Attractors and stochastic attractors of motion in a magnetic field

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(Submitted 12 May 1988)
Zh. Eksp. Teor. Fiz. **94**, 170–187 (October 1988)

We investigate the influence of dissipation on the regular and chaotic dynamics of charged particles in the field of a wave packet that propagates across an external magnetic field. We obtain the conditions for randomization of the motion and find the upper bound of the energy for stochastic acceleration of the particles. The importance of particle trapping for their acceleration and heating is determined in the limit of a single-mode wave packet. For the limiting case of an infinitely broad packet, the equations of motion are reduced to a Poincaré mapping. The stochastic attractors resulting from the randomization of the particle motion are investigated analytically and numerically.

1. INTRODUCTION

Resonant interaction between particles and waves in a plasma located in an external magnetic field has numerous applications, mainly for plasma heating.[1] The latter is of particular interest when the wave propagates perpendicularly to the magnetic field. Such waves can be damped not only by collisionless cyclotron resonance and by dissipation due to particle collisions, but also by more complicated nonlinear processes involving, in particular, randomization of the charged-particle trajectories. This internal chaos of the particle motion can occur even when the plasma wave is planar and monochromatic,[2–6] has an electric-field strength

$$E_x = E_0 \sin(k_0 x - \omega t), \quad E_y = E_z = 0, \tag{1.1}$$

and propagates across an external magnetic field $\mathbf{B}_0$ parallel to the z axis. The problem reduces formally to an investigation of stochastic dynamics of particles whose equations of motion have the simple form

$$\ddot{x} + \omega_H^2 x = \varepsilon \sin(k_0 x - \omega t) \tag{1.2}$$

and describe linear-oscillator motion induced by a plane harmonic wave. By ω_H is meant here the cyclotron frequency ($\omega_H = eB_0/mc$), while the parameter $\varepsilon = eE_0/m$ is proportional to the amplitude E_0 of the potential-wave field strength. The seeming simplicity of Eq. (1.2) is illusory, and it has already been the subject of many analytic and numerical investigations. References 2 and 3 deal with the case of strong magnetic fields, when the particle passes through the resonance region twice in each cyclotron period and collides briefly with the wave. It has been shown[3] that increasing the wave amplitude randomizes the motion if

$$k_0 \varepsilon \sim (\omega \omega_H^2)^{2/3}. \tag{1.3}$$

The opposite case of a weak magnetic field is considered in Refs. 4 and 5. The dynamics here is stochastic. An important role is played by the trapped particles: it is they which govern the absorption of the plasma waves.

Stochastic acceleration of particles by a wave of frequency equal to one of the cyclotron harmonics

$$\omega = N \omega_H, \tag{1.4}$$

where N is an integer, was considered in Refs. 2 and 6. An interesting feature of the nonlinear dynamics of the particles in this situation is the formation of a stochastic web on the $(x, \dot{x})$ phase plane, with unbounded stochastic acceleration in the web channels for arbitrarily low amplitude of the wave's electric field strength.[6] The unbounded stochastic web in (1.2) is due to the degeneracy of this equation when there is no perturbation ($\varepsilon = 0$), i.e., to the independence of the cyclotron frequency of the oscillation amplitude or of the Larmor radius. In nondegenerate systems, a similar stochastic acceleration (Arnol'd diffusion) is produced when the number of degrees of freedom is larger.

We investigate here the influence of dissipation on stochastic acceleration of particles. Dissipation is always present in a real situation and has various physical causes, such as particle collisions, emission processes, and others. The influence of dissipation is investigated in Sec. 2 using as an example the model equation

$$\ddot{x} + \gamma \dot{x} + \omega_H^2 x = \varepsilon \sin(k_0 x - \omega t), \tag{1.5}$$

which has, in contrast to Eq. (1.2), the dissipative term $\gamma \dot{x}$ in the left-hand side. An important property of (1.5) is that it remains degenerate for $\varepsilon = 0$ in the absence of a perturbation. We shall show that dissipation always restricts the stochastic acceleration of particles. A certain limiting invariant set is then produced in phase space and attracts all the trajectories. If the nonlinearity is weak, such a set contains as a rule a limited number of attracting points, and an averaging method is used to describe typical bifurcations of the particle phase trajectories. When the electrostatic wave amplitude is increased, the motion becomes random and tends to the strange attractor. A physical interpretation is given for the particle-motion picture revealed by the numerical calculations. These topics are the subject of Sec. 3 of the paper.

In the next two sections (Secs. 4 and 5) we consider the dissipative dynamics of charged particles in a magnetic field and in a wave-packet field, when the electrostatic field in the right-hand side of (1.5) comprises a set of plane waves with different amplitudes ε_n, frequencies ω, and wave numbers k_n:

$$\ddot{x} + \gamma \dot{x} + \omega_H^2 x = \sum_n \varepsilon_n \sin(k_n x - \omega_n t). \tag{1.6}$$

We shall assume that the wave packet in (1.6) is fairly broad

 0038-5646/88/102053-10$04.00

and homogeneous. It is possible then to change over from the differential equation (1.6) to the corresponding Poincaré mapping. In the limit of zero magnetic field ($\omega_H = 0$) this mapping corresponds to the nondegenerate case and reduces to the Zaslavskiĭ mapping[7] studied in detail in a number of papers. The corresponding mapping for $\gamma = 0$ was investigated in Refs. 8 and 9, which revealed the many remarkable properties brought about by the structure of the phase space in the resonance case, for rational ratios of the electrostatic-field and cyclotron frequencies.

A kinetic equation of the Fokker-Planck-Kolomogorov type is derived in Sec. 5 on the basis of a dissipative mapping for particles in a magnetic field. This equation has a stationary solution, owing to the presence of friction. The energy limit for stochastic heating of particles is found and the size of the strange attractor is estimated

2. INFLUENCE OF DISSIPATION ON THE AVERAGED MOTION IN THE RESONANCE CASE

We transform Eq. (1.5) for the dynamics of a charged particle in the field of a harmonic wave propagating across an electric field into the set of equations

$$\dot{v}=-\gamma v-\omega_H^2 x+\varepsilon \sin(k_0 x-\omega t), \qquad \dot{x}=v. \tag{2.1}$$

We restrict our investigation to resonant interaction of a particle with the wave, i.e., we consider a situation in which the electrostatic-wave frequency is close to a multiple of the natural frequency of the unperturbed problem:

$$\omega=N\Omega+\delta, \quad \delta<\Omega, \tag{2.2}$$

where N is an integer and

$$\Omega=(\omega_H^2-\gamma^2/4)^{1/2} \tag{2.3}$$

is the friction-shifted natural frequency for $\varepsilon = 0$. The damping γ is assumed further to be relatively weak:

$$\gamma<2\omega_H. \tag{2.4}$$

We change to new independent variables I and φ in a frame rotating with angular velocity ω/N:

$$x=(2NI/\Omega)^{1/2}\sin\left(\frac{\varphi+\omega t}{N}\right),$$

$$v=(2NI\Omega)^{1/2}\cos\left(\frac{\varphi+\omega t}{N}\right). \tag{2.5}$$

This change of variables transforms (2.1) into

$$\dot{I}+2\gamma I\cos^2\left(\frac{\varphi+\omega t}{N}\right)=\varepsilon\left(\frac{2I}{N\Omega}\right)^{1/2}\cos\left(\frac{\varphi+\omega t}{N}\right)\sin(k_0x-\omega t),$$

$$\dot{\varphi}+\delta-\frac{1}{2}N\gamma\sin\left(\frac{2\varphi+2\omega t}{N}\right)=-\varepsilon\left(\frac{N}{2I\Omega}\right)^{1/2}\sin\left(\frac{\varphi+\omega t}{N}\right)\sin(k_0x-\omega t). \tag{2.6}$$

