Tricritical point in a disordered system

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It is shown that the tricritical behavior of a pure thermodynamic system is unstable against the introduction of disorder in the form of weak fluctuations of T_c. In the neighborhood of the "pure" tricritical point (TCP) a phase transition of the percolation type is realized. The true TCP of the disordered system is displaced into the region of a first-order transition of the pure system. In the region of the first-order transition the width $\Delta\tau$ of the region of hysteresis associated with pinning of the interphase boundaries is found. In the three-dimensional case $\Delta\tau \propto \Delta S$, where ΔS is the entropy discontinuity in the transition, and in a two-dimensional system $\Delta\tau \propto (\Delta S)^{2/3}$.

1. INTRODUCTION

We are interested in the behavior of a system near a tricritical point (TCP) in the presence of frozen impurities. We assume that such a system is described by a Ginzburg-Landau functional with a random temperature:

$$\mathcal{F}=\mathcal{F}_0+\int\left[\frac{1}{2}c(\nabla\varphi)^2+\frac{1}{2}\alpha(\tau+\delta\tau(\mathbf{x}))\varphi^2+\frac{1}{4}b\varphi^4+\frac{1}{6}d\varphi^6\right]d^Dx, \tag{1.1}$$

where the spatial dimensionality is $D = 2$ or 3.

To simplify the writing of the formulas we confine ourselves to the case of a scalar order parameter, although our analysis can be generalized without difficulty to systems with a multicomponent order parameter.

The fluctuations of the temperature in (1.1) are assumed to be Gaussian and uncorrelated:

$$\langle\delta\tau(\mathbf{x})\delta\tau(\mathbf{x}')\rangle=\gamma\delta(\mathbf{x}-\mathbf{x}'). \tag{1.2}$$

We shall consider the case of weak disorder:

$$\gamma(\alpha/c)^{D/2}\ll 1. \tag{1.3}$$

The second-order phase transition in the disordered system was investigated in Refs. 1 and 2, in which it was shown that frozen impurities give rise to an asymptotic "dirty scaling" regime with changed critical indices. The renormalization-group (RG) method used in Refs. 1 and 2 is applicable only for $b \gtrsim \gamma\alpha^2/T_c$ (T_c is the transition temperature). Moving along the second-order transition curve toward the TCP we inevitably reach the region of small $b \lesssim \gamma\alpha^2/T_c$, in which the RG equations lead to an increase of the effective charges and lose their applicability. Below (in Sec. 2) we investigate this region and show that here a phase transition of the percolation type occurs, analogous to that studied previously in the theory of very dirty superconductors and dilute ferromagnets. We show too that the percolation transition occurs also in the region $b < 0$, $|b| \lesssim b^*$, where

$$b^*\sim\gamma\alpha^2(d/c^3)^{1/2},\quad D=3,\qquad b^*\sim\alpha(\gamma d/c)^{1/2},\quad D=2. \tag{1.4}$$

Thus, the disorder displaces the TCP into the region of negative b, in which, in the pure system, a first-order transition occurs.

Below, in Sec. 3, we investigate the stability of the domain boundary between the ordered and disordered phases in the region $b < 0$, $b \gg b^*$. It will be shown that the domain boundaries in this region are stable, i.e., in thermodynamic equilibrium a first-order phase transition, associated with discontinuities of the entropy S and other thermodynamic quantities (which we denote collectively by O), should occur. In fact, equilibrium is difficult to reach near the transition line, since a first-order transition occurs by way of the creation of domains of another phase and subsequent motion of the domain boundaries, which is strongly braked by frozen disorder. In Sec. 4 we estimate the width $\Delta\tau$ of the hysteresis region associated with the pinning of domain boundaries at defects. It is shown that for $D = 3$ the ratio $\Delta O/\Delta\tau$ in the region $|b| \gg b^*$ remains constant as the TCP is approached along the first-order transition line, while for $D = 2$ we have $\Delta O/\Delta\tau \propto (\Delta O)^{1/3}$.

2. THE PHASE TRANSITION OF THE PERCOLATION TYPE

We consider the region of small $|b|$ in the functional (1.1). Following Ref. 3, we expand the order parameter φ in the eigenfunctions of the equation

$$-c\nabla^2\psi_n+\alpha(E_n+\delta\tau(\mathbf{x}))\psi_n=0, \tag{2.1}$$

$$\varphi=\sum_n a_n\psi_n. \tag{2.2}$$

Then the Ginzburg-Landau functional (1.1) can be written in the form

$$\mathcal{F}=\mathcal{F}_0+\frac{1}{2}\alpha\sum_n a_n^2(\tau-E)+\frac{1}{4}b\int\left(\sum_n a_n\psi_n\right)^4 d^Dx+\frac{1}{6}d\int\left(\sum_n a_n\psi_n\right)^6 d^Dx. \tag{2.3}$$

