Theory of emission of radiation and of the production of electron-positron pairs by ultrahigh-energy particles in crystals at angles of incidence greater than the Lindhard angle

N. K. Zhevago

I. V. Kurchatov Institute of Atomic Energy
(Submitted 3 December 1987)
Zh. Eksp. Teor. Fiz. **94**, 23–36 (October 1988)

It is shown that, at high particle energies, the existing theory of coherent bremsstrahlung and e^+e^- pair production in crystals may break down even at relatively large angles of incidence θ_0 relative to the principal crystallographic directions. These angles may substantially exceed the critical Lindhard angle θ_L for channeling. The theory developed in this paper provides a satisfactory description of electromagnetic processes in crystals in a relatively broad range of angles of incidence $\theta_0 \gtrsim \theta_L$. Theoretical results are compared with the existing theories and with experimental data.

1. INTRODUCTION

It is well-known (see, e.g., Ref. 1) that the probability of bremsstrahlung and of electron-positron pair production by a photon in a crystal may be significantly different from the corresponding probability in an amorphous medium. The reason for this difference is the coherence of the crystal atoms, which occurs for particle energies and angles of incidence relative to the principal crystallographic axis for which the coherence length is given by

$$l_c = |u| E^{-1}$$

is close to the path traversed by a particle between successive collisions with the crystal atoms ($u = \omega/(E-\omega)$; E and ω are, respectively, the energy of the charged particle and of the photon, and $\hbar = m = c = 1$).[1] The standard theory of coherent bremsstrahlung and of pair production[1,2] is based on the Born approximation for the interaction between charged particles and crystal atoms. This approach implicitly assumes that the change in the direction of the charged-particle momentum over the coherence length is small in comparison with the effective angle ($\sim E^{-1}$) for the emission of bremsstrahlung or for pair production. The effect of the curvature of the charged-particle trajectory during the time between collisions is also neglected.

Several workers have shown (see Ref. 3 and the references sited therein) that the latter assumption is not valid for angles for incidence relative to the crystal axes or planes that approach the Lindhard critical angle θ_L for channeling. When this is so, the deflection of a charged particle by the crystal field within the coherence length is comparable to the angle of incidence of the particles relative to the crystal axes or planes, or may even exceed it, and this leads to a significant change in the mean time between collisions. Even for zero angle of incidence, the time between collisions with the axes (planes) remains finite because of the channeling effect.

However, it will be shown below that the standard theory of coherent bremsstrahlung and pair production may cease to be valid even for angles of incidence much greater than the Lindhard angle, for which the influence of the curvature of the trajectory between collisions can be completely neglected. The point is that the probability of these processes in the case of relativistic charged particles is very sensitive to the ratio $p = E\theta_d$ between the angle θ_d of deflection of the particle by the field and the effective emission angle $\theta_{\rm eff} \sim E^{-1}$ (the angle at which the pair is emitted). Dipole bremsstrahlung is emitted when the parameter p is small ($p \ll 1$). In the opposite limit, $p \gg 1$, the radiation is emitted by a segment of the trajectory that is small in comparison with its radius of curvature, so that the field can be assumed to be constant within this segment, and the emission spectrum is similar to the synchrotron spectrum. This general discussion is given, for example, in Ref. 4 and can be applied to our case for which the external field is the crystal field. When a fast charged particle travels through the crystal at a sufficiently small angle $\theta_0 \ll 1$ to the crystal axes or planes, the effective field acting on the particle is determined by the atomic potential on the crystal axes or planes, averaged in the longitudinal direction.[5,3] Suppose that U_0 is the amplitude of energy E, incident at an angle $\theta_0 \gtrsim \theta_L$ to the crystal axes or planes, is given by $\theta_d \sim U/E\theta_0$. The parameter p in which we are interested here is then given by $p \sim U_0/\theta_0$ and, consequently, the existing theory of coherent bremsstrahlung[1,2] ceases to be valid even for angles of incidence $\theta_0 \sim \theta_c \equiv U_0$. Since we are considering the case of particles traveling above the barrier, $\theta_0 \gtrsim \theta_L$ and the inequality $\theta_c \gtrsim \theta_L$ should be satisfied, i.e., the particle energy should be sufficiently high, $E \gg E_1 \equiv 1/U_0$. It follows that, at ultrahigh energies, the parameter p for particles above the barrier ($\theta_0 \gtrsim \theta_L$) can vary between small values ($p \ll 1$) for $\theta_0 \gg \theta_c$ and relatively large values ($p \gtrsim 1$) for $\theta_0 \lesssim \theta_c$. Accordingly, there should be a considerable change in the way these processes depend on the angle of incidence and the particle energy. Recent CERN experiments[7] at 150 GeV have indeed revealed a considerable difference between the measured probability of production of electron-positron pairs by photons incident at small angles to the $\langle 110 \rangle$ axis in germanium and the predictions of the standard theory of coherent pair production, even for angles of incidence much greater than the Lindhard angle.

The aim of this paper is to develop a more universal theory of radiation and pair production by particles above the barrier ($\theta_0 \gtrsim \theta_L$), which will be free from the restrictions that apply to the standard theory of coherent bremsstrah-

 0038-5646/88/101967-08$04.00

lung[1,2] and the theory based on the constant-field approximation.[8-11] This will enable us to define more precisely the range of validity of the usual approximations, and also to explain some of the experimental results[7] at ultrahigh energies that do not fit existing ideas.

2. EMISSION OF RADIATION AND PAIR PRODUCTION IN THE FIELD OF CRYSTAL PLANES AT ANGLES OF INCIDENCE $\theta_0 \gtrsim \theta_L$

Suppose that a particle of charge e and energy $E \gg 1$ enters the crystal at an angle $\theta_0 \ll 1$ to the crystallographic planes, but is quite distant from the principal axes lying on these planes. It can then be assumed that the particle experiences the potential $U(x)$ due to continuously charged planes, where the coordinate x is measured along the normal to the planes. The probability of emission of radiation by a particle above the barrier will be calculated using the general approach developed in Ref. 12 in which this probability is expressed in terms of transitions between states of transverse motion of the particle. Channeled particles ($\theta_0 < \theta_L$) correspond to bound states of transverse motion, whereas particles above the barrier ($\theta_0 \gtrsim \theta_L$) correspond to states in the continuous spectrum, with transverse energy $\varepsilon > U_0$.