We expand the right-hand sides of the equations in (2.6) in a series

$$\dot{I}+2\gamma I\cos^2\left(\frac{\varphi+\omega t}{N}\right)=\frac{\varepsilon}{k_0}\sum_{n=-\infty}^{+\infty}\frac{n}{N}J_n(k_0\rho)\sin\left[\left(\frac{n}{N}-1\right)\omega t+\frac{n}{N}\varphi\right]$$

$$\dot{\varphi}+\delta-\frac{1}{2}N\gamma\sin\left(\frac{2\varphi+2\omega t}{N}\right)=\frac{\varepsilon N}{\Omega\rho}\sum_{n=-\infty}^{+\infty}J_n'(k_0\rho)\cos\left[\left(\frac{n}{N}-1\right)\omega t+\frac{n}{N}\varphi\right], \tag{2.7}$$

where $J_n(k_0)$ is a Bessel function. The prime denotes differentiation of the Bessel function with respect to its argument, and

$$\rho=(2NI/\Omega)^{1/2} \tag{2.8}$$

is the particle Larmor radius. The terms in the right-hand sides of (2.7) are of two types: resonant with $n = N$ and nonresonant with $n \neq N$, which oscillate rapidly if

$$k_0\varepsilon/\omega_H^2\ll 1. \tag{2.9}$$

Under the condition (2.9) the influence of the nonresonant terms is weak and the dynamics of the system (2.7) can be investigated in the framework of an equation set averaged over the cyclotron period

$$\dot{I}+\gamma I=\frac{\varepsilon}{k_0}J_N(k_0\rho)\sin\varphi,$$

$$\dot{\varphi}+\delta=(\varepsilon N/\Omega\rho)J_N'(k_0\rho)\cos\varphi. \tag{2.10}$$

We consider next the phase portrait of Eqs. (2.10) for exact resonance, $\delta = 0$.

The phase plane of the averaged system (2.10) has stationary points defined by the condition $\dot{I} = 0$ and $\dot{\varphi} = 0$. These points can be found from the set of equations

$$\gamma I=(\varepsilon/k_0)J_N(k_0\rho)\sin\varphi,$$

$$J_N'(k_0\rho)\cos\varphi=0. \tag{2.11}$$

This set determines the stable (foci and nodes) and unstable (saddles) fixed points. In the saddle points we have $\cos\varphi_* = 0$, i.e.,

$$\varphi_*=\pm\pi/2. \tag{2.12}$$

The corresponding values of the variable I_* are obtained from the equation

$$\pm\gamma(\Omega/2N\varepsilon k_0)\chi_*^2=J_N(\chi_*), \tag{2.13}$$

where $\chi_* = k_0\rho_*$. It follows from this equation that the number of saddle points is limited and they are concentrated in a bounded region around the origin.

The stable stationary points of the system (2.11) to which the phase trajectories can be attracted, correspond to $I_0^{(s)}$ values determined from the equation

$$J_N'(k_0\rho_0^{(s)})=0. \tag{2.14}$$

Designating by $j_{N,s'}$ the root of the derivative of the sth Bessel function of order N, we write down the equations for the

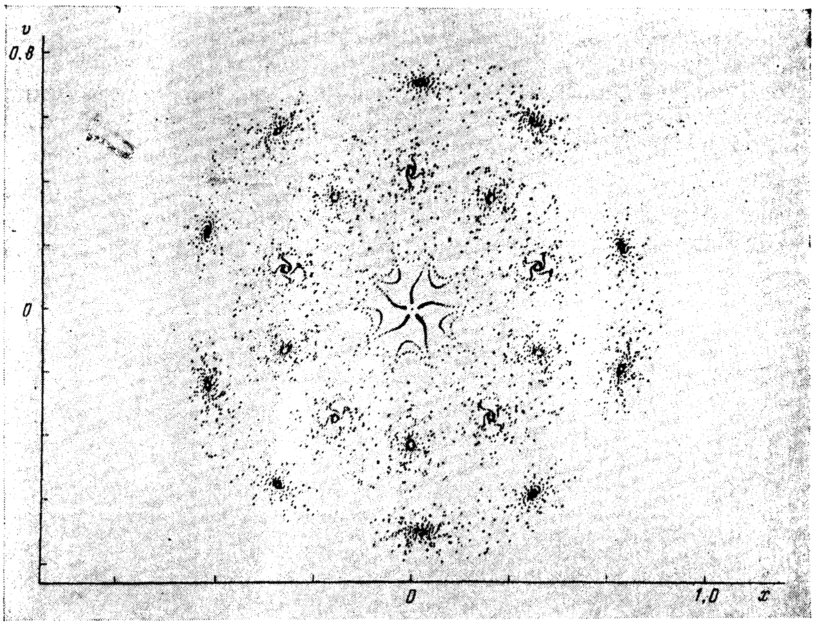


FIG. 1. Poincaré section of the phase space of Eq. (1.5) for $\varepsilon = 1/80$, $\gamma = 8\cdot 10^{-4}$, $k_0 = 15$, $\omega = 5\omega_H$.

angle coordinates $\varphi_0^{(s)}$ of the stable stationary points:

$$\sin\varphi_0^{(s)} = (\gamma k_0/\varepsilon)\frac{I_0^{(s)}}{J_N(k_0\rho_0^{(s)})}, \tag{2.15}$$

where

$$I_0^{(s)} = j_{N,s}'^2 \Omega/2Nk_0^2. \tag{2.16}$$

It follows from (2.15) that the stable points, like the unstable ones, are localized around the origin in a region

$$k_0\rho < (k_0\varepsilon N/\gamma\Omega)^{2/5}. \tag{2.17}$$

Equation (2.17) shows that as the damping increases the localization region, and the number of the stationary points decrease until a single stable fixed point—the origin—is left. Let us determine the damping coefficient γ for which this occurs. We put $s = 1$, and then

$$j_{N,1}' \approx N + 0.81N^{1/3}, \qquad J_N(j_{N,1}') \approx 0.67N^{-1/3}. \tag{2.18}$$

The condition for the existence of a stationary point for the action value $I_0^{(0)} \leqslant 1$ is the inequality $\sin\varphi_0^{(1)} \leqslant 1$ or

$$(\gamma\Omega/\varepsilon k_0)(0.75N^{4/3} + 1.21N^{2/3}) \leqslant 1. \tag{2.19}$$

For $N = 5$, in particular, we find that if

$$\gamma > 0.1(\varepsilon k_0/\Omega) \tag{2.20}$$

the only remaining stationary point is the origin.

