Propagation of long light pulses in a two-level resonant medium

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(Submitted 25 January 1988)
Zh. Eksp. Teor. Fiz. **94**, 159–169 (October 1988)

Whitham's equations for the slow variation of the parameters of a periodic solution are derived for the self-induced-transparency equations. The theory developed is applied to the problem of soliton formation on the front of a long light pulse.

1. INTRODUCTION

Sufficiently short and intense light pulses are known to propagate in a resonant medium in the form of solitons, here called 2π-pulses.[1] Longer and more intense pulses decay ultimately into several solitons. Experiments in which the initial pulse breaks up into several (~ 10) solitons have by now been reported.[2] A description of the processes whereby solitons are formed from a long light pulse is therefore called for and is the subject of the present paper.

From the theoretical viewpoint, the problem posed is related to the familiar theory of collisionless shock waves in a plasma: the leading edge of a light pulse entering a resonant medium is similar to the shock-wave front on which solitons are created. Gurevich and Pitaevskiĭ[3] developed a theory for such a process on the basis of Whitham's method,[4] and described the dynamics of the medium through which a shock wave propagates by using the Korteweg–de Vries (KdV) equation. The inverse scattering problem method was not yet in use when Refs. 3 and 4 were written. It is clear at present, however, that Whitham's method is fully effective only for equations which are integrable by the inverse scattering problem method (and which include the KdV equation). In fact, the latter method leads directly to Whitham's equations in an invariant Riemann form, as demonstrated in Ref. 5 for the KdV equation. Although the substitution needed to transform to the Riemann variable was inferred by Whitham,[4] it is more difficult to find Riemann variables for other equations, and the use of the inverse scattering problem is almost obligatory. It was just on this basis that Whitham's method was developed in Ref. 6 for the physically important sine-Gordon equation.

Whitham's theory is a variant of the averaging method: the complete solution is subdivided into sections, each of which is described by a single-period or multiperiod solution with slowly varying parameters. The parameter variation within each section is determined by Whitham's equations, which can be obtained in principle by substituting periodic solution with variable parameters into the initial dynamic equations, and by averaging them under the assumption that the parameters change little in one wavelength and in one period. A more effective method of obtaining Whitham's equations is to average the conservation laws[4–6] and to use the inverse scattering problem to select the natural parameters on which the periodic solutions depend, as well as to determine the solutions themselves.

In view of the foregoing, we must find first, by the inverse scattering method, periodic solutions of the self-induced-transparency (SIT) equations that describe the propagation of a light pulse in a resonant medium, then obtain Whitham's equations by the averaging method, and finally solve them for the specific situations of interest.

2. PERIODIC SOLUTIONS OF THE SIT EQUATIONS

The SIT equations take the form[1,7,8]

$$\frac{\partial p}{\partial t}=-\Delta q,\quad \frac{\partial n}{\partial t}=-\mathscr{E}q,\quad \frac{\partial q}{\partial t}=\Delta p+\mathscr{E}n,$$

$$\frac{\partial \mathscr{E}}{\partial t}+\frac{\partial \mathscr{E}}{\partial x}=-q, \tag{1}$$

where $\mathscr{E}$ is the envelope of the light wave, Δ is the difference between the frequency of the applied field and the frequency of the transitions in the medium, p, q, and n are the elements of the medium's dipole–moment operator matrix

$$\begin{pmatrix} n & p+iq \\ p-iq & -n \end{pmatrix} \tag{2}$$

and the light pulse propagates along the x axis. The matrix element n in (2) describes the level population, while $p+iq$ describes the amplitude of the transition between the levels of the medium. The integral of the motion

$$p^2+q^2+n^2=1 \tag{3}$$

reflects conservation of probability, so that its value is fixed. Equations (1) have been written under the frequently used assumption that inhomogeneous line broadening and relaxation processes can be neglected. In our problem this means that the field $\mathscr{E}$ is strong enough to make the duration of one 2π pulse much shorter than all the relaxation times, and to make Eqs. (1) valid for the description of a train of many 2π pulses.

Periodic solutions of the system (1) date back to Ref. 9, but their form there does not lend itself readily to the use of Whitham's method. As indicated in the Introduction, convenient and natural solutions are obtained by the inverse scattering problem method, to which we now turn.