Equation (2.1) is the Schrödinger equation with a random potential. In any realization $\tau(\mathbf{x})$ and for large E_n [namely, $E_n \gg E_0 \sim \gamma^{2/(4-D)}(\alpha/c)^{D/(4-D)}$; see Ref. 5], regions in which levels with energy E_n exist are exponentially rare. The density of states for such values of E have, to within a numerical constant ρ_0, the form[5,6]

$$\rho(E)=\rho_0\gamma^{-2}E^{3/2}\left(\frac{c}{\alpha}\right)^{3/2}\exp\left[-38\frac{E^{1/2}}{\gamma}\left(\frac{\alpha}{c}\right)^{-3/2}\right],\quad D=3, \tag{2.4a}$$

$$\rho(E)=\rho_0\gamma^{-1}E^{1/2}\left(\frac{c}{\alpha}\right)^{1/2}\exp\left[-11\frac{E}{\gamma}\frac{c}{\alpha}\right],\quad D=2. \tag{2.4b}$$

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As can be seen from (2.3), the eigenmodes with $E_n > \tau$ turn out to be unstable against the appearance of nonzero values of a_n. For $\tau \gg E_0$ these modes have an energy E_n in the region of the "tail" of (2.4), and the corresponding wavefunctions are strongly localized. Below it will be seen that the reduced percolation-transition temperature τ_c does satisfy the inequality $\tau_c \gg E_0$, and therefore, to first order, we can neglect the interaction of different localized modes and write the free energy corresponding to the amplitude a_n, and the corresponding equilibrium equation, in the form

$$\mathcal{F}_n = {}^1/_2 \alpha a^2(\tau - E) + {}^1/_4 b a^4 \Psi_4 + {}^1/_6 d a^6 \Psi_3, \tag{2.5}$$

$$\alpha(\tau - E) + b a^2 \Psi_4 + d a^6 \Psi_6 = 0, \tag{2.6}$$

where

$$\Psi_4 = \int \psi^4 d^D x \sim l^{-D} \sim (\alpha E/c)^{D/2}, \quad \Psi_6 = \int \psi^6 d^D x \sim (\alpha E/c)^D,$$

in which $\psi \equiv \psi(x)$ is the localized solution of (2.1) in the region of the tail and l is the characteristic scale of this solution. Thus, we obtain a system of exponentially rare "droplets" with a nonzero order parameter; the energy of such a droplet is given by (2.5). In (2.5) we have neglected completely the interaction between the droplets (interaction due to overlap of the functions ψ_n; at the same time, the transition to the ordered state in such a system is determined by precisely this interaction. It is not difficult to show (see Appendix A) that in the case of infrequent droplets this interaction (or, more precisely, the difference of the energies of interaction of two droplets with oppositely and with identically oriented order parameters) is given by the expressions

$$\begin{aligned} V_{ij} &= 8\pi c \frac{a_i a_j}{\xi^3} \frac{\xi^2}{R_{ij}} \exp\left(-\frac{R_{ij}}{\xi}\right), \quad D=3, \\ V_{ij} &= 4\pi^{1/2} c \frac{a_i a_j}{\xi^2} \left(\frac{\xi}{R_{ij}}\right)^{1/2} \exp\left(-\frac{R_{ij}}{\xi}\right), \quad D=2, \end{aligned} \tag{2.7}$$

where R_{ij} is the distance between droplets i and j, a_i is the solution of (2.6) for the ith droplet, and $\xi = (c/\alpha\tau)^{1/2}$ is the correlation length.

Finally, we shall assume that droplets for which V_{ij} is greater than the temperature are "coupled," i.e., the order parameter in them is of the same sign; otherwise, the droplets are independent. As is well known,[7] a percolation transition in such a system of randomly arranged droplets occurs when their concentration reaches the value

$$N = (\beta/R_c)^D, \tag{2.8}$$

where $\beta = 0.95$ for $D = 2$ and $\beta = 0.89$ for $D = 3$, and R_c is determined by the equation

$$T_c = V_{ij}(R_c). \tag{2.9}$$

The equations (2.8) and (2.9) determine the phase-transition temperature; here it is necessary to remember that N itself depends on the temperature.

Strictly speaking, the above picture of a percolation transition is valid when the magnitude of the interaction depends only on the distance R_{ij} between the droplets. In our case there is also a dependence on the amplitude a_i of the order parameter in the droplet, but the dependence on R_{ij} is much sharper (exponential). Therefore, we shall assume that the spread of values of a_i can be disregarded, and shall consider certain average (characteristic) droplets.