Figure 1 shows the Poincaré cross section for trajectories with several different initial conditions. This figure was obtained by numerically solving the set (2.1) for $\varepsilon = 1/80$, $\gamma = 0.0008$, $k_0 = 15$ and $\omega = 5\omega_H$. The values of $x(t)$ and $v(t)$ were marked as points on the (x,v) plane after equal time intervals $\Delta t = 2\pi/\omega$. Since the damping is rather weak, stable stationary points other than the origin are present. One can see the trajectories that are attracted, depending on the initial conditions, to different stationary points.

Let us examine in greater detail the behavior of the phase trajectories near the origin. For $\gamma = 0$ the origin is a degenerate unstable stationary point that has stable and unstable invariant manifolds. A weak dissipation lifts the degeneracy, and as a result new stationary saddle points appear on the unstable invariant manifold. The origin turns then into a node to which the trajectories are attracted.

3. STOCHASTIC DYNAMICS OF CHARGED PARTICLES IN A WEAK MAGNETIC FIELD

The analysis in the preceding section pertains to the case of weak nonlinearity $k_0\varepsilon \ll \omega_H^2$. As the parameter ε is increased, the situation becomes more complicated and the averaging method is inconvenient for further analysis. Numerical estimates show that if ε is large enough a strange attractor is produced on the phase plane. Figure 2a shows the Poincaré section $t = 2\pi n/\omega$, $n = 0, 1,...$ of the set (2.1), obtained by numerical integration for one initial condition and for the parameter values $\varepsilon = 8.0$, $k_0 = 15.0$, $\gamma = 0.16$, $\omega = 5$ and $\omega_H = 1$. It is seen from this figure that the dynamics tends to a strange attractor having a rather complicated structure comprising a "spiral" and a "beak." The explanation of this attractor shape is the following. Motion over the beak means motion in which the particle is trapped by the wave. The particle phase oscillates relative to the phase of the wave $u = k_0x - \omega t$, while x increases on the average by $2\pi/k_0$ after each iteration of the Poincaré mapping. After a number of iterations the particle reaches a region where the term $\omega_H^2 x$ in (2.1), which reflects the influence of the magnetic field, becomes large, and ejects the particle from the wave's potential well. The main influence on the particle motion is now that of the magnetic field and of the friction. As a result the particles approach the origin along a spiral. Near the origin, where x and $\dot{x}$ are small, the conditions become again favorable for particle trapping by a wave. The trapping takes place, the particle moves again over the beak, and the process is repeated. We present below an analytic description of the trapping regime that corroborates the foregoing qualitative scheme.

We introduce

$$u = k_0x - \omega t - 2\pi n, \tag{3.1}$$

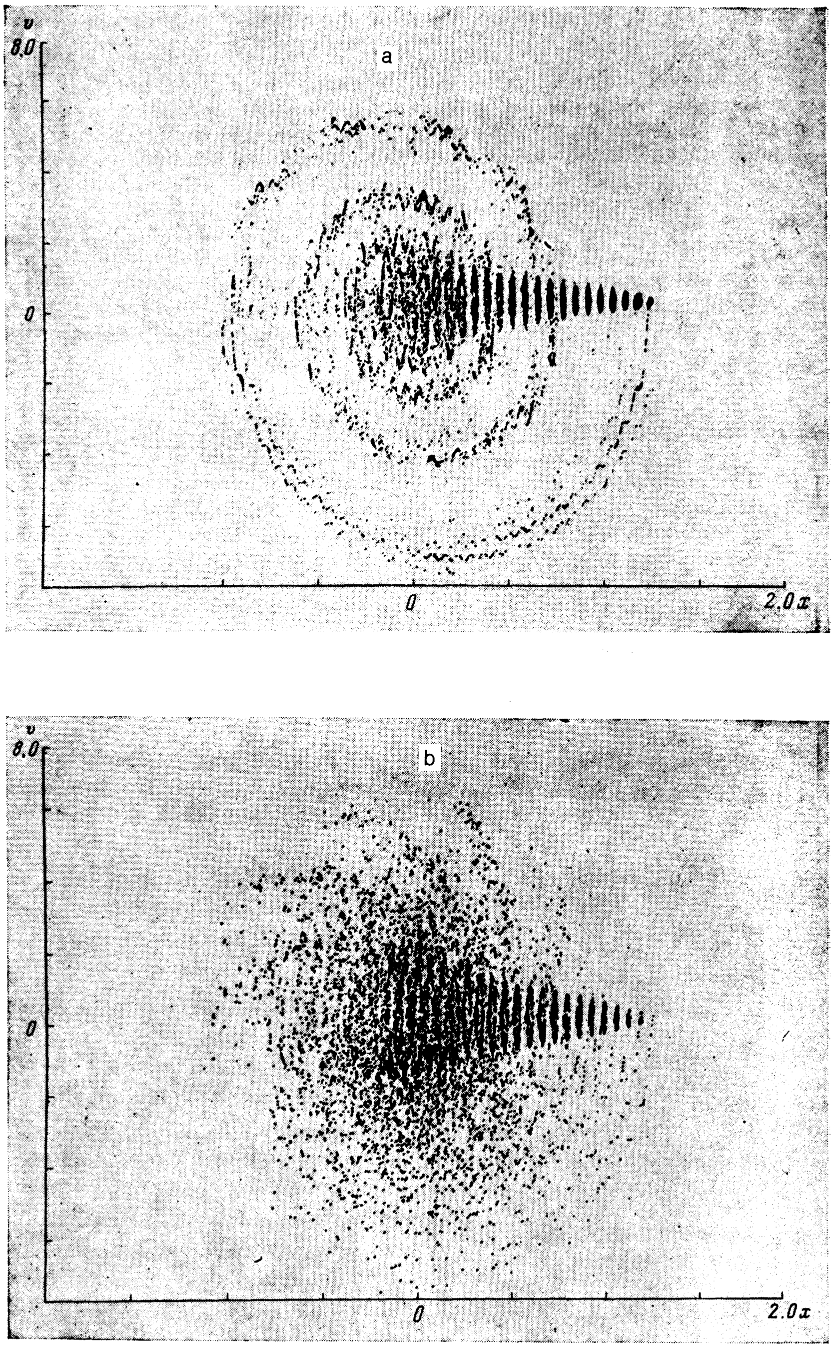


FIG. 2. a) Poincaré cross section of the stochastic attractor of Eq. (1.5) for $\gamma = 0.16$, $\varepsilon = 8.0$, $k_0 = 15$, $\omega = 5\omega_H$; b) the same for $\gamma = 0.08$.

where n is chosen so that $0 \leqslant u \leqslant 2\pi$ at the initial instance $t = t_0$. We obtain

$$\ddot{u} = k_0 \varepsilon \sin u - \omega_H^2 (2\pi n + \omega t) - \omega_H^2 u - \gamma(\dot{u} + \omega). \qquad (3.2)$$

Let the condition $k_0 \varepsilon \gg \max\{\omega_H^2, \gamma\omega, \gamma^2, (\omega_H^2 \omega)^{2/3}\}$ be met. Equation (3.2) with $|\dot{u}| \lesssim (k_0 \varepsilon)^{1/2}$ can then be regarded as a perturbation of a torsion pendulum:

$$\ddot{u} = k_0 \varepsilon \sin u - L, \quad L = \text{const}. \qquad (3.3)$$

The perturbation causes L to vary slowly with time: $L = \omega_H^2 (2\pi n + \omega t)$. The phase portraits of Eq. (3.3) for different fixed values of L are shown in Figs. 3a–3d. For $|L| < k_0 \varepsilon$ each portrait has an oscillatory region whose area we designate by $S(L)$. This area decreases with $|L|$.