The inverse scattering problem method is based on the possibility of representing the system (1) as the compatibility condition of two systems of linear equations, for arbitrary values of the spectral parameter ζ (see, e.g., Ref. 9):

$$\frac{\partial \psi_1}{\partial t}=\frac{1}{2}(\zeta^2-\Delta^2)^{1/2}\psi_1-\frac{\mathscr{E}}{2}\psi_2,$$

$$\frac{\partial \psi_2}{\partial t}=\frac{\mathscr{E}}{2}\psi_1-\frac{1}{2}(\zeta^2-\Delta^2)^{1/2}\psi_2; \tag{4}$$

$$\frac{\partial \psi_1}{\partial x}=-\frac{1}{2}(\zeta^2-\Delta^2)^{1/2}\left(1+\frac{n}{\zeta^2}\right)\psi_1$$

$$+\frac{1}{2}\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p-\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)\psi_2, \tag{5}$$

 0038-5646/88/102047-06$04.00

$$\frac{\partial\psi_2}{\partial x}=-\frac{1}{2}\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p+\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)\psi_1+\frac{1}{2}(\zeta^2-\Delta^2)^{1/2}\left(1+\frac{n}{\zeta^2}\right)\psi_2.$$

These linear systems have two basic solutions (φ_1, φ_2) and (ψ_1, ψ_2) with different boundary conditions. An important role in the theory of finite-domain integration[10] of the KdV equation is played by the square of the eigenfunction. In our case this role is assumed by a system of "squared basic functions" (see, e.g., Refs. 6 and 11):

$$f=\varphi_1\psi_2+\varphi_2\psi_1,\quad g=\varphi_1\psi_1,\quad h=\varphi_2\psi_2. \tag{6}$$

Their dependence on x and t is determined by systems of linear equations that follow from (4) and (5):

$$\frac{\partial f}{\partial t}=\mathscr{E}g-\mathscr{E}h,\quad \frac{\partial g}{\partial t}=-\frac{\mathscr{E}}{2}f+(\zeta^2-\Delta^2)^{1/2}g,\quad \frac{\partial h}{\partial t}=\frac{\mathscr{E}}{2}f-(\zeta^2-\Delta^2)^{1/2}h; \tag{7}$$

$$\frac{\partial f}{\partial x}=-\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p+\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)g+\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p-\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)h,$$

$$\frac{\partial g}{\partial x}=\frac{1}{2}\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p-\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)f-(\zeta^2-\Delta^2)^{1/2}\left(1+\frac{n}{\zeta^2}\right)g, \tag{8}$$

$$\frac{\partial h}{\partial x}=-\frac{1}{2}\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p+\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)f+(\zeta^2-\Delta^2)^{1/2}\left(1+\frac{n}{\zeta^2}\right)h.$$

We readily find from this that

$$\frac{\partial}{\partial t}(f^2-4gh)=\frac{\partial}{\partial x}(f^2-4gh)=0,$$

so that the value of f^2-4gh depends only on ζ:

$$f^2-4gh=R(\zeta). \tag{9}$$

Periodic (finite-range) solutions are distinguished by the condition that $R(\zeta)$ be a polynomial in ζ. The simplest nontrivial solution is obtained when $R(\zeta)$ is a fourth-degree polynomial:

$$R(\zeta)=\prod_{i=1}^{4}(\zeta-\zeta_i), \tag{10}$$

where ζ_i are the zeros of the polynomial. For our purposes it suffices to know only this solution, and we therefore investigate it in greater detail. It is natural to seek the solution of Eqs. (7) and (8) in the form

$$f=f_0+f_1(\zeta^2-\Delta^2)^{1/2}-(\zeta^2-\Delta^2),\quad g=g_0+g_1(\zeta^2-\Delta^2)^{1/2},\quad h=h_0+h_1(\zeta^2-\Delta^2)^{1/2}. \tag{11}$$

Substitution of (11) in (7) and (8) results in a set of equations for the coefficients; these equations are easily solved and lead to the expressions

$$f=An-\zeta^2,\quad g=-\frac{A}{2}q-\frac{\mathscr{E}}{2}(\zeta^2-\Delta^2)^{1/2},\quad h=\frac{A}{2}q-\frac{\mathscr{E}}{2}(\zeta^2-\Delta^2)^{1/2}, \tag{12}$$

with the requirement that the desired solutions satisfy the relation

$$p=-\Delta\mathscr{E}/A. \tag{13}$$

Here A is an integration constant easily expressed in terms of the zeros of the polynomial $R(\zeta)$. Substituting (12) in (9) and (19) we obtain

$$\zeta^4-(\mathscr{E}^2+2An)\zeta^2+A^2(p^2+q^2+n^2)=\prod_{i=1}^{4}(\zeta-\zeta_i),$$

whence follows, with allowance for (3),

$$A^2=\prod_{i=1}^{4}\zeta_i,\quad A=\left(\prod_{i=1}^{4}\zeta_i\right)^{1/2} \tag{14}$$

(the choice of the sign of the square root will be discussed below). In addition, we have obtained the integral of the motion

$$\mathscr{E}^2+2An=-\sum_{i<k}\zeta_i\zeta_k,$$

as well as the relations between the zeros

$$\sum\zeta_i=0,\quad \sum_{i<j<k}\zeta_i\zeta_j\zeta_k=0,$$

from which it follows that $\zeta_3=-\zeta_1$, $\zeta_4=-\zeta_2$ (apart from a change of notation), so that

$$A=\zeta_1\zeta_2,\quad \mathscr{E}^2+2\zeta_1\zeta_2 n=\zeta_1^2+\zeta_2^2,\quad \zeta_1,\ \zeta_2>0. \tag{15}$$