According to Ref. 12 (see also Refs. 3 and 13), the differential probability (per unit length) of emission of a photon of energy ω by an electron (positron) in the field of the planes can be written in the following general form, independently of whether or not the particle is channeled or is above the barrier:

$$\frac{d^2w}{d\omega do} = \frac{e^2\omega}{2\pi}\sum_f\left\{\left(1+u+\frac{u^2}{2}\right)\left|j_x-\alpha j_z\right|^2 + \left[(1+u)\beta^2+\frac{u^2}{2}(\beta^2+E^{-2})\right]|j_z|^2\right\} \times\delta\left\{\frac{\omega}{2}\left[\alpha^2+(1+u)(\beta^2+E^{-2})\right]-\omega_{if}\right\}. \quad (1)$$

where $u = \omega/(E-\omega)$, $\omega_{if} = \varepsilon_i - \varepsilon_f$ is the energy difference between the transverse states of motion between which the radiative transition takes place, α, β are the angular variables related to the polar and azimuthal angles θ, φ by the formulas $\alpha = \theta\cos\varphi$, $\beta = \theta\sin\varphi$, and $do = d\alpha d\beta$ is the solid angle element into which the emission takes place. The new angular variables that we have just introduced will be found more convenient because α is the angle between the momentum of the photon and the crystallographic planes and β is the angle between the momentum of the photon and the plane containing the initial momentum of the particle and the normal to the crystallographic planes ($\alpha \ll 1, \beta \ll 1$). The Dirac delta-function in (1) represents the conservation of energy and of the component of the particle momentum component on the crystallographic plane during the emission process. The current matrix element for the radiative transition can be expressed in terms of the wave functions $\psi_i(x,E)$, $\psi_f(x,E-\omega)$ describing the transverse motion of the particles as follows:

$$j_z = \int_{-\infty}^{\infty} e^{i\omega\alpha x}\psi_i^*\psi_f\,dx, \qquad j_x = \frac{i}{E}\int_{-\infty}^{\infty} e^{i\omega\alpha x}\frac{d\psi_i^*}{dx}\psi_f\,dx \quad (2)$$

The wave functions ψ_i, ψ_f satisfy the Schroedinger equation with the potential $U(x)$, where the quantity E or, correspondingly, $E-\omega$ play the part of the mass, aand ε_i or, correspondingly, ε_f, plays the part of the transverse energy eigenvalue.

In the discussion given below, we shall be interested in the emission process in which the energy of the emitted photon can be of the same order as the particle energy ($u \sim 1$), and also in the closely related process of electron-positron pair production by the photon in the field of the crystal planes $\tilde{u} = \omega/(\omega - E) > 1$). The probability e^+e^- pair production can be deduced by a cross-transformation from the emission probability given by (1). Because of the change in the density of final states, the right hand side of (1) must be multiplied by $2\pi E/\omega^2$, followed by the replacements $E \to -E, \omega \to -\omega, \varepsilon_i \to -\varepsilon_i$. The quantity E is then looked upon as the total energy of the transformed positron, ε_i is its transverse energy, ε_f is the transverse energy of the electron, α is the angle of emission of the photon, and β the angle between the projections of the momenta of the photon and positron onto the crystallographic plane. The left hand side of (1) is then the probability of pair production per unit time per unit positron energy E per unit angle β, where the positron is produced in the state of transverse motion ψ_i. Similar replacements must be introduced in the equation for the transverse wave function ψ_i with the result that the Schroedinger equation for the resulting positron differs from the corresponding equation for the electron by the sign of the potential (and also by the value of the relativistic mass).

The transverse-motion wave functions above the barrier ($\varepsilon > U_0$) at high particle energies $E \gg 1$ can be written in the quasiclassical form

$$\psi_q(x) = B_q\exp\left[i\int^x p_q(\xi)\,d\xi\right], \qquad q=i,f, \quad (3)$$

where $B_q = [NT_q v_q(x)]^{-1/2}$ is the normalizing constant, N is the number of planes in the crystal, T_q are the time intervals between particle collisions with the planes, and

$$p_q(x) = [2E_q(\varepsilon_q - U(x))]^{1/2}, \quad E_i = E, \quad E_f = E-\omega.$$

Since particles above the barrier travel in the periodic potential $U(x)$ of the planes, their wave functions can be conveniently written in the Bloch form

$$\psi_q = \exp(i\varkappa_q x)\varphi_q(x), \quad (4)$$

where

$$\varkappa_d = (1/d)\int_0^d p_q(\xi)\,d\xi$$

are the transverse crystal momenta of the particles in the initial ($q=i$) and final ($q=f$) states, and

$$\varphi_q(x) = B_q\exp\left[\int_0^x p_q(\xi)\,d\xi - i\varkappa_q x\right]$$

are periodic functions of period d equal to the separation between the planes. When we evaluate the matrix element j_z of the radiative transition [see (2)], we take the product $\varphi_i^*\varphi_f$ in the form of the Fourier series

$$\varphi_i^*\varphi_f=(Nd)^{-1}\sum_{n=-\infty}^{\infty}C_n^{(1)}\exp(2\pi inx/d),$$

$$C_n^{(1)}=(T_iT_f)^{-1/2}\int_0^d\varphi_i^*\varphi_f(v_iv_f)^{-1/2}\exp(-2\pi inx/d)\,dx. \qquad (5)$$

The result is

$$j_z=2\pi\sum_{n=-\infty}^{\infty}C_n^{(1)}\delta(\varkappa_f-\varkappa_i+k_x+2\pi n/d),$$

where the Dirac delta function represents the conservation of transverse crystal momentum to within the reciprocal lattice vector $2\pi n/d$.