Let the phase point $(u, \dot{u})$ be located at $t = t_0$ in the oscillatory region of the pendulum (3.3) with $L \simeq \omega_H^2 (2\pi n + \omega t_0)$. To describe the motion we average over the phase of the unperturbed oscillations. We denote by J the action variable of the pendulum (3.3) in the oscillatory region. The averaged equation for J yields

$$\dot{J} = -\gamma J, \quad J = J(t_0) \exp[-\gamma (t - t_0)], \qquad (3.4)$$

so that the point remains in the oscillatory region of pendulum (3.3) with $L = \omega_H^2 (2\pi n + \omega t)$ so long as the condition

$$2\pi J(t_0) \exp[-\gamma(t - t_0)] < S(\omega_H^2 (2\pi n + \omega t)) \qquad (3.5)$$

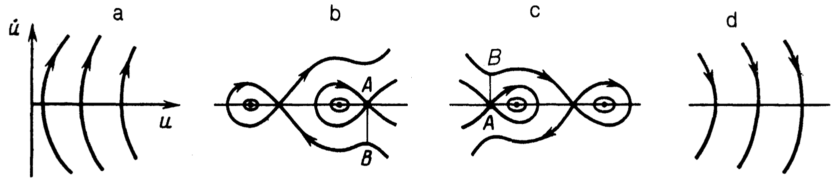


FIG. 3. Phase portraits of Eq. (3.4) for different fixed values of L: a) $L < -k_0\varepsilon$, b) $-k_0\varepsilon < L < 0$, c) $0 < L < k_0\varepsilon$, d) $L > k_0\varepsilon$.

is met (here we note that $2\pi J$ is the area bounded by the unperturbed-pendulum trajectory passing through the phase point). Equations (3.4) and (3.5) describe the motion of a particle trapped by the wave. The phase point moves in this case over the beak on the Poincaré cross section.

The times $t_- < t_0$ and $t_+ > t_0$ at which (3.5) turns into an equality are respectively the instants of trapping into and ejection from the oscillation region. According to (3.5), the relation between them is

$$\exp[-\gamma(t_+-t_-)]=\frac{S(\omega_H^2(2\pi n+\omega t_+))}{S(\omega_H^2(2\pi n+\omega t_-))}. \tag{3.6}$$

Since u varies in a limited range, we obtain from (3.1) $k_0x \simeq \omega t + 2\pi n$, and from (3.6) we get an approximate connection between the values $x_\mp$ of the coordinate x of the particle trapping and ejection points:

$$\exp[-(\gamma k_0/\omega)(x_+-x_-)]=\frac{S(\omega_H^2 k_0 x_+)}{S(\omega_H^2 k_0 x_-)}. \tag{3.7}$$

Equations (3.4) and (3.7) show that the narrowing of the beak with increase of x is due to the interaction of two effects: the decrease of the action J due to friction, and the decrease of the area $S(\omega_H^2 k_0 x)$ of the oscillation region as x increases for $x > 0$. If the friction coefficient is large enough, the first effect predominates almost over the entire length of the beak. Most points then travel along the entire beak and are ejected nears its end point, where $x \simeq \omega_H^2/\varepsilon$ and the oscillation region vanishes (Fig. 2a). For a smaller friction coefficient the second effect becomes substantial earlier and some of the points are ejected from the beak at smaller x (Fig. 2b).

Let us examine in greater detail the trapping of a particle by a wave. Let the point $(u, \dot{u})$ reach at the instant t_0 the segment AB (Fig. 3). We denote by AC the segment from which trapping into the oscillation region adjacent to the point A is possible. The phase-space flux through AC is approximately equal to $\gamma S - S'\omega_H^2\omega$ (we have calculated the phase space freed inside the oscillation region on account of compression of the volumes by the friction and by the change of the area bounded by the separatrix loop; here S' is the derivative of S with respect to its argument; trapping is impossible if the calculated flux is negative). The phase-space flux through the entire segment AB is equal to $2\pi|L|$.

For $L \simeq k_0\varepsilon$ it is natural to define the trapping probability as the ratio of these fluxes:

$$P=(\gamma S-S'\omega_H^2\omega)/2\pi|L|. \tag{3.8}$$

If

$$\frac{\omega_H^2\omega}{(k_0\varepsilon)^{3/2}}\ln\frac{\omega_H^2\omega}{(k_0\varepsilon)^{3/2}} \ll \frac{|L|}{k_0\varepsilon} \ll 1$$

we have the familiar problem of excitation of pendulum oscillations in the presence of a small torque and low friction. The probability of this excitation is[10]

$$P=\begin{cases}\dfrac{8\gamma(k_0\varepsilon)^{1/2}}{\pi|L|+4\gamma(k_0\varepsilon)^{1/2}}, & |L|>4\gamma(k_0\varepsilon)^{1/2}/\pi\\ 1, & |L|\leqslant 4\gamma(k_0\varepsilon)^{1/2}/\pi\end{cases}. \tag{3.9}$$

In these equations, $L = \omega_H^2 k_0 x$. It can be seen that trapping is most probable in the region of small x, i.e., near the origin [since trapping also calls for $|\dot{x} - \omega/k_0| \sim (\varepsilon/k_0)^{1/2}$].

The trapping probability for large x is small but is nonetheless present and is according to (3.8) larger for negative x, where $S' < 0$, than for positive x. The corresponding trapped points are seen near the negative x axis on Fig. 2a.