Knowing the integrals (3) and (15) it is easy to find the solution of the system (1) by the usual method. Bearing in mind, however, the averaging that follows, it is more convenient to use procedures developed in the inverse scattering method. The dynamics of finite-domain solutions is described by the motion of the zeros and poles of the functions (6) (see Refs. 10 and 11). Let γ be a zero of the function g; then

$$q=-\frac{\mathscr{E}}{\zeta_1\zeta_2}(\gamma^2-\Delta^2)^{1/2} \tag{16}$$

and

$$g=\frac{\mathscr{E}}{2}((\gamma^2-\Delta^2)^{1/2}-(\zeta^2-\Delta^2)^{1/2}). \tag{17}$$

Putting $\zeta=\gamma$ in (9) we get

$$f=-\gamma^2+\zeta_1\zeta_2 n=\pm R^{1/2}(\gamma),\quad R(\gamma)=(\gamma^2-\zeta_1^2)(\gamma^2-\zeta_2^2). \tag{18}$$

It is clear now that all the quantities can be expressed in terms of only one function $\gamma(x,t)$. It follows from (18) that

$$n=\frac{1}{\zeta_1\zeta_2}(\gamma^2\pm R^{1/2}(\gamma)), \tag{19}$$

from (15) and (19) we get

$$\mathscr{E}=[\zeta_1^2+\zeta_2^2-2(\gamma^2\pm R^{1/2}(\gamma))]^{1/2}, \quad (20)$$

and, finally, (13) and (16) yield

$$p=-\frac{\Delta}{\zeta_1\zeta_2}[\zeta_1^2+\zeta_2^2-2(\gamma^2\pm R^{1/2}(\gamma))]^{1/2}, \quad (21)$$

$$q=-\frac{(\gamma^2-\Delta^2)^{1/2}}{\zeta_1\zeta_2}[\zeta_1^2+\zeta_2^2-2(\gamma^2\pm R^{1/2}(\gamma))]^{1/2}. \quad (22)$$

To find the equations that determine the function $\gamma(x,t)$ we substitute expressions (17) and (18) in the second equations of the systems (7) and (18), and then put $\zeta=\gamma$. Equation (17) yields

$$\left.\frac{\partial g}{\partial t}\right|_{\zeta=\gamma}=\frac{\mathscr{E}}{2}\frac{\gamma}{(\gamma^2-\Delta^2)^{1/2}}\frac{\partial\gamma}{\partial t}=-\frac{\mathscr{E}}{2}f(\gamma)=\mp\frac{\mathscr{E}}{2}R^{1/2}(\gamma),$$

so that

$$\frac{\partial\gamma}{\partial t}=\mp\frac{(\gamma^2-\Delta^2)^{1/2}}{\gamma}R^{1/2}(\gamma). \quad (23)$$

We obtain similarly from (8) (taking (13) and (16) into account)

$$\frac{\partial\gamma}{\partial x}=\pm\left(1+\frac{1}{\zeta_1\zeta_2}\right)\frac{(\gamma^2-\Delta^2)^{1/2}}{\gamma}R^{1/2}(\gamma)=-\left(1+\frac{1}{\zeta_1\zeta_2}\right)\frac{\partial\gamma}{\partial t}. \quad (24)$$

From the viewpoint of general theory, it is interesting to note that the variable in these equations is in fact γ^2 (and not γ), and that γ^2 moves over a Riemann surface defined by the equation $y^2=(\gamma^2-\Delta^2)R(\gamma)$, i.e., an additional factor $\gamma^2-\Delta^2$ has appeared here, in contrast to the KdV theory.

It follows from (24) that γ depends only on the variable

$$\xi=x-vt, \quad v=(1+1/\zeta_1\zeta_2)^{-1}. \quad (25)$$

Since we are considering equations for the envelope of the electromagnetic field $\mathscr{E}$, the wave velocity should be less than that of light: $v<1$. To meet this requirement, the constant $A=\zeta_1\zeta_2$ in (14) was chosen positive.