Similarly, the other component of the current is

$$j_x=2\pi\sum_{n=-\infty}^{\infty}C_n^{(2)}\delta(\varkappa_f-\varkappa_i+k_x+2\pi n/d),$$

$$C_n^{(2)}=(T_iT_f)^{-1/2}\int_0^d\varphi_i^*\varphi_f(v_i/v_f)^{1/2}\exp(-2\pi inx/d)\,dx. \qquad (5')$$

The sum of (1) over the final quantum numbers f of the particles above the barrier must be interpreted as an integral over $(Nd/2\pi)\varkappa_f$. Integration over the final crystal momenta $\varkappa_f$ can then be reduced to integration over the transverse energy ε_f, since $d\varkappa_f=d\varepsilon_j/\langle v_f\rangle$, where $\langle v_f\rangle$ is the average velocity of the particle, evaluated over the time between the planes (in the final state).

The integral with respect to the final energy ε_f reduces the following substitution when the delta function in (1) is taken into account:

$$\varepsilon_f=\varepsilon_i-\frac{\omega}{2}\,[\alpha^2+(1+u)(\beta^2+E^{-2})], \qquad (6)$$

since the effective value of the square of the angle of emission can be large ($\sim v_i^2$) compared with $1/[E(E-\omega)]$, it is more convenient to transform to the new angle variable $\alpha'=\alpha-\langle v_i\rangle$. The transverse kinetic energy in the final state can then be written in the form

$$\varepsilon_f-U(x)=[(E-\omega)/2]\,v_i^2(x)+(\omega/2)\,[v_i^2(x)-\langle v_i\rangle^2]$$
$$-\omega\alpha'\langle v_i\rangle-(\omega/2)\,[\alpha'^2+(1+u)(\beta^2+E^{-2})].$$

Suppose now that

$$v_i^2(x)\gg u\,[v_i^2(x)-\langle v_i\rangle^2]. \qquad (7)$$

When the energy ω of the photon is of the same order as the particle energy E, the inequality (7) signifies that the angle of incidence of the particles should be several times greater than the Lindhard angle, whereas for softer photons ($u\ll 1$) the inequality is satisfied for practically any angles of incidence $\theta_0>\theta_L$.

Under condition(7), and confining our attention to terms of order higher than $\sim(\theta_d/\langle v_i\rangle)^2$ in the expansion for p_f, we obtain

$$\varkappa_f-\varkappa_i+k_x=-[\omega(1+u)/2\langle v_i\rangle](\alpha'^2+\beta^2+E^{-2}+\langle\theta_d^2\rangle),$$
$$p_f-p_i-(\varkappa_f-\varkappa_i)=-[\omega(1+u)/2v_i](\theta_d^2-2\alpha'\theta_d-\langle\theta_d^2\rangle).$$

The angle brackets in this expression represent time averages over the motion of the classical particle of energy E within the collision period. This time is related to the transverse coordinate x by the classical equation of motion $dx/dt=v_i(x)$. The angle of deflection $\theta_d(t)$ due to the field of the planes is given by $\theta_d(t)=v_i(x)-\langle v_i\rangle$.

Substituting the matrix elements (5) and (5') into the general expression given by (1), we obtain the following differential probability for emission of an electron (positron) above the barrier per unit path in the crystal:

$$\frac{d^2w}{d\omega do}=\frac{e^2\omega}{2\pi}\sum_{n=1}^{\infty}\Big\{\Big(1+u+\frac{u^2}{2}\Big)|I_n^{(x)}-\alpha'I_n^{(z)}|^2$$
$$+\Big[(1+u)\beta^2+\frac{u^2}{2}(\beta^2+E^{-2})\Big]|I_n^{(z)}|^2\Big\}$$
$$\times\,\delta\Big[\frac{\omega'}{2}(\theta'^2+E^{-2}+\langle\theta_d^2\rangle)-\omega_n\Big], \qquad (8)$$

$$I_n^{(z)}=\frac{1}{T}\int_0^T\exp[-if(t)]\,dt,\qquad I_n^{(x)}=\frac{1}{T}\int_0^T\theta_d(t)\exp[-if(t)]\,dt,$$

$$f(t)=\omega'\Big\{-\alpha'\int_0^t\theta_d(\tau)\,d\tau+\frac{1}{2}\int_0^t[\theta_d^2(\tau)-\langle\theta_d^2\rangle]\,d\tau\Big\}-\omega_nt,$$
$$\omega_n=2\pi n/T,\qquad \omega'=\omega(1+u). \qquad (9)$$

The collision period T between the particles and the planes and the average transverse velocity $\langle v_i\rangle$ are given by

$$T=(E/2)^{1/2}\int_{-d/2}^{d/2}[\varepsilon_i-U(x)]^{-1/2}\,dx,\qquad \langle v_i\rangle=d/T.$$

Next, $dO=\theta'd\theta'd\varphi'$, $\alpha'=\theta'\cos\varphi'$, $\beta=\theta'\sin\varphi'$, and θ',φ' are the polar and azimuthal angles of emission in the new coordinate frame in which the polar axis coincides with the mean velocity vector of the particle in the field of the planes.

In the limit of relatively soft photons ($u\to 0$), the expression given by (8) becomes identical with the result obtained earlier in Refs. 3, 14, and 15 in the classical theory of emission if we take into account the fact that $\theta'=(\alpha'^2+\beta^2)^{1/2}$ is the angle between the direction of emission and the mean velocity vector of a particle above the barrier.

The differential e^+e^- pair production probability per unit length per unit energy E per unit solid angle of emission of the positron (electron) is obtained by multiplying the right-hand side of (8) by a factor $(E/\omega)^2$ and introducing the replacements $E\to -E$, $\omega\to-\omega$, $u\to-\tilde{u}$, where $\tilde{u}=\omega/(\omega-E)$, in all the expressions following the summation sign in which T and $\theta_d(t)$ must now be regarded as the period of motion and the angle of deflection of the positron (electron) in the field of the planes, respectively. All this yields the following differential probability:

$$\frac{d^2w_p}{dE\,do}=\frac{e^2E^2}{2\pi\omega}\sum_{n=1}^{\infty}\left\{\left(1-\tilde{u}+\frac{\tilde{u}^2}{2}\right)|J_n^{(x)}-\alpha'J_n^{(z)}|^2\right.$$

$$\left.+\left[(1-\tilde{u})\beta^2+\frac{\tilde{u}^2}{2}(\beta^2+E^{-2})\right]|J_n^{(z)}|^2\right\}$$

$$\times\delta\left[\frac{\omega}{2}(\tilde{u}-1)(\theta'^2+E^{-2}+\langle\theta_d^2\rangle)-\omega_n\right]. \qquad (8')$$

The quantity J_n in this expression is obtained from I_n [see (9)] by introducing the above replacement. We note that, when the pairs are produced by a photon, the parameter $\tilde{u}$ is greater than unity, so that the pair production probability can be obtained from the probability of emission of soft photons ($\omega > E/2$) alone.