4. PARTICLE DYNAMICS IN A LOW-INTENSITY WAVE-PACKET FIELD

We turn to Eq. (1.6), i.e., assume that the electrostatic field comprises a set of plane waves with different amplitudes, frequencies, and wave vectors. The packet structure is assumed to satisfy the expressions

$$k_n=k_0,\quad \omega_n=n\omega,\quad \varepsilon_n=\varepsilon. \tag{4.1}$$

Expressions (4.1) mean that the dispersion effects are weak and that the spectral characteristics of the wave packet are homogeneous and symmetrical enough. Substituting (4.1) in (1.6) we get

$$\ddot{x}+\gamma\dot{x}+\omega_H^2x=\varepsilon\sum_{n=-\infty}^{+\infty}\sin(k_0x-n\omega t). \tag{4.2}$$

Using the equation

$$\sum_{n=-\infty}^{+\infty}\cos n\omega t=2\pi\sum_{n=-\infty}^{+\infty}\delta(\omega t-2\pi n),$$

we rewrite (4.2) in the form

$$\ddot{x}+\gamma\dot{x}+\omega_H^2x=2\pi\varepsilon\sin k_0x\sum_{n=-\infty}^{+\infty}\delta(\omega t-2\pi n). \tag{4.3}$$

In Eq. (4.3), unlike Eq. (1.5) considered in the preceding sections, the continuous perturbation of the damped oscillator in (1.5) is replaced by periodic δ pulses, i.e., instantaneous jolts.

We confine ourselves in the present section to an investigation of the phase portrait of Eq. (4.2) for weak nonlinearity $k_0\varepsilon \ll \omega_H^2$ and at resonance, $\omega = N\Omega$. Changing over in (4.2) to the variable I and φ [Eq. (2.5)] in a frame rotating with frequency ω/N, we obtain the following set of equations:

$$\dot{I}+2\gamma I\cos^2\left(\frac{\varphi}{N}+\Omega t\right)$$

$$=\varepsilon\left(\frac{2I}{N\Omega}\right)^{1/2}\cos\left(\frac{\varphi}{N}+\Omega t\right)\sum_{n=-\infty}^{+\infty}\sin(k_0x-n\omega t),$$

$$\dot{\varphi}-\frac{1}{2}N\gamma\sin\left(\frac{2\varphi}{N}+2\Omega t\right)$$

$$=-\varepsilon\left(\frac{N}{2I\Omega}\right)^{1/2}\sin\left(\frac{\varphi}{N}+\Omega t\right)\sum_{n=-\infty}^{+\infty}\sin(k_0x-n\omega t). \quad (4.4)$$

Averaging in (4.4) over time, we obtain

$$\dot{I}+\gamma I=-\frac{\varepsilon}{k_0}\frac{\partial}{\partial\varphi}\left[\sum_{n=-\infty}^{+\infty}J_{nN}(k_0\rho)\cos n\varphi\right],$$

$$\dot{\varphi}=\frac{\varepsilon N}{\Omega\rho}\frac{\partial}{\partial\rho}\left[\sum_{n=-\infty}^{+\infty}J_{nN}(k_0\rho)\cos n\varphi\right]. \quad (4.5)$$

The series in the right-hand sides of (4.5) is summed with the aid of the identity

$$\sum_{n=-\infty}^{+\infty}J_{nN}(k_0\rho)\cos n\varphi=\frac{1}{N}\sum_{j=1}^{N}\cos\left[k_0\rho\sin\frac{\varphi+2\pi j}{N}\right].$$

We confine ourselves next for simplicity to $N=4$ and return to the old variable x and $y=\dot{x}/\Omega$. In these variables, the averaged system (4.4) takes the form

$$\dot{x}+(\gamma/2)x=-(\varepsilon/2\Omega)\sin k_0y,$$
$$\dot{y}+(\gamma/2)y=(\varepsilon/2\Omega)\sin k_0x. \quad (4.6)$$

The stationary points are obtained from the equations

$$\sin k_0x=(\gamma\Omega/\varepsilon)y, \quad (4.7)$$
$$\sin k_0y=-(\gamma\Omega/\varepsilon)x.$$

For the variable x we obtain the transcendental equation

$$\sin[(\varepsilon k_0/\gamma\Omega)\sin k_0x]=-(\gamma\Omega/\varepsilon)x. \quad (4.8)$$

In the strong-dissipation limit,

$$\gamma\gg\varepsilon k_0/\Omega, \quad (4.9)$$

it follows from (4.8) that the only stationary point is the origin. For weak dissipation,

$$\gamma\ll\varepsilon k_0/\Omega, \quad (4.10)$$

there are many stationary points [roots of Eq. (4.8)] located in the bounded region

$$\rho\lesssim\varepsilon/\gamma\Omega. \quad (4.11)$$

The size of this region does not increase when the parameter γ is decreased, but the locations of the stationary points approach those of the stationary points (x_n, y_m) of a conservative system:

$$x_n=\pi\pm\pi n, \qquad y_m=\pi\pm\pi m. \quad (4.12)$$

The number of stationary points decreases with the parameter $\varepsilon k_0/\gamma\Omega$, and the last stationary points other than the origin vanish at $\varepsilon k_0/\gamma\Omega\approx 3.57$.

Let us investigate the stability of the stationary points for this purpose. We linearize the set (4.6) in the vicinity of the fixed point (x_0, y_0). The tangent matrix is

$$\begin{pmatrix}-\gamma/2 & -(\varepsilon k_0/2\Omega)\cos k_0y_0\\ (\varepsilon k_0/2\Omega)\cos k_0x_0 & -\gamma/2\end{pmatrix} \quad (4.13)$$

and its eigenvalues are

$$\lambda_{1,2}=\frac{\gamma}{2}[-1\pm(1-(\varepsilon k_0/\gamma\Omega)^2\cos k_0x_0\cos k_0y_0)^{1/2}]. \quad (4.14)$$

Depending on the value of the parameter $\varepsilon k_0/\gamma\Omega$ and on the position of the point (x_0, y_0), the following cases are possible:

1. $\cos k_0x_0\cos k_0y_0>(\gamma\Omega/\varepsilon k_0)^2$

i.e., both eigenvalues have negative real parts and nonzero imaginary parts, so that the point (x_0, y_0), is an attracting focus;

2. $(\gamma\Omega/\varepsilon k_0)^2>\cos k_0x_0\cos k_0y_0>0$

i.e., both eigenvalues are real and negative. The point (x_0, y_0) is consequently an attracting node;

3. $\cos k_0x_0\cos k_0y_0<0$

i.e., both eigenvalues are positive and of opposite sign, so that (x_0, y_0) is a saddle point.

5. DISSIPATIVE MAPPING WITH "TWISTING"

The differential equation (4.3) can be replaced by a finite-difference equation. Between two successive applications of a δ function the particle trajectory satisfies the equation

$$\ddot{x}+\gamma\dot{x}+\omega_H^2x=0. \quad (5.1)$$

Its solutions on passage through the δ function at the instant $t_n=nT$ $(T=2\pi/\omega)$ should satisfy the boundary conditions

$$x(t_n+0)=x(t_n-0),\ \dot{x}(t_n+0)=\dot{x}(t_n-0)+\varepsilon T\sin k_0x(t_n);$$

using these conditions, we obtain from (4.3)

$$x_{n+1}=\frac{1}{\Omega}e^{-\gamma T/2}\left\{(p_n+\varepsilon T\sin k_0x_n)\sin\Omega T+\Omega x_n\left(\cos\Omega T+\frac{\gamma}{2\Omega}\sin\Omega T\right)\right\},$$

$$p_{n+1}=e^{-\gamma T/2}\left\{(p_n+\varepsilon T\sin k_0x_n)\left(\cos\Omega T-\frac{\gamma}{2\Omega}\sin\Omega T\right)-\Omega x_n\sin\Omega T\left(1+\frac{\gamma^2}{4\Omega^2}\right)\right\}, \quad (5.2)$$

where

$$p_n=\dot{x}(t=nT-0),$$

i.e., the subscript n corresponds to the instant of time immediately preceding the action of the δ function at $t=nT$ and $\Omega=(\omega_H^2-\gamma^2/4)^{1/2}$.