To find $\gamma(\xi)$ we must integrate the equation

$$\frac{d\gamma}{d\xi}=\pm\frac{1}{v}\frac{(\gamma^2-\Delta^2)^{1/2}}{\gamma}R^{1/2}(\gamma), \quad (26)$$

where $R(\gamma)$ is the polynomial (18). We assume that $\Delta<\zeta_1<\zeta_2$, so that the function γ varies in the interval $\Delta\leqslant\gamma\leqslant\zeta_1$. It is easy to express the solution of Eq. (26) in terms of elliptic functions:

$$\gamma^2=\Delta^2+(\zeta_1^2-\Delta^2)\,\mathrm{sn}^2\left(\frac{(\zeta_1^2-\Delta^2)^{1/2}}{v}\xi,s\right), \quad (27)$$

where

$$s=\left(\frac{\zeta_1^2-\Delta^2}{\zeta_2^2-\Delta^2}\right)^{1/2} \quad (28)$$

is the parameter (modulus) of the elliptic function. Substituting (27) in (19)–(22) we obtain ultimately expressions for the periodic solution:

$$n=\frac{1}{\zeta_1\zeta_2}\left[\Delta^2+(\xi_1^2-\Delta^2)\,\mathrm{sn}^2\left(\frac{(\xi_2^2-\Delta_2)^{1/2}}{v}\xi,s\right)\pm((\zeta_1^2-\Delta^2)\right.$$
$$\left.\times(\zeta_2^2-\Delta^2))^{1/2}\,\mathrm{cn}\left(\frac{(\zeta_2^2-\Delta^2)^{1/2}}{v}\xi,s\right)\mathrm{dn}\left(\frac{(\zeta_2^2-\Delta^2)^{1/2}}{v}\xi,s\right)\right], \quad (29)$$

$$\mathscr{E}=(\zeta_2^2-\Delta^2)^{1/2}\,\mathrm{dn}\left(\frac{(\zeta_2^2-\Delta^2)^{1/2}}{v}\xi,s\right)$$
$$\mp(\zeta_1^2-\Delta^2)^{1/2}\,\mathrm{cn}\left(\frac{(\zeta_2^2-\Delta^2)^{1/2}}{v}\xi,s\right) \quad (30)$$

(we shall not write out the obvious equations for p and q, which we do not need).

3. WHITHAM'S EQUATIONS

All the quantitites in the propagating nonlinear single-period wave investigated in the preceding section depend on the phase

$$\theta=\frac{2\pi}{\lambda}(x-vt), \quad (31)$$

so normalized that a shift equal to the wavelength or to the period changes the phase by 2π. If the wave properties vary slowly in time and in space, the local values of the wave number and of the frequency are given by

$$k=\partial\theta/\partial x, \quad \omega=-\partial\theta/\partial t. \quad (32)$$

Hence follows the kinematic condition

$$\partial k/\partial t+\partial\omega/\partial x=0, \quad (33)$$

which takes in our case the form

$$\frac{\partial}{\partial t}\left(\frac{1}{\lambda}\right)+\frac{\partial}{\partial x}\left(\frac{v}{\lambda}\right)=0. \quad (34)$$

In the calculation of k and ω from Eqs. (32) we have taken into account here only the most rapidly varying terms of (31). The phase velocity v is given by Eq. (25), and the wavelength can be easily calculated:

$$\lambda=2v\int_{\Delta}^{\zeta_1}\frac{\gamma\,d\gamma}{((\gamma^2-\Delta^2)R(\gamma))^{1/2}}=\frac{2vK(s)}{(\zeta_2^2-\Delta^2)^{1/2}}, \quad (35)$$

where $K(s)$ is a complete elliptic integral of the first kind. The parameters ζ_1 and ζ_2 must thus satisfy the equation

$$\frac{\partial}{\partial t}\left[\left(1+\frac{1}{\zeta_1\zeta_2}\right)\frac{(\zeta_2^2-\Delta^2)^{1/2}}{K(s)}\right]+\frac{\partial}{\partial x}\left[\frac{(\zeta_2^2-\Delta^2)^{1/2}}{K(s)}\right]=0. \quad (36)$$

The other equation, following Whitham,[4] can be obtained by averaging the conservation law

$$\frac{\partial}{\partial t}(\mathscr{E}^2-2n)+\frac{\partial}{\partial x}\mathscr{E}^2=0 \quad (37)$$

in accordance with the rule

$$\langle F(\theta)\rangle=\frac{1}{2\pi}\int_0^{2\pi}F(\theta)\,d\theta. \quad (38)$$

Substitution here of the solutions (29) and (30) and calculation of the integrals are not particularly difficult and yield the averaged conservation law

$$\frac{\partial}{\partial t}\left[(\zeta_2^2-\Delta^2)\left(1+\frac{1}{\zeta_1\zeta_2}\right)\left(\frac{E(s)}{K(s)}-1\right)+\frac{\zeta_1^2+\zeta_2^2}{2}-\frac{\Delta^2}{\zeta_1\zeta_2}\right]$$
$$+\frac{\partial}{\partial x}\left[(\zeta_2^2-\Delta^2)\frac{E(s)}{K(s)}-\frac{\zeta_2^2-\zeta_1^2}{2}\right]=0, \quad (39)$$

where $E(s)$ is a complete elliptic integral of the second kind.