For angles of incidence $\theta_0 \gtrsim \theta_L$, in which we are interested here, and for which the transverse charged-particle energy $\varepsilon_i \approx E\theta_0^2/2$ is much greater than the barrier height U_0 between neighboring channels, the transverse velocity $v_i(x)$ can be represented by the first two terms in the expansion of powers of U/ε:

$$v_i\approx\theta_0(1-U/2\varepsilon).$$

The deflection angle is then given by

$$\theta_d(t)\approx(E\theta_0)^{-1}[\langle U\rangle-U(x)], \quad x\approx t/\theta_0,$$
$$\langle\theta_d^2\rangle\approx(E\theta_0)^{-2}[\langle U^2\rangle-\langle U\rangle^2], \quad T\approx d/\theta_0. \qquad (10)$$

where the averages are evaluated over the transverse coordinate x within the interval between the planes. For angles of incidence $\theta_0^2 \gg \theta_L^2$, the probability of emission by positrons is obtained from (8) by introducing $\alpha' \to -\alpha'$ from the corresponding expression for electrons. The probability integrated over the angles of emission of the photon is then found to be the same for both electrons and positrons, i.e., it is independent of the sign of the charge of the radiating particle.

The results that we have obtained become very much simpler in two limiting cases. We can then use the more general theory (8) to establish the connection between the standard theory of coherent bremsstrahlung[1,2] and the constant-field approximation.[8–11] If the angle of incidence is sufficiently large in comparison with the angle $\theta_c \equiv U_0$, the mean square of the angle of deflection $\langle \theta_d^2\rangle$ in the argument of the delta function in (8) can be neglected. We can also neglect the second term in the phase $f(t)$ [see (9)] as compared with the first in which case the first term is found to be much smaller than the last, so that the corresponding exponent in I_n can be expanded into a series, and only the first significant terms are retained. The result of all this is that the differential emission probability assumes the form corresponding to the dipole approximation:

$$\frac{d^3w}{d\omega d\alpha' d\beta}=\frac{e^2\omega}{2\pi}\sum_{n=1}^{\infty}|x_n|^2\left\{\left(1+u+\frac{u^2}{2}\right)[\,|\omega'\alpha'-\omega_n|^2\right.$$

$$+(\omega'\alpha'\beta)^2]$$

$$\left.+\alpha'^2u^4/2\right\}\delta[(\omega'/2)(\theta'^2+E^{-2})-\omega_n] \qquad (11)$$

where we have used the following notation:

$$\omega_n=2\pi n\theta_0/d, \quad x_n=(U_n/E\theta_0^2)(d/2\pi n),$$

$$U_n=\frac{1}{d}\int_{-d/2}^{d/2}U(x)\exp[-2\pi inx/d]dx. \qquad (12)$$

It is clear that the Fourier component of the dipole moment x_n can be expressed in terms of the corresponding Fourier component U_n of the average potential of the planes. In its turn, U_n is related to the three-dimensional Fourier component of the potential $\varphi(k_x, k_y, k_z)$ of a crystal atom by the well-known expression

$$U_n=V^{-1}\varphi\left(\frac{2\pi n}{d},0,0\right)\exp\left[-\left(\frac{2\pi n}{d}\right)^2 u_t^2\right], \qquad (13)$$

where V is the volume of the unit cell of the (monatomic) crystal and u_t^2 is the mean square amplitude of the thermal vibrations of the atoms. Integrating (11) with respect to the angles, we obtain the differential probability of dipole emission per unit length per unit photon energy:

$$\frac{dw}{d\omega}=\frac{e^2}{(E\theta_0)^2}\sum_{n=1}^{\infty}|U_n|^2\left[1+\frac{u^2}{2(1+u)}-2\Omega_n+2\Omega_n^2\right]\eta(1-\Omega_n), \qquad (14)$$

where $\Omega_n = u/(2E\omega_n)$ and η is the Heaviside step function. When (13) is taken into account, the expression given by (14) becomes identical with the analogous expression for the emission probability in the standard theory of coherent bremsstrahlung in the Born approximation [see, for example, equation (4.9) in Ref. 16]. The standard theory of coherent bremsstrahlung and e^+e^- pair production is thus seen to remain valid for angles of incidence θ_0 significantly greater than $\theta_c \equiv U_0$ (and also the critical Lindhard angle for charged-particle channeling).

In the limit $\theta_0 \ll \theta_c$, which is the opposite of the dipole case, the probability of emission by particles with ultrahigh energies $E \gg E_1$ contains the contribution due to the relatively large number of the higher harmonics. We can then neglect the relatively fine structure of the spectrum, due to the coherence of the planes, and replace summation over the harmonics n by integration over the quasicontinuous quantity ω_n. The method of stationary phase can then be used to evaluate the components of the current. The phase $f(t)$ [see (9)] is then represented by an expansion around the stationary point t_s, defined by the condition $\alpha' = \theta_d(t_s)$, and has the form

$$f(t)\approx\frac{\omega'}{2}\left[(\beta^2+E^{-2})(t-t_s)+\frac{1}{3}\left(\frac{d\theta_d(t_s)}{dt_s}\right)^2(t-t_s)^3\right],$$

and the integration with respect to the time $\tau = t - t_s$ is performed between $-\infty$ and $+\infty$. The components of the current I given by (9) can then be expressed in terms of the modified Bessel function of the second kind, K_ν, and the differential emission probability (8) assumes the form

$$\frac{d^3w}{d\omega d\alpha' d\beta}=\frac{e^2\omega}{3\pi^2T}\frac{1+\delta^2}{(E^2a)^2}\left\{\left(1+u+\frac{u^2}{2}\right)(1+\delta^2)K_{2/3}^2(\xi)\right.$$

$$\left.+\left[(1+u)\delta^2+\frac{u^2}{2}(1+\delta^2)\right]K_{1/3}^2(\xi)\right\}, \qquad (15)$$

where $\xi = |u/(3aE^2)|(1+\delta^2)^{-3/2}$, $\delta = E\beta$, and $a = (dU/dx)E^{-1}$ is the acceleration of the particle at the stationary point x_s. The emission angle α' for given transverse particle energy ε_i defines, via the expression $\theta_d(x_s) = \alpha'$, the point x_s on the trajectory and, hence, the acceleration $a = a(x_s)$. Integration of (15) with respect to α' then reduces to integration with respect to the time of motion within the collision period, since $d\alpha' = adt$.

