mn}=\varepsilon_n(k)\delta_{nm}\tau_3+\tau_1\mathbf{k}\mathbf{\Delta}_{1mn}+\tau_2\mathbf{k}\mathbf{\Delta}_{2mn}. \tag{5.10}$$

The topological invariant which for the case of independent levels was equal to $N = \Sigma_n N_n$ [see Eq. (5.6)] can be expressed in the general case in terms of the matrix H_{mn} in the following manner:

$$N=\pi\int\frac{d^2k\,d\omega}{(2\pi)^3}\,\mathrm{tr}\,\mathcal{G}\partial_\omega\mathcal{G}^{-1}[\mathcal{G}\partial_{k_x}\mathcal{G}^{-1},\mathcal{G}\partial_{k_y}\mathcal{G}^{-1}], \tag{5.11}$$

where

$$(\mathcal{G}^{-1})_{mn}=i\omega\delta_{mn}-H_{mn}. \tag{5.12}$$

A change of N occurs when one passes through the diabolic point, where the determinant det H vanishes (see Refs. 9 and 10).

The parameter α_{xy} again consists of two contributions, one of which can be expressed in terms of N according to Eq. (1.2), and the other leads to the generalized expression (4.8) which no longer vanishes. The latter contribution is, however, small of order Δ/μ and is effectively nonzero only in a narrow region near the discontinuity of N, where it smoothes the discontinuity in σ_{xy}.

6. CONCLUSION

The quantization of the Hall conductivity σ_{xy} in a ^{3}He-A film is, according to Eq. (5.6), the result of the presence of the topological invariant N, Eq. (2.6) for each of the zones which appear as a result of the dimensional quantization. Each of the quantities N_n takes the values 1 or 0 depending on whether the level n of the transverse quantization lies below or above the Fermi surface. In principle systems with other values of the topological charge are possible for each of the zones. As we have seen, in QED $N = 1/2$. In superfluid Fermi systems the topological invariant N_n can also have values which differ from one or zero. For example, if a Cooper pair has a projection of the orbital momentum m on the z axis (for ^{3}He-A $m = +1$ or $m = -1$ depending on the orientation of the vector $\mathbf{l}$), then the Bogolyubov Hamiltonian has the form

$$H=\begin{pmatrix} \varepsilon & [(k_x+ik_y)/k]^m \\ [(k_x-ik_y)/k]^m & -\varepsilon \end{pmatrix}, \tag{6.1}$$

which leads to the topological invariant (2.6)

$$N_n=m. \tag{6.2}$$

As a result of this, the contribution of each level situated below the Fermi surface equals

$$\sigma_{xy}^{(1)}=m/2h, \tag{6.3}$$

i.e., in units $1/h$ the Hall conductivity of one level in a superfluid Fermi liquid of the type of ^{3}He-A is equal to the orbital angular momentum per atom of fluid.

One can imagine superfluid systems in which the angular momentum per atom may take on arbitrary values. If, for example, the bosons which form a coherent superfluid state are not Cooper pairs, but sextuplets of atoms, and each boson carries, e.g., an orbital angular momentum $m = 2$, then each atom will have an orbital angular momentum of 1/3, which yields $\sigma_{xy} = 1/3h$. Such states could be formed near discontinuities in the half-integer quantum Hall effect in ^{3}He-A, when one of the levels α_n of the transverse motion has just barely passed below the Fermi surface and the order parameter $\mathbf{\Delta}_{1n} + i\mathbf{\Delta}_{2n}$ corresponding to the formation of Cooper pairs at that level has not yet been finally formed. In principle, near a half-integer jump, analogues of the Laughlin states (Ref. 21) of the fractional quantum Hall effect could also be formed.

For an experimental observation of the quantum Hall effect in a ^{3}He-A film, the film must be sufficiently thin; otherwise the singularities in the parameter σ_{xy} would be washed out by thermal effects, since the gap in the fermionic spectrum decreases as the thickness of the film increases (see the figure). In thin films of size smaller than the coherence length, the superfluid triplet state of ^{3}He-A is suppressed by diffuse scattering on the surface roughness of the substrate. Therefore a high degree of smoothness of the substrate is required. In this direction a hope-inspiring fact is the observation (Ref. 22) that covering the substrate with monolayers of adsorbed ^{4}He increases the specularity of surface reflection. leading to an increase in the superfluid density of a ^{3}He-A film.

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