The system (36) and (39) for the variables ζ_1 and ζ_2 is quite complicated and can be greatly simplified by using in place of (39) the averaged generating function of the conservation laws, an infinite set of which is possessed by the integrable system (1). It is easy to find two forms of the generating function:

$$\frac{\partial}{\partial t}\left[\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p-\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)\frac{1}{g}\right]+\frac{\partial}{\partial x}\left(\frac{\mathscr{E}}{g}\right)=0,$$
$$\frac{\partial}{\partial t}\left[\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p+\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)\frac{1}{h}\right]+\frac{\partial}{\partial x}\left(\frac{\mathscr{E}}{h}\right)=0. \qquad (40)$$

To simplify the calculations we sum these expressions and use the normalization $f^2-4gh=1$ rather than Eq. (9) (see Ref. 6). We then obtain a generating function in the form

$$\frac{\partial}{\partial t}\left\{R^{1/2}(\zeta)\left[\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p-\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)\frac{1}{g}+\left(\mathscr{E}-\frac{\Delta}{\zeta^2}p+\frac{(\zeta^2-\Delta^2)^{1/2}}{\zeta^2}q\right)\frac{1}{h}\right]\right\}+\frac{\partial}{\partial x}\left\{R^{1/2}(\zeta)\mathscr{E}\left(\frac{1}{g}+\frac{1}{h}\right)\right\}=0. \qquad (41)$$

The functions $\mathscr{E}, p, q, g$, and h depend on the parameters ζ_i, while expansion in powers of $1/\zeta$ with $\zeta\to\infty$ generates an infinite aggregate of conservation laws. Averaging over the phase variables, in accordance with (37) and (38), transforms Eq. (41) into the generating function of the Whitham equations for the slow variables ζ_i. In our case of a single-phase solution, Eq. (41) can be easily expressed in terms of the function γ:

$$\frac{\partial}{\partial t}\left\{R^{1/2}(\zeta)\left[\left(1+\frac{1}{\zeta_1\zeta_2}\right)\frac{1}{\zeta^2-\gamma^2}-\frac{1}{\zeta_1\zeta_2\zeta^2}\right]\right\}+\frac{\partial}{\partial x}\left\{\frac{R^{1/2}(\zeta)}{\zeta^2-\gamma^2}\right\}=0. \qquad (42)$$

Averaging over θ reduces to integration with respect to γ:

$$\left\langle\frac{1}{\zeta^2-\gamma^2}\right\rangle=\frac{1}{\lambda}\int_{\Delta^2}^{\zeta_1^2}\frac{1}{\zeta^2-\gamma^2}\frac{d\gamma^2}{((\gamma^2-\Delta^2)R(\gamma))^{1/2}}$$
$$=\frac{1}{(\zeta^2-\Delta^2)K(s)}\Pi(\nu,s), \qquad (43)$$

where

$$\nu=-(\zeta_1^2-\Delta^2)/(\zeta^2-\Delta^2) \qquad (44)$$

and

$$\Pi(\nu,s)=\int_0^{\pi/2}\frac{d\varphi}{(1+\nu\sin^2\varphi)(1-s^2\sin^2\varphi)^{1/2}} \qquad (45)$$

is a complete elliptic integral of the third kind. After averaging, Eq. (42) finally takes the form

$$\frac{\partial}{\partial t}\left\{R^{1/2}(\zeta)\left[\left(1+\frac{1}{\zeta_1\zeta_2}\right)\frac{\Pi(\nu,s)}{K(s)}-\frac{1}{\zeta_1\zeta_2}\frac{\zeta^2-\Delta^2}{\zeta^2}\right]\right\}+\frac{\partial}{\partial x}\left\{R^{1/2}(\zeta)\frac{\Pi(\nu,s)}{K(s)}\right\}=0. \qquad (46)$$

Expansion in powers of $1/\zeta$ near the singular point $\zeta=0$ leads to a sequence of averaged conservation laws, the first of which coincides with (39). Real new information is obtained by investigating the other singular points $\zeta=\zeta_1$ and ζ_2. At the point $\zeta=\zeta_2$ the function $\Pi(\nu,s)$ is regular, so that the singularity contains only derivatives of

$$R^{1/2}(\zeta)=[(\zeta^2-\zeta_1^2)(\zeta^2-\zeta_2^2)]^{1/2},$$

and equating the coefficient of $(\zeta-\zeta_2)^{-1/2}$ to zero leads directly to an equation for ζ_2 in Riemann form:

$$\left[\left(1+\frac{1}{\zeta_1\zeta_2}\right)\frac{E(s)}{(1-s^2)K(s)}-\frac{1}{\zeta_1\zeta_2}\frac{\zeta_2^2-\Delta^2}{\zeta_2^2}\right]\frac{\partial\zeta_2}{\partial t}+\frac{E(s)}{(1-s^2)K(s)}\frac{\partial\zeta_2}{\partial x}=0 \qquad (47)$$