The mapping (5.2) goes over as $\omega_H\to 0$ into the dissipative standard mapping[7]:

$$x_{n+1}=x_n+(p_n+\varepsilon T\sin k_0x_n)\frac{1-e^{-\gamma T}}{\gamma},$$
$$p_{n+1}=e^{-\gamma T}(p_n+\varepsilon T\sin k_0x_n), \quad (5.3)$$

the connection of which with the kinetic description of particle dynamics is quite well known.[11] In particular, Eqs. (5.3) correspond to the equation of motion (4.3) with $\omega_H=0$, and stochasticity sets in in this case[12] if

$$k_0|\varepsilon|T^2\frac{1-e^{-\gamma T}}{\gamma T}>1. \tag{5.4}$$

We change over in (5.2) to dimensionless variables

$$k_0x=-v,\ k_0p/\Omega=u,\ \alpha=\Omega T. \tag{5.5}$$

The mapping (5.2) then takes the form $\hat{D}_\alpha$:

$$\hat{D}_\alpha:\begin{cases} u_{n+1}=e^{-\gamma T/2}\left\{(u_n+K_H\sin v_n)\left(\cos\alpha-\frac{\gamma}{2\Omega}\sin\alpha\right)+v_n\sin\alpha\left(1+\frac{\gamma^2}{4\Omega^2}\right)\right\} \\ v_{n+1}=e^{-\gamma T/2}\left\{-(u_n+K_H\sin v_n)\sin\alpha+v_n\left(\cos\alpha+\frac{\gamma}{2\Omega}\sin\alpha\right)\right\}, \end{cases} \tag{5.6}$$

where

$$K_H=-Tk_0\varepsilon/\Omega.$$

The mapping $\hat{D}_\alpha$ as $\gamma\to 0$ goes over into a measure-conserving mapping with "twisting"[8]:

$$\hat{M}_\alpha:\begin{cases} u_{n+1}=(u_n+K_H\sin v_n)\cos\alpha+v_n\sin\alpha \\ v_{n+1}=-(u_n+K_H\sin v_n)\sin\alpha+v_n\cos\alpha' \end{cases} \tag{5.7}$$

which generates structures if $\alpha=2\pi p/q$ (p and q are integers). In particular, it was shown in Refs. 8 and 9 that for arbitrarily small K_H the unbounded part of the phase plane of the set (5.7) is covered by a stochastic web, in the channels of which the particles move randomly in analogy with Arnol'd diffusion, but for the minimum dimensionality (three) of the phase space. For small K_H the thickness of the stochastic web is exponentially small, but if K_H is large, the stochastic-web channels broaden, the chaos becomes strong, and a large fraction of the particles participate in the diffusion.

A criterion for strong chaos in the dissipative problem (5.6) can be roughly estimated from the condition that the origin be stable. The mapping (5.6) has a stationary point $u=v=0$ for all values of the parameters. Linearizing (5.6) in the vicinity of this point, we obtain the characteristic equation

$$\lambda^2-\lambda e^{-\gamma T/2}(2\cos\alpha-K_H\sin\alpha)+e^{-\gamma T}=0. \tag{5.8}$$

The point (0, 0) becomes unstable when

$$K_H\sin\alpha>2[\cos\alpha+\mathrm{ch}(\gamma T/2)],\ K_H>0. \tag{5.9}$$

For $\gamma=0$, in particular, (5.9) leads to the condition

$$K_H>2\,\mathrm{ctg}(\alpha/2), \tag{5.10}$$

obtained earlier in Ref. 8. As $\omega_H\to 0$ we have accordingly from (5.9) the inequality

$$T^2k_0|\varepsilon|/\gamma T>2\,\mathrm{cth}(\gamma T/2), \tag{5.11}$$

which corresponds to the criterion (5.4). When conditions (5.9)–(5.11) are met the dynamics of the particles is chaotic and tends to a strange attractor. A numerical analysis confirms the criterion (5.9), and the set of points of the $\hat{D}_\alpha$ on one trajectory has the form typical of a strange attractor (Fig. 4). With increase of scale, each line on Fig. 4a split into a family of straight lines (Fig. 4b). This pattern repeats as the scale is further increased (Fig. 4c), i.e., the strange attractor has the structure of a Cantor set. The arrangement of the lines in phase space is determined by the twist angle α and by the damping γ. Figure 5 illustrates the results of a numerical analysis of the mapping $\hat{D}_\alpha$ for different values of the angle α. The slopes of the straight lines in Fig. 5 can be estimated by putting $K_H\gg 1$ and $\gamma T\gtrsim 1$ in (5.6). The dynamics of the system is then described by the mapping

$$\begin{gathered} u_{n+1}=e^{-\gamma T/2}K_H\sin v_n\left(\cos\alpha-\frac{\gamma}{2\Omega}\sin\alpha\right), \\ v_{n+1}=-e^{-\gamma T/2}K_H\sin v_n\sin\alpha. \end{gathered} \tag{5.12}$$

For the slopes of the straight lines we obtain

$$\mathrm{tg}\,\varphi=u_{n+1}/v_{n+1}=\gamma/2\Omega-\mathrm{ctg}\,\alpha. \tag{5.13}$$

In particular, for $\gamma T=1$ and $\Omega T=2\pi/5$ we obtain $\varphi\approx 4°$, in good agreement with the results of a numerical analysis (Fig. 5b).

6. FOKKER-PLANCK-KOLMOGOROV EQUATION

If condition (5.9) is met, it is easiest to understand the simplest properties of stochastic dynamics by starting with the FPK kinetic equation. We assume the dissipation to be weak, i.e.,

$$\gamma T\ll 1,\ \gamma/\omega_H\ll 1. \tag{6.1}$$

If the inequalities (6.1) are satisfied, the mapping $\hat{D}_\alpha$ can be approximately written in the form

$$\begin{aligned} u_{n+1}&=\left(1-\frac{\gamma T}{2}\right)\left\{(u_n+K_H\sin v_n)\left(\cos\alpha-\frac{\gamma}{2\Omega}\sin\alpha\right)+v_n\sin\alpha\right\}, \\ v_{n+1}&=\left(1-\frac{\gamma T}{2}\right)\left\{-(u_n+K_H\sin v_n)\sin\alpha+v_n\left(\cos\alpha+\frac{\gamma}{2\Omega}\sin\alpha\right)\right\}. \end{aligned} \tag{6.2}$$