We have derived (15) in the limit of relatively large deflection angles ($p \gg 1$), but its form is the same as that of the expression for the probability of emission by a relativistic particle in a constant magnetic field (see, for example, Ref. 17). The constant-field approximation is usually used in theory of bremsstrahlung and pair production by channeled particles,[10,11] but it is clear that it can also be used in the case of particles above the barrier (see for example, Ref. 9). It must be remembered, however, that the condition for the validity of this approximation for particles above the barrier is satisfied at higher energies E than for channeled particles, and the necessary energies increase with increasing angle of incidence θ_0.

In the above limit, in which $p \gg 1$, we can use (6) to obtain a relation between the total energy loss per unit time averaged over all ω and the transverse-energy loss:

$$\frac{d\varepsilon}{dt} = \frac{v_x^2}{2}\frac{dE}{dt}. \qquad (16)$$

Since $E \gg E_1$, this result is identical with the analogous result obtained in classical electrodynamics by evaluating the work done by radiation reaction (see, for example, Sec. 7.1 in Ref. 3). However, (16) has a greater range of validity because it is not restricted by the condition $\omega \ll E$, i.e., the condition that the energy carried off by an individual photon must be small in comparison with the energy of the radiating particle. Expression (16) is important for the investigation of the radiation reaction of the motion of high-energy particles in a crystal field.

To illustrate our theory, the figure shows the integrated probability (per unit length of e^+e^- pair production by a photon with energy of $\omega = 3$ TeV in a germanium crystal at 20 °C as a function of the angle of emission of the photon, θ_0, relative to the (110) planes. The potential of the planes in germanium was approximated by the parabola $U(x) = \pm U_0(2x/d)^2$, where $|x| < d/2$, $U_0 = 39$ eV, $d = 2.0$ Å. The components of the current (9) can then be written in the form

$$I_n^{(z)} = \int_0^1 \cos f(\eta)\, d\eta, \qquad I_n = \frac{p}{E}\int_0^1 (1-3\eta^2)\cos f(\eta)\, d\eta,$$

$$f(\eta) = \mp\frac{3}{2}u\alpha' E_2 p^2(\eta-\eta^3) + \frac{27uE_2}{4E}p^3\left(\frac{\eta}{45} - \frac{2}{9}\eta^3 + \frac{\eta^5}{5}\right) + \pi n\eta,$$

$$\langle\theta_d^2\rangle = \frac{4p^2}{5E^2}, \qquad p \equiv \frac{U_0}{3\theta_0}, \qquad E_2 \equiv \frac{d}{U_0}.$$

Curve 1 shows the results obtained using the standard theory of coherent pair production, curve 2 shows the results obtained in the theory of pair production in a strong constant

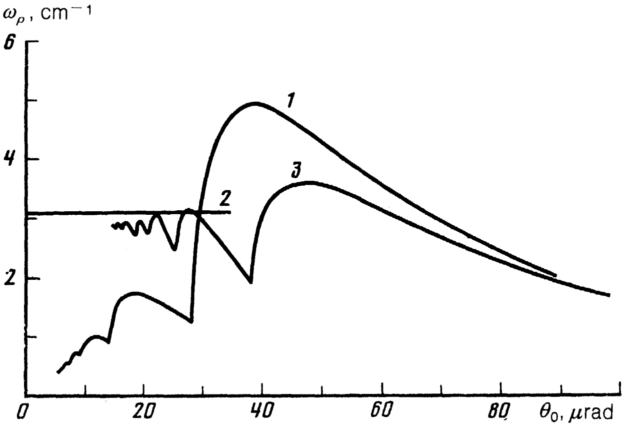

FIG. 1. Integrated probability of e^+e^- production by 3-TeV photons in germanium as a function of the angle of incidence relative to the (110) planes: 1—standard theory of coherent pair production, 2—constant-field approximation, 3—more rigorous theory.

field, and curve 3 was calculated from the more rigorous expression given by (8).

The standard theory of coherent pair production predicts that, on average, the probability will decrease linearly with decreasing angle of incidence, with the coherent maxima superimposed on this variation. The theory of pair production based on the constant-field approximation leads to a probability that is independent of the angle of incidence. The more rigorous theory again leads to the appearance of coherent maxima at particular angles of the incidence, but the positions of these peaks are strongly shifted, especially those corresponding to the high-order harmonics, relative to the positions predicted by the standard coherent theory. Moreover, the integrated probability calculated from the general theory does not decrease with decreasing angle of incidence θ_0 but, on average, approaches the level typical for the constant-field approximation. It is important to remember that, in practical situations, the photon beam has a certain angular spread in θ_0 and in the energy ω. The result of this may be that the frequent and low maxima that correspond to the higher ($n \gtrsim 1$) harmonics on curve 3 may disappear altogether after the additional averaging over the narrow ranges of θ_0 and ω.