(where the equation $\Pi(-s^2,s)=E(s)/(1-s^2)$ is used). The singular point $\zeta=\zeta_1$ is more difficult to investigate, but there is no need for this, since (36) and (47) readily lead to

$$\left[\left(1+\frac{1}{\zeta_1\zeta_2}\right)\frac{E(s)-(1-s^2)K(s)}{(1-s^2)K(s)}-\frac{1}{\zeta_1\zeta_2}\frac{\zeta_1^2-\Delta^2}{\zeta_2^2}\right]\frac{\partial\zeta_1}{\partial t}+\frac{E(s)-(1-s^2)K(s)}{(1-s^2)K(s)}\frac{\partial\zeta_1}{\partial x}=0. \qquad (48)$$

Equations (47) and (48) constitute the desired set of Whitham's equations. We write them in the form

$$\frac{\partial\zeta_1}{\partial t}+v_1\frac{\partial\zeta_1}{\partial x}=0,\qquad \frac{\partial\zeta_2}{\partial t}+v_2\frac{\partial\zeta_2}{\partial x}=0, \qquad (49)$$

where the group velocities are

$$v_1=\frac{E-(1-s^2)K}{(1+1/\zeta_1\zeta_2)E-(1-\Delta^2/\zeta_1^3\zeta_2)(1-s^2)K}$$
$$v_2=\frac{E}{(1+1/\zeta_1\zeta_2)E-(1-\Delta^2/\zeta_2^2)(1-s^2)K/(\zeta_1\zeta_2)^{-1}}. \qquad (50)$$

It is easy to verify with the aid of these equations the conservation law (39), which is thus found to be a consequence of the equations of motion (49) and (50). We consider now a specific application of the derived equations.[1)]

4. SOLITON CREATION ON A PULSE FRONT

We consider the evolution of a light pulse which is step-like at the initial instant:

$$\mathscr{E}=\mathscr{E}_0\quad\text{for}\quad x<0,\qquad \mathscr{E}=0\quad\text{for}\quad x>0 \qquad (51)$$

(a similar problem for the KdV equation was solved in Refs. 3 and 10). After a certain time there is produced between the zero-field region and the region with constant amplitude $\mathscr{E}_0$ an intermediate region (the analog of a simple wave) described by the solution (30) with variables ζ_1 and ζ_2. The field amplitude vanishes on the leading edge of this region, i.e.,

$$\zeta_1=\zeta_2,\quad s=1. \qquad (52)$$

On the trailing edge the field oscillations should be damped, i.e.,

$$s=0,\quad \zeta_1=\Delta,\quad \zeta_2=(\mathscr{E}_0^2+\Delta^2)^{1/2}. \qquad (53)$$

After a sufficient time, when the number of field oscillations in the intermediate region is large, the variation of the parameters ζ_1 and ζ_2 in this region can be described by Whitham's equations (49) and (50). On the edges of the region, the parameters ζ_1 and ζ_2 must satisfy the boundry conditions (52) and (53).

The problem formulated contains no characteristic dimensions whatever, so that all the quantities depend only on the self-similar variable $\tau = x/t$:

$$\zeta_1=\zeta_1(\tau), \quad \zeta_2=\zeta_2(\tau).$$

Equations (49) take therefore the form

$$\frac{d\zeta_1}{d\tau}(v_1-\tau)=0, \quad \frac{d\zeta_2}{d\tau}(v_2-\tau)=0. \tag{54}$$

The solution satisfying the boundary conditions (52) and (53) is

$$v_1=\tau, \quad \zeta_2=\text{const}=(\mathscr{E}_0^2+\Delta^2)^{1/2}. \tag{55}$$

The expressions

$$\frac{x}{t}=\frac{E(s)-(1-s^2)K(s)}{(1+1/\zeta_1\zeta_2)E(s)-(1+\Delta^2/\zeta_1^3\zeta_2)(1-s^2)K(s)},$$

$$s=\frac{(\zeta_1^2-\Delta^2)^{1/2}}{\mathscr{E}_0}, \tag{56}$$

where ζ_2 is given by (55), determine the desired dependence on $\tau = x/t$.

On the leading edge, where $s\to 0$, we obtain by using the known asymptotic forms of elliptic integrals

$$\frac{x}{t}\approx\left(1+\frac{1}{\mathscr{E}_0^2+\Delta^2}\right)^{-1}-\frac{1-\Delta^2/(\mathscr{E}_0^2+\Delta^2)^2}{(1+1/(\mathscr{E}_0^2+\Delta^2))^2}\frac{1-s^2}{2}\ln\frac{16}{1-s^2}. \tag{57}$$