In the mapping (6.2) we change over to the new variables

$$I=u^2+v^2,\ \varphi=\mathrm{arctg}(u/v). \tag{6.3}$$

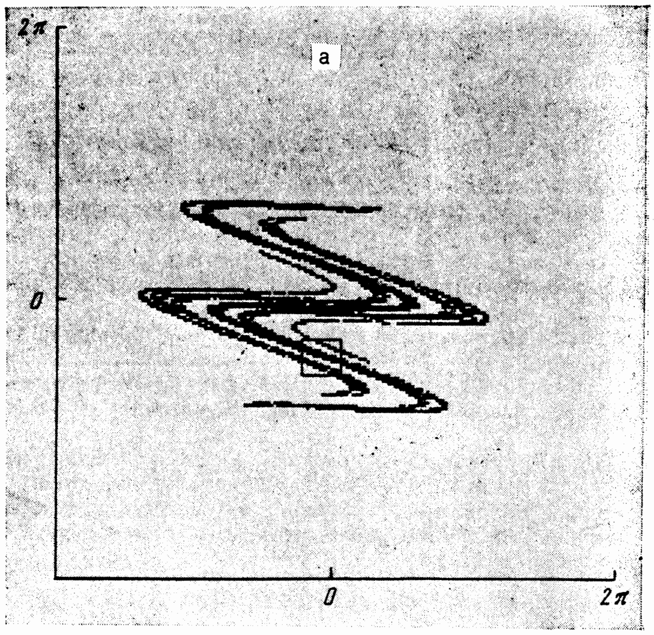
The FPK-equation approximation becomes valid when the variable I is large enough to meet the condition

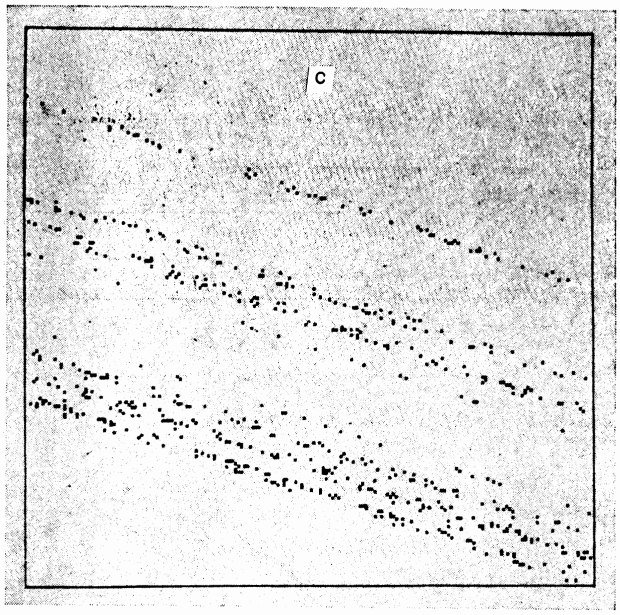
$$|\Delta I|=|I_{n+1}-I_n|\ll I_n. \tag{6.4}$$

The phase φ can be assumed here to be randomly distributed in the interval $(0, 2\pi)$. The condition for this assumption is the inequality

$$K_H>2\,\mathrm{ctg}(\alpha/2).$$

We use (6.2) and (6.3) to calculate the friction coefficient





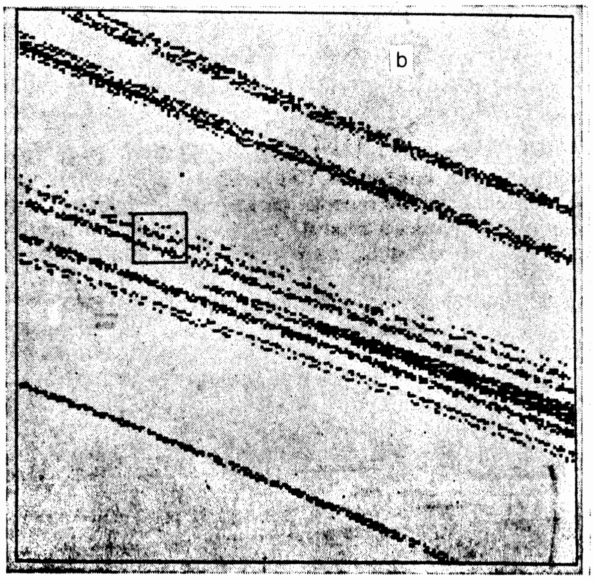


FIG. 4. Stochastic attractor specified by the mapping (5.6) (the abscissa and ordinate are respectively u and v): a) result of 10^4 iterations of the mapping $\hat{D}_\alpha$ for one initial condition at $K_H = 6.0$, $\alpha = 1.57$, $\gamma T = 0.7$; b) magnified attractor section in the box on Fig. 4a, with 10^6 interactions; c) further magnification of the box on Fig. 4b, with $5 \cdot 10^6$ iterations. The fractal structure typical of a strange attractor is evident.

$$B = \frac{1}{T}\langle\langle \bar{I} - I \rangle\rangle = \frac{K_H^2}{2\pi T}\left(1 - \gamma T - \frac{\gamma}{2\Omega}\sin 2\alpha\right)\int_0^{2\pi} d\varphi \sin^2(I^{1/2}\cos\varphi)$$

$$-\gamma I = \frac{K_H^2}{2T}\left(1 - \gamma T - \frac{\gamma}{2\Omega}\sin 2\alpha\right)(1 - J_0(2I^{1/2})) - \gamma I. \qquad (6.5)$$

For $I \gg 1$ we obtain from (6.5)

$$B \approx \frac{K_H^2}{2T}\left(1 - \gamma T - \frac{\gamma}{2\Omega}\sin 2\alpha\right) - \gamma I. \qquad (6.6)$$

Similar calculations yield for the diffusion coefficient

$$D = \frac{1}{T}\langle\langle (\bar{I} - I)^2 \rangle\rangle = \frac{2K_H^2 I}{\pi T}\int_0^{2\pi} d\varphi \sin^2(I^{1/2}\cos\varphi)\Big[\sin^2\varphi$$

$$- \frac{\gamma}{\Omega}\sin 2\alpha \sin^2\varphi + \frac{\gamma}{\Omega}\sin 2\alpha\cos 2\varphi - 2\gamma T\sin^2\varphi - \frac{\gamma}{2}T\Big]$$

$$= \frac{IK_H^2}{T}\Big[\left(1 - \frac{\gamma}{\Omega}\sin 2\alpha - 2\gamma T\right)(1 - I^{-1/2}J_1(2I^{1/2}))$$

$$- \frac{\gamma}{2\Omega}\sin 2\alpha J_2(2I^{1/2}) - \gamma T(1 - J_0(2I^{1/2}))\Big], \qquad (6.7)$$