We must now examine the difference between the more general formula for the emission probability (8) and the dipole limit (11) corresponding to the standard coherent bremsstrahlung theory. As noted in the Introduction, the deviations from the standard theory depend on the parameter $p \approx E\langle\theta_d^2\rangle^{1/2}$, i.e., on the ratio of the angle of deflection of the particle to the effective emission angle. Both in the dipole limit and in the general case, the emission probability consists of harmonics that correspond to crystal momenta transferred during the emission process in different multiples of the reciprocal lattice vector $2\pi/d$. For a given harmonic, photons with maximum energy are always emitted at zero angle ($\theta' = 0$), i.e., in the direction of the mean velocity of the particle in the field of the planes. The maximum energy associated with the nth harmonic is

$$\omega_{max} = E\frac{\xi_n}{1+\xi_n}, \qquad \xi_n = \frac{2E\omega_n}{1+E^2\langle\theta_d^2\rangle},$$

and is in general smaller than predicted by the standard theory of coherent bremsstrahlung because of the presence of the term $E^2\langle\theta_d^2\rangle$ in the denominator of ξ_n.

In the dipole approximation (14), the probability of emission of the nth harmonic is proportional to the square of the corresponding Fourier component of the potential (13), and decreases relatively rapidly (as n^{-4} if we ignore the Debye-Waller factor) with increasing number of the harmonic. However, in general, the probability is determined by the Fourier components of functions that are more complicated that $U(x)$, even when the transverse energy is high ($\varepsilon \gg U_0$) and (10) is valid. This means that, when the dipole approximation ($p \gtrsim 1$) breaks down, all the high-order harmonics begin to play a significant part, whereas the intensity of the first harmonics gradually declines.

3. BREMSSTRAHLUNG AND PAIR PRODUCTION IN THE FIELD OF THE CRYSTALLOGRAPHIC AXES

Let us now consider the other characteristic case in which the initial momentum of the particles entering the crystal is close to the direction of a principal crystallographic axis. The angle θ_0 to the axis will be assumed to be large in comparison with the critical Lindhard angle $\theta_L^{(s)}$ for axial channeling. In the planar case, this restriction will enable us to neglect the influence of the curvature of the particle trajectory on the characteristic collision time.

We take the Cartesian coordinate frame so that the z axis coincides with the direction of the initial particle momentum and the yz plane lies in the plane containing the crystallographic axis and the initial momentum. When the charged particle collides with the continuously charged axis passing through the origin, and the impact parameter is x, the particle is deflected through the angle θ_x in the xy plane and the angle θ_y in yz plane. If we assume that the angle of incidence to the crystal axis is relatively large ($\theta_0^2 \gg \theta_L^{(s)^2}$), the deflection angles are given by [see (10)]

$$\theta_x(x,y) = -(E\theta_0)^{-1}\int_{-\infty}^{y}\frac{\partial U(\rho)}{\partial x}dy, \qquad \theta_y(x,y) = -(E\theta_0)^{-1}U(\rho),$$

where $U(\rho)$ is the average potential of the axis and ρ is the distance from the axis. Suppose that the direction of incidence of the particle does not lie in any of the principal crystallographic planes. Planar channeling effects can then be neglected. Moreover, the periodic disposition of the axes is then unimportant and, in the first approximation, we can neglect the coherence of the probability amplitudes due to different axes. The problem thus reduces to the evaluation of the probability of bremsstrahlung and e^+e^- pair production in the field of an isolated axis. This approximation is also possible when the initial direction of incidence lies along the direction of one of the principal crystallographic axes, but the angle of incidence θ_0 with respect to the chosen axis is not too large in comparison with the critical $\theta_L^{(s)}$ for axial channeling. The point is that, under these conditions, the scattering of the particle by even one or two axes may take it out of the initial planar channel (see for example Refs. 6 and 18), thus upsetting the coherence of the process.

If we use the general quantum theory of emission in the field of an axis,[6,3] under the above restrictions, we obtain by analogy with the planar case the following expression for the differential probability of emission of a photon of energy ω (per unit length) by an electron (positron) incident at an angle θ_0 ($\theta_0^2 \gg \theta_L^{(s)2}$) to the crystallographic axes:

$$\frac{d^2w}{d\omega do} = \frac{e^2\omega}{2\pi^2}\frac{\theta_0}{S}\int_0^\infty\left\{\left(1+u+\frac{u^2}{2}\right)|\mathbf{I}_\rho - I_z\mathbf{n}_\rho|^2 + (u^2/2E^2)|I_z|^2\right\}dx, \tag{17}$$

$$I_z = \int_{-\infty}^{\infty}\exp[-ig(\rho)]dy, \qquad \mathbf{I}_\rho = \int_{-\infty}\boldsymbol{\theta}_d(\rho)\exp[-ig(\rho)]dy,$$

$$g(\rho) = \frac{\omega'}{2\theta_0}\int_0^y[|\boldsymbol{\theta}_d(\rho) - \mathbf{n}_\rho|^2 + E^{-2}]dy, \tag{18}$$

where S is the area per crystallographic axis in the plane perpendicular to the axes, $\omega' = \omega(1+u)$, $\theta_d = (\theta_x, \theta_y)$ is the two-dimensional vector representing the angle deflection of the particle, $\mathbf{n}_\rho = (\theta\cos\varphi, \theta\sin\varphi)$, and θ,φ are the polar and azimuthal angles of emission. Integration with respect to x in (17) is equivalent to averaging the probability over all the impact parameters.

The emission probability given by (17) and the analogous result for the pair production probability become much simpler in two limiting cases. For relative small angles of the deflection of the particle by the field of the axis ($\theta_0 \gg U_0$), we can use the dipole approximation which, as shown above, corresponds to the standard theory of coherent bremsstrahlung:

$$\mathbf{I}_\rho \approx \int_{-\infty}^{\infty}\boldsymbol{\theta}_d(\rho)\exp\left[-\frac{i\omega'}{2\theta_0}(\theta^2+E^{-2})y\right]dy,$$

$$I_z \approx [2/(\theta^2+E^{-2})]\mathbf{n}_\rho\mathbf{I}_\rho.$$

In the opposite limit, in which $\theta_0 \ll U_0$, the current components (18) can be evaluated by the method of stationary phase. The particle trajectory can be regarded as planar in the neighborhood of the stationary point y_s, so that the analysis of this limit in the axial case is completely analogous to the planar case and leads to an expression analogous to (15).