Thus it is clear that the propagation velocity of the pulse front in the resonance region is equal to

$$c=\left(1+\frac{1}{\mathscr{E}_0^2+\Delta^2}\right)^{-1}. \tag{58}$$

The soliton on the front is of the form

$$\mathscr{E}=\frac{2\mathscr{E}_0}{\operatorname{ch}[\mathscr{E}_0(t-x/c)]}, \tag{59}$$

i.e., its amplitude is double the amplitude of the incident wave. This result is similar to that obtained in the KdV theory.[4] The distance between the solitons is determined by the wavelength. Let us determine the wavelength variation with increasing distance from the leading edge, where the self-similar variable is obviously $\tau^+ = c$. Introducing the variable $\tau' = c - \tau$ we obtain from (57)

$$c^2\left[1-\frac{\Delta^2}{(\mathscr{E}^2+\Delta^2)^2}\right]\frac{1-s^2}{2}\ln\frac{16}{1-s^2}=\tau',$$

whence, with logarithmic accuracy,

$$1-s^2=\frac{1}{1-\Delta^2/(\mathscr{E}_0^2+\Delta^2)^2}\frac{2\tau'}{c^2\ln(8c^2/\tau')},$$

so that the wavelength is equal to

$$\lambda=cT=\frac{4\pi cK(s)}{(\zeta_2^2-\Delta^2)^{1/2}}=\frac{2\pi}{\mathscr{E}_0}\left(1+\frac{1}{\mathscr{E}_0^2+\Delta^2}\right)^{-1}\ln\frac{8c^2}{\tau'}. \tag{60}$$

The wavelength increases logarithmically as $\tau'\to 0$.

On the trailing edge of the intermediate section, where $s = 0$ and $\zeta_1 = \Delta$, we obtain from (56)

$$\frac{x}{t}=\left(1+\frac{1}{\Delta(\mathscr{E}_0^2+\Delta^2)^{1/2}}\right)^{-1}=c'. \tag{61}$$

For a frequency difference $\Delta = 0$ the trailing edge remains in place.

A comparison of these equations with the experiment of Ref. 2 is difficult because in the experiment the leading front of the initial pulse is not steep enough. The slow increase (60) of the distance between the produced solitons is apparently confirmed, so that qualitative agreement with experiment can be claimed.

I am grateful to V. G. Nosov and A. L. Chernyakov for a discussion of the results.

APPENDIX

For $\Delta = 0$, the SIT equation

$$\frac{\partial\mathscr{E}}{\partial t}+\frac{\partial\mathscr{E}}{\partial x}=-q, \quad \frac{\partial q}{\partial t}=\mathscr{E}n, \quad \frac{\partial n}{\partial t}=-\mathscr{E}q \tag{A1}$$

can be reduced to a sine-Gordon equation by introducing a function u such that

$$\mathscr{E}=\frac{\partial u}{\partial t}, \quad n=\cos u, \quad q=\sin u. \tag{A2}$$

The last two equations of (A1) are then automatically satisfied, together with the probability conservation condition $n^2 + a^2 = 1$, while the first equation of (A1) takes the form

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^2 u}{\partial t\,\partial x}+\sin u=0.$$

The change of variables

$$t=\tau, \quad x=\tfrac{1}{2}(\xi+\tau) \tag{A3}$$

reduces this equation to the canonical form

$$\frac{\partial^2 u}{\partial\tau^2}-\frac{\partial^2 u}{\partial\xi^2}+\sin u=0, \tag{A4}$$

studied in numerous papers by the inverse scattering problem method. A finite-domain integration of this equation was carried out in Refs. 6 and 11 for $N = 1$ and 2 zones, and the corresponding Whitham equations were derived. We shall show that our Eqs. (49) and (50) go over at $\Delta = 0$ into the corresponding equations of Ref. 6.

We must first find the connection between the spectral parameter ζ and the spectral parameter E of Refs. 6 and 11. It is shown in these references that at $N = 1$ the periodic solution depends on the phase

$$\left(1+\frac{1}{16(E_1E_2)^{1/2}}\right)\xi+\left(1-\frac{1}{16(E_1E_2)^{1/2}}\right)\tau$$
$$=2\left[\left(1+\frac{1}{16(E_1E_2)^{1/2}}\right)x-\frac{1}{16(E_1E_2)^{1/2}}t\right],$$

so that in the variables (x, t) the phase velocity is given in terms of the spectral parameter $E = E_1, E_2$ by

$$v=\frac{1}{1+16(E_1E_2)^{1/2}}.$$

Comparison with Eq. (25) with allowance for the locations of E_1, E_2 and ζ_1, ζ_2 on the corresponding complex planes leads to

$$\zeta_1 = {}^1/_4(-E_1)^{1/2}, \quad \zeta_2 = {}^1/_4(-E_2)^{1/2}. \quad \text{(A5)}$$