The magnitude of the order parameter in a droplet and the smallest value of E for which droplets exist are determined from (2.5) and (2.6):

$$a^2 = \frac{-b\Psi_4 + (b^2\Psi_4 - 4\alpha d \Psi_6(\tau - E))^{1/2}}{2d\Psi_6}, \tag{2.10}$$

$$E_{min} = \begin{cases} \tau, & b > 0 \\ -\dfrac{3}{16}\dfrac{b^2}{\alpha d}\dfrac{\Psi_4^2}{\Psi_6} + \tau, & b < 0 \end{cases}. \tag{2.11}$$

In the remainder of this section we shall not consider the region $-b \gtrsim \tau\alpha d$ (which corresponds, as will be shown below, to a first-order transition), and so we neglect the difference between $E_{\min}$ and τ. In this case, this averaging implies that we are considering droplets with

$$\overline{a^2} = \left\{ \int_\tau^\infty \rho(E)\, dE \right\}^{-1} \int_\tau^\infty a^2 \rho(E)\, dE \tag{2.12}$$

and with concentration

$$N = \int_\tau^\infty \rho(E)\, dE,$$

or

$$\begin{aligned} N &= \frac{\rho_0}{11} \frac{\tau^{3/2}}{\gamma^{1/2}} \left(\frac{\alpha}{c}\right)^{1/2} \exp\left(-11 \frac{\tau}{\gamma}\frac{c}{\alpha}\right), \quad D=2, \\ N &= \frac{2}{38} \rho_0 \frac{\tau^2}{\gamma} \exp\left[-38 \frac{\tau^{1/2}}{\gamma}\left(\frac{c}{\alpha}\right)^{3/2}\right], \quad D=3. \end{aligned} \tag{2.13}$$

The expressions (2.13) are valid if

$$\tau \gg \begin{cases} \dfrac{1}{11}\gamma\dfrac{\alpha}{c}, & D=2 \\ \left(\dfrac{\gamma}{38}\right)^2 \left(\dfrac{\alpha}{c}\right)^3, & D=3 \end{cases} \tag{2.14}$$

We shall consider the two-dimensional and three-dimensional cases separately.

1) $D = 3$. Using (2.8), (2.9), and (2.13) we obtain equations for τ at the transition point:

$$\tau_c^{1/2} = \frac{3}{38}\gamma \left(\frac{\alpha}{c}\right)^{3/2} \ln\left[\left(\frac{c}{\alpha}\right)^{1/2} \gamma^{-1/3} \tau_c^{1/6} \ln \frac{K\tau_c^{-1/2}}{T}\right], \tag{2.15}$$

where

$$K \sim 10 \frac{c^2}{\alpha} \gamma^{-1/3} \frac{a^2}{\xi^3}.$$

This expression gives the possibility of obtaining a criterion for applicability of the assumption of exponentially infrequent droplets [or, equivalently, of the requirement (2.14)]. Since the factor multiplying the inner logarithm is ~ 1, this criterion can be formulated as

$$K\tau_c^{-1/2}/T \gg 1. \tag{2.16}$$

First we consider positive $b \gg \frac{1}{5}\gamma\alpha^2(d/c^3)^{1/2}$, when the term with d in (2.5) and (2.6) can be neglected (everywhere below in such inequalities and order-of-magnitude estimates numerical factors ~ 1 are omitted; otherwise, we give a number that is only an estimate and gives an idea of the magnitude of this factor). Taking into account that the average

value of a^2 in (2.10) corresponds to $(\overline{E-\tau})\sim 10^{-2}\gamma^2(\alpha/c)^3$ [which, incidentally, gives us the right to replace $l=(c/\alpha E)^{1/2}$ by $\xi=(c/\alpha\tau)^{1/2}$], we obtain

$$a^2/\xi^3\sim 10^{-2}\gamma^2\alpha^4/bc^3,$$

and the condition (2.16) gives

$$b\ll\gamma\alpha^2/T. \tag{2.17}$$

(We recall that the region of dirty scaling[1,2] corresponds to $b\gtrsim\gamma^2\alpha/T$).