4. ANALYSIS OF RESULTS AND COMPARISON WITH EXPERIMENTAL DATA

Analysis of the general result given by (17), and further numerical calculations in the intermediate range of the angles of incidence $\theta_0 \sim U_0$, are considerably simplified if we use the potential of an axis in the form $U(\rho) = U_0a/\rho$, where $\rho > \rho_{\min}$. The potential parameters U_0, a and the cutoff parameter $\rho_{\min}$ are usually determined from the condition that the well depth $U(\rho_{\min})$ and the maximum potential gradient $U(\rho_{\min})/\rho_{\min}$ must agree with the corresponding values for the more general potential $U(\rho)$. In the first approximation, which is sufficient for simple estimates, we may suppose that $U_0 \approx \pi Ze^2/2d_s$, $\rho_{\min} \approx a \approx a_{TF} = 0.88e^{-2}Z^{-1/3}$, where Z is the nuclear charge of the crystal atoms and d_s is the separation between neighboring atoms on the axis.[1)]

Using this model together with (17) and the cross-transformation of the emission probability, we find that the probability per unit length of the production of the e^+e^- pair by a photon is

$$\frac{d^2w_p}{dEdo}=\frac{2e^2}{\pi^2}E_2p^3\frac{a^2}{S}\int\limits_{\rho_{min}/a}^{\infty}e^{\nu\pi}\left\{\left[(1-\tilde{u})k_1+\frac{\tilde{u}^2}{2}\left(1-\frac{\nu^2}{\zeta^2}\right)\right]K_{i\nu}^2(\zeta)+\left[(1-\tilde{u})k_2+\frac{\tilde{u}^2}{2}\right]K_{i\nu}'^2(\zeta)\right\}dq. \quad (19)$$

where $K_{i\nu}(\zeta)$, $K'_{i\nu}(\zeta)$ are, respectively, the modified Bessel function and its derivative with respect to ζ, and

$$k_1=1-\frac{\nu^2}{\zeta^2}\left[1+\left(\frac{\delta q}{p\zeta}\right)^2\right],\quad k_2=1-\left(\frac{\eta q}{p\zeta}\right)^2,$$
$$\nu=(\omega'E_2/E^2)p^2\psi\sin\varphi,\quad \eta=(\omega'E_2/E^2)p^2\psi\cos\varphi,$$
$$\delta=(\omega'E_2/2E^2)pq[\psi^2+1+(p/q)^2],\quad \zeta^2=\delta^2-\eta^2,$$
$$p=U_0/\theta_0,\quad E_2=a/U_0,\quad q=x/a,\quad \psi=E\theta,$$
$$\omega'=\omega(\tilde{u}-1),\quad \tilde{u}=\omega/(\omega-E).$$

The usual procedure is to measure the pair production probability integrated with respect to the energy and the angle of emission (see, for example, Refs. 1, 2, and 7). The corresponding integration in (19) in the general case of arbitrary p can be carried out only by numerical methods. However, the basic features of the behavior of the integrated probability w_b as a function of the photon emission angle θ_0 can be understood by analyzing the integrands in (19) for different parameter ratios.

According to (19), the total probability of e^+e^- pair production by a photon depends on two parameters, namely, the ratio $1/p$ of the angle of incidence θ_0 to the angle $\theta_0\equiv U_0$, and the ratio of the photon energy ω to the energy $E_2=a/U_0$. Using the above expressions for the parameters of the potential of an axis, we obtain the analytic expression $E_2\approx 0.6e^{-4}Z^{4/3}d_s$. The critical energy e_2 is thus seen to decrease with increasing nuclear charge Z of the crystal atoms, and to increase together with the Miller indices of the axis in proportion to the separation between neighboring atoms on the axis. We note that the energy E_2 is greater by a factor of about $10^2Z^{-1/3}$ than the critical energy $E_1=1/U_0$ introduced earlier.

For angles of incidence $\theta_0\lesssim U_0$, the total probability is practically independent of the angle θ_0 and its value is determined exclusively by the ratio ω/E_2. In this range of angles of incidence, the probability at first rapidly (exponentially) increases with increasing ω/E_2, and when the ratio ω/E_2 reaches unity, the increase slows down and is subsequently described by the expression $(\omega/E_2)^{1/3}$. When $\omega/E_2\sim 1$, the pair production probability is $w_p\sim(e^2/\pi^2E_2)\times(a^2/S)$. If we compare this result with the probability w_{pam} for the amorphous medium, we find that

$$w_p/w_{pam}\sim\pi^{-2}e^{-4}Z^{-4/3}[\ln(183Z^{-1/3})]^{-1}. \quad (20)$$

This ratio increases with decreasing Z, but it must be remembered that the photon energy $\omega\sim E_2$ for which (20) is still valid is then also higher. The above probabilities depend significantly on the ratio ω/E_2 in the range $\theta_0\gtrsim U_0$. For $\omega/E_2\lesssim 1$, the probability increases with increasing ratio θ_0/U_0, reaching its maximum value for

$$\theta_0^{(m)}\approx(2U_0E_2/\omega)(1+\omega^2/4E_2^2).$$

The value of the probability at this point is

$$w_p(\theta_0^{(m)})\sim e^2\omega E_2^{-2}(1+\omega^2/4E_2^2)^{-1}.$$

Further increase in the angle of incidence is accompanied by a slow ($\sim 1/\theta_0$) decrease in the probability.

On the other hand, when the photon energy is relatively high, so that $\omega\gtrsim 2E_2$, the behavior of the probability for angles of incidence $\theta_0\gtrsim U_0$ differs significantly. The probability then decreases monotonically with increasing θ_0/U_0.

The table lists the calculated parameters E_1, E_2, $\theta_c\equiv U_0$ for different crystals and different crystallographic directions. The potentials of the axes and planes were calculated using the Doyle-Turner model of the atomic potential, taking into account thermal vibrations at 100°. The quantity E_2 was then determined as $1/|\nabla U|_{max}$, where $(\nabla U)_{max}$ is the maximum potential gradient due to the axis or plane, and is a function of temperature.

The integrated pair-production probability was measured in Ref. 7 for protons with energy between 22 and 150 GeV as a function of the angle of incidence relative to the ⟨110⟩ axis in a germanium crystal at 100°. Calculations show that, in this case, $E_2=47$ GeV for $\theta_c=5.3\times10^{-4}$.