where $J_m(2I^{1/2})$ is a Bessel function. In the region $I \gg 1$ we

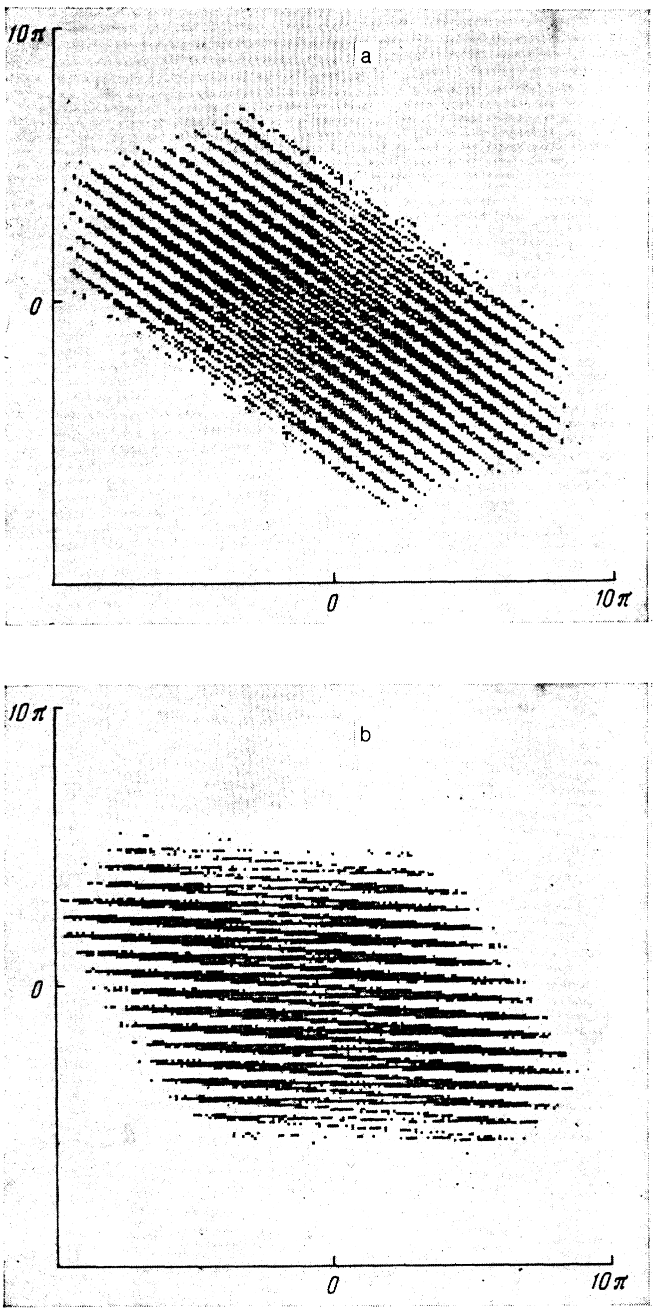


FIG. 5. Result of $2.5\cdot 10^4$ iterations of the mapping $\hat{D}_\alpha$ in the case of strong nonlinearity. The abscissa and ordinate are v and u, respectively; $K_H = 50$, $\gamma T = 1.0$: a) $\alpha = 2.093$; b) $\alpha = 1.256$.

obtain from (6.7) an approximate expression for the diffusion coefficient

$$D \approx \frac{IK_H^2}{T}\left(1-\frac{\gamma}{\Omega}\sin 2\alpha - 3\gamma T\right). \tag{6.8}$$

The FPK equation for the distribution function $F(I, t)$ is[12]

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial I}(BF) + \frac{1}{2}\frac{\partial^2}{\partial I^2}(DF). \tag{6.9}$$

The stationary solution F_s of (6.9) satisfies the following ordinary differential equation:

$$-BF_s + \frac{1}{2}\frac{\partial}{\partial I}(DF_s) = 0. \tag{6.10}$$

Substituting expressions (6.6) for the friction coefficient and (6.8) for the diffusion in (6.10) and solving the latter, we get

$$F_s = CI^{\gamma(2T+(\sin 2\alpha)/2\Omega)} \exp(-2\gamma TI/K_H^2). \tag{6.11}$$

It follows from (6.11) that the stationary distribution function is localized in the region

$$I < I_\gamma \equiv K_H^2/2\gamma T. \tag{6.12}$$

The quantity I_γ can be used as an estimate of the size of the strange attractor if the latter exists for the given parameters of the problem. The constant C in (6.11) is determined by the normalization conditions. For small γ the low-energy region in which the FPK equation approximation is not valid is small compared with I and can be neglected if F_s is normalized. Consequently we obtain from (6.11)

$$C = \left[\int_0^\infty dI\, e^{-I/I_\gamma}\right]^{-1} \sim I_\gamma^{-1}. \tag{6.13}$$

Let us now examine the time dependence of the approach to a stationary distribution. The equation for the first moment of the distribution function

$$\langle I\rangle = \int_0^\infty IF\, dI$$

can be obtained by multiplying (6.9) and I and integrating over I from zero to infinity:

$$\frac{d}{dt}\langle I\rangle = \frac{K_H^2}{2T}\left[1-\gamma T - \frac{\gamma}{2\Omega}\sin 2\alpha\right] - \gamma\langle I\rangle. \tag{6.14}$$

The solution of (6.14) is

$$\langle I\rangle = \frac{K_H^2}{2\gamma T}\left[1-\gamma T - \frac{\gamma}{2\Omega}\sin 2\alpha\right](1-e^{-\gamma T}) + I_0 e^{-\gamma T}, \tag{6.15}$$

where I_0 is the initial value of the variable I. It follows from (6.15) that for $\gamma t \ll 1$ we have

$$\langle I\rangle = I_0 + \frac{K_H^2}{2T}\left[1-\gamma T - \frac{\gamma}{2\Omega}\sin 2\alpha\right]t, \tag{6.16}$$

i.e., the particle energy increases linearly with time, but at $\gamma t \gg 1$ the solution becomes stationary and $\langle I\rangle$ approaches the value

$$\langle I\rangle_s = \frac{K_H^2}{2\gamma T}\left[1-\gamma T - \frac{\gamma}{2\Omega}\sin 2\alpha\right] \approx I_\gamma. \tag{6.17}$$

The corresponding maximum energy to which the particles are heated is

$$\mathscr{E}_s = (\Omega^2/2k_0^2)\langle I\rangle_s \approx (\Omega^2/2k_0^2) I_\gamma. \tag{6.18}$$

The physical content of (6.18) is that it determines not only the feasibility of stochastic heating of magnetized particles by a wave-packet field in the presence of dissipation, but also the heating limit as a function of the packet parameters (amplitude, wave-mode spacing).

CONCLUSION

An investigation of the dissipative dynamics of charged particles in the field of a wave packet propagating across a magnetic field has made it possible to identify two typical limiting cases that are different for narrow (single-mode) and broad packets. A common feature of both cases is degeneracy of the unperturbed problem, and an important physical consequence of the dissipation is the limit imposed on the increase of the particle energy by stochastic heating. The particle dynamics, however, is substantially different in these limiting cases. In the single-mode approximation, if the particles are strongly perturbed by the electrostatic-wave field, the chaotic dynamics of the particles is determined to a considerable degree by the trapped particles, whereas in the limit of an infinitely broad wave packet the stochastic dynamics tends to a strange attractor typical of nonintegrable dissipative systems.

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Translated by J. G. Adashko