From this and from (49) and (50) follow Whitham's equations for E_1 and E_2 in the variables (x, t):

$$\frac{\partial E_1}{\partial t} + v_1 \frac{\partial E_1}{\partial x} = 0, \quad \frac{\partial E_2}{\partial t} + v_2 \frac{\partial E_2}{\partial x} = 0,$$

$$v_1 = -\frac{E(s) - (1-s^2)K(s)}{(1+16(E_1E_2)^{1/2})E(s) - (1-s^2)K(s)}, \quad \text{(A6)}$$

$$v_2 = \frac{E(s)}{(1+16(E_1E_2)^{1/2})E(s) - 16(E_1E_2)^{1/2}(1-s^2)K(s)},$$

where

$$s = \zeta_1/\zeta_2 = (E_2/E_1)^{1/2}. \quad \text{(A7)}$$

In the variables (ξ, τ) Eqs. (A6) yield

$$\frac{\partial E_1}{\partial \tau} - S^{(1)} \frac{\partial E_1}{\partial \xi} = 0, \quad \frac{\partial E_2}{\partial \tau} - S^{(2)} \frac{\partial E_2}{\partial \xi} = 0, \quad \text{(A8)}$$

$$S^{(1)} = 1 - 2v_1 = \frac{(16(E_1E_2)^{1/2} - 1)E(s) + (1-s^2)K(s)}{(16(E_1E_2)^{1/2}+1)E(s) - (1-s^2)K(s)}, \quad \text{(A9)}$$

$$S^{(2)} = 1 - 2v_2 = \frac{(16(E_1E_2)^{1/2} - 1)E(s) - 16(E_1E_2)^{1/2}(1-s^2)K(s)}{(16(E_1E_2)^{1/2}+1)E(s) - 16(E_1E_2)^{1/2}(1-s^2)K(s)}. \quad \text{(A10)}$$

The very same group velocities were obtained in Ref. 6 in the form

$$S^{(l)} = \frac{E_l - (E_1E_2)^{1/2}E_l^{-1}/16 - C^{(+)} + (E_1E_2)^{1/2}C^{(-)}/16}{E_l + (E_1E_2)^{1/2}E_l^{-1}/16 - C^{(+)} - (E_1E_2)^{1/2}C^{(-)}/16}, \quad l = 1, 2. \quad \text{(A11)}$$

The constants $C^{(\pm)}$ are determined from the conditions that the differentials

$$\Omega_{+1} = -\frac{1}{2}(E - C^{(+)})\frac{dE}{R(E)},$$

$$\Omega_{-1} = -\frac{(E_1E_2)^{1/2}}{2}\left(\frac{1}{E} - C^{(-)}\right)\frac{dE}{R(E)}, \quad \text{(A12)}$$

where $R(E) = [E(E-E_1)(E-E_2)]^{1/2}$ have zero periods on the b-cycle (the differentials (A12) are defined on a two-sheet Riemann surface obtained by joining together two complex planes along the cuts (E_1, E_2) and $(0, \infty)$, so that the b-cycle starting from an arbitrary point on one of the planes passes through one cut to the second plane and returns through the second cut to the starting point on the first plane). Calculations of the integrals over the b-cycle yields

$$C^{(+)} = \left(1 - \frac{E(s)}{K(s)}\right)E_1, \quad C^{(-)} = \left(1 - \frac{E(s)}{K(s)}\right)E_2^{-1}, \quad \text{(A13)}$$

and substitution of these quantities in (A11) leads to expressions that coincide with (A9) and (A10).

It is interesting to note that in Ref. 6 Whitham's equations were obtained by averaging the conservation-law generating function

$$\frac{\partial}{\partial \tau}\left[\left(1 + \frac{e^{iu}}{16E}\right)g - \left(1 + \frac{e^{-iu}}{16E}\right)h\right] - \frac{\partial}{\partial \xi}\left[\left(1 - \frac{e^{iu}}{16E}\right)g - \left(1 - \frac{e^{-iu}}{16E}\right)h\right] = 0, \quad \text{(A14)}$$

in which the "squares of the basic functions" g and h are multipliers rather than divisors as in (40) and (41). For the sine-Gordon equation, however, it is easy to obtain conservation-law generating functions similar to (40):

$$\frac{\partial}{\partial \tau}\left[\left(1 - \frac{e^{-iu}}{16E}\right)\frac{1}{g}\right] - \frac{\partial}{\partial \xi}\left[\left(1 + \frac{e^{-iu}}{16E}\right)\frac{1}{g}\right] = 0, \quad \text{(A15)}$$

$$\frac{\partial}{\partial \tau}\left[\left(1 - \frac{e^{iu}}{16E}\right)\frac{1}{h}\right] - \frac{\partial}{\partial \xi}\left[\left(1 + \frac{e^{iu}}{16E}\right)\frac{1}{h}\right] = 0. \quad \text{(A16)}$$

It is easy to verify that averaging any of expressions (A14)–(A16) results in the same Whitham equations.

[1)] The SIT equations (1) can be reduced for $\Delta = 0$ to the sine-Gordon equation for which the Whitham equations were obtained in Ref. 6. The connection between (49), (50), and the equations of Ref. 6 is made clear in the Appendix.

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Translated by J. G. Adashko