The angle θ_c is close to the value at which a sharp difference is observed between $w_p(\theta_0)$ and the predictions of the standard theory at all photon energies.

The different behavior of the measured probability as a function of photon energy for angles of incidence $\lesssim 5\cdot10^{-4}$ and $\gtrsim 5\cdot10^{-4}$ is also in agreement with the predictions of the theory developed here. In particular, the maximum that is clearly observed at the lower energies is found to disappear altogether for photon energies in the range 90–120 GeV. This is in agreement with the theoretical value $\omega\approx 2E_2=94$ GeV for which this effect should be observed.

5. CONCLUDING REMARKS

The characteristic features of electromagnetic processes in crystal at high incident-particle energies can thus be explained only by a theory that takes into account the nondipole nature of the processes. The following point must also be taken into account in detailed calculations and in

TABLE I.

	(110) C_d	⟨110⟩ C_d	(110) Si	⟨110⟩Si	(110) Ge	⟨110⟩Ge
E_1, GeV	10.9	2.44	11.6	1.91	7.13	0.960
E_2, GeV	834	93.7	1018	94.6	545	46.8
$\theta_c\cdot10^5$	4.7	20.9	4.42	26.8	7.16	53.2

comparisons between calculations and measurements. The fact that the photon beam is not strictly monochromatic is equivalent to an additional averaging of the pair production probability (19) over the photon energy. Thermal vibrations of the atoms lead to a change in the potential parameters and to an additional background in the probabilities, which is due to the interaction between the particles and the individual crystal atoms. The background differs from the pair production probability in the amorphous medium only by the factor $\sim u_t/a_{TF}$ under the sign of the radiation logarithm [see for example Refs. 1 and 3] and, according to the estimate given by (20), it is particularly significant at relatively low energies $\omega \ll E_2$. The influence of the curvature of the particle trajectory on the characteristic time of their interaction with the axes or planes must be taken into account for angles of incidence approaching the Lindhard angle.[6,9,14,15]

The above theoretical analysis of the e^+e^- production by ultrahigh energy photons shows that, for small angles of incidence $\theta_0 \lesssim \theta_c$, the production probability may be higher by an order of magnitude as compared with the amorphous medium (θ_c does not depend on the energy and is determined exclusively by the properties of the crystal lattice). An estimate analogous to (20) is valid[6,3] for the ratio of the emission probabilities in a crystal and in the amorphous medium. As a result, the effective avalanche length in the crystal for particles entering at angles of incidence $\theta_0 \lesssim \theta_c$ should also be lower by an order of magnitude.

For particles heavier than the electron (for example, muons), the angle θ_c is found to be significantly smaller because it is inversely proportional to the particle mass M, whereas the critical energies E_1, E_2 are greater by a factor of M^2. The production probability is then lower by a factor of M^2.

I am indebted to V. I. Glebov for help with numerical calculations and for useful discussions.

[1)]Better precision is achieved if the cut off parameter is taken to be temperature dependent in the form $\rho_{\min} = (2a_{TF}u_t)^{1/2}$, where u_t is the amplitude of thermal vibrations of the atoms.

[1]M. L. Ter-Mikaelyan, *The Influence of the Medium on Electromagnetic Processes at High Energies*, [in Russian], Academy of Sciences of the Armenian SSR, Erevan, 1969.

[2]G. Diambrini-Pallazzi, Rev. Mod. Phys. **40**, 611 (1968).

[3]V. A. Bazylev and N. K. Zhevago, *The Emission of Radiation by Fast Particles in a Medium and in External Fields*, [in Russian], Nauka, Moscow, 1982.

[4]L. D. Landau and E. M. Liftshitz, Teoriya Blya, Nauka, Moscow, 1967, p. 460 (The Classical Theory of Fields, (Pergamon, Oxford, 1989). (English translation by Pergamon Press).

[5]J. Lindhard, Phys. Lett. **12**, 126.

[6]A. L. Avakyan, N. K. Zhevago, and Yan Shi, Zh. Eksp. Teor. Fiz. **82**, 584 (1982).

[7]A. Belkacem, G. Bologna, M. Chevallier *et al.*, Phys. Rev. Lett. **54**, 2667 (1985); *ibid.* **58**, 1196 (1987); Phys. Lett. **B117**, 211 (1986).

[8]N. F. Shul'ga, Pis'ma Zh. Eksp. Fiz. **32**, 179 (1980) [JETP Lett. **32**, 166 (1980)].

[9]A. L. Avakian, C. Yang, and N. K. Zhevago, Radiat. Eff. **56**, 39 (1981).

[10]C. J. Kimball and N. Cue, Nucl. Instr. Meth. **230** [B2], 25 (1984), Phys. Rep. **125**, 69 (1985).

[11]A. H. Sorenson, E. Uggerhoj, J. Bak, and S. P. Moller, Phys. Scripta. **32**, 149 (1985).

[12]N. K. Zhevago, Zh. Eksp. Teor. Fiz. **75**, 1389 (1978) [Sov. Phys. JETP **48**, 701 (1978)].

[13]V. A. Bazylev and N. K. Zhevago, Usp. Fiz. Nauk. **137**, 605 (1982) [Sov. Phys.—Usp. **25**, 565 (1982)].

[14]V. A. Bazylev, V. V. Beloshitsky, V. I. Glebov *et al.*, Radiat. Eff. **56**, 87 (1981).

[15]V. A. Bazylev, V. I. Glebov, and N. K. Zhevago, Zh. Eksp. Teor. Fiz. **78**, 62 (1980) [Sov. Phys.—JETP **51**, 31 (1980)].

[16]A. I. Akhiezer and N. F. Shul'ga, Usp. Fiz. Nauk, **137**, 561 (1982) [Sov. Phys.—Usp. **25**, 541 (1982)].

[17]V. B. Berestetskiĭ, E. M. Liftshitz, and L. P. Pitaevskiĭ, Relativistic Quantum Theory, Part 1, Pergamon, Oxford (1971).

[18]H. R. Avakian, I. V. Glebov, V. V. Goloviznin *et al.*, Radiat. Eff. **80**, 141 (1984).

Translated by S. Chomet