For smaller values of b, lying in the region $|b|\ll\frac{1}{5}\gamma\alpha^2(d/c^3)^{1/3}$, the criterion (2.16) [with allowance for (2.10)] leads to the expression

$$0.1T(d/c^3)^{1/3}\ll 1. \tag{2.18}$$

Here it is necessary to take into account that the replacement of the real system by randomly distributed average droplets also implies averaging over scales smaller than ξ. This leads to a renormalization of the coefficients of the powers of φ in (1.1). The corrections to the quantities d, b, and τ were calculated with logarithmic accuracy in Ref. 8:

$$d=d_0\left(1+\frac{25}{4\pi^2}\frac{T^2d_0}{c^3}\ln\frac{1}{\tau}\right)^{-1}, \tag{2.19}$$

$$b=b_0\left(1+\frac{25}{4\pi^2}\frac{T^2d_0}{c^3}\ln\frac{1}{\tau}\right)^{-2/5}. \tag{2.20}$$

The bare quantities b_0 and d_0 are taken from $\tau\sim 1$. It is to be understood that such renormalizations have been performed everywhere after (2.16). It can be seen that the condition (2.18) is always fulfilled. As regards the corrections to τ and γ, we can neglect them.

For values of b that are large in magnitude and negative it is necessary, as already indicated, to take into account the amount by which $E_{\min}$ deviates from τ [see (2.11)]; in this case the approximation of infrequent droplets becomes inapplicable. However, it will be shown below that for $-b\gg\gamma\alpha^2(d/c^3)^{1/2}$ a first-order transition occurs.

Thus, the transition occurs in the region of values of b defined by the inequalities

$$\begin{aligned}
&b_0\ll\gamma\alpha^2\frac{1}{T}\left(1+\frac{25}{4\pi^2}\frac{T^2d_0}{c^3}\ln\frac{1}{\tau_c}\right)^{2/5}, \quad b_0>0,\\
&|b_0|\ll\gamma\alpha^2\left(\frac{d_0}{c^3}\right)^{1/2}\left(1+\frac{25}{4\pi^2}\frac{T^2d_0}{c^3}\ln\frac{1}{\tau_c}\right)^{-1/10}, \quad b_0<0,\\
&\tau_c\approx\frac{3}{38}\gamma^2\left(\frac{\alpha}{c}\right)^3\ln\ln\left[\max\left(2\frac{\gamma\alpha^2}{T|b|};10\left(\frac{c^3}{T^2d}\right)^{1/2}\right)\right].
\end{aligned} \tag{2.21}$$

The formation of droplets gives rise to a correction to the specific heat of the homogeneous system. This correction is given by the expression

$$C=\frac{\alpha}{T}\frac{\partial}{\partial\tau}\left[\frac{1}{\mathcal{V}}\int\varphi^2\,d^Dx\right]=\frac{\alpha}{T}\frac{\partial}{\partial\tau}\left[\int_\tau^\infty a^2\rho(E)\,dE\right],$$

where $\mathcal{V}$ is the volume of the system. Thus, for $D=3$,

$$C\propto\frac{1}{T}\exp\left(-38\frac{\tau^{1/2}}{\gamma}\left(\frac{c}{\alpha}\right)^{3/2}\right)$$

$$\begin{cases}
\dfrac{\alpha^{1/2}c^{3/2}}{b_0}\dfrac{\tau^{1/2}}{\gamma}\left(1+\dfrac{25}{4\pi^2}\dfrac{T^2d_0}{c^3}\ln\dfrac{1}{\tau}\right)^{2/5}, & \gamma\alpha^2\left(\dfrac{d}{c^3}\right)^{1/2}\ll b\ll\gamma\alpha^2\\
\dfrac{c^{3/4}}{d_0^{1/2}\alpha^{3/4}}\dfrac{\tau^{1/4}}{\gamma^{3/2}}\left(1+\dfrac{25}{4\pi^2}\dfrac{T^2d_0}{c^3}\ln\dfrac{1}{\tau}\right)^{1/2}, & |b|\ll\gamma\alpha^2\left(\dfrac{d}{c^3}\right)^{1/2}
\end{cases} \tag{2.22}$$

[expressions for b and d are given in (2.19) and (2.20)].

Above, in the derivation of (2.17), we used the estimate $(\overline{E-\tau})\sim\gamma^2(\alpha/c)^3$; however, in (2.22) the more accurate expression $(\overline{E-\tau})\sim\tau^{1/2}\gamma(\alpha/c)^{3/2}$ was used. It was also taken into account that $\psi_4\sim 1/l^3$ and $\Psi_6\sim 1/l^6$ (l is the droplet radius, $l\sim\xi$). We recall once again that the expressions (2.22) are valid under the condition (2.14), which, as we have established, is fulfilled in the vicinity of the percolation-transition point. The expression (2.22) for the specific heat due to the droplets must be compared with the classical expression of the Landau theory (for $\tau<0$ and $b=\text{const}$)

$$C_0=\frac{1}{2T}\frac{\alpha^2}{(b^2+4\alpha|\tau|d)^{1/2}} \tag{2.23}$$

(which is valid even when fluctuations are taken into account, if renormalized quantities b and d are used) and with the fluctuational contribution to the specific heat (for $\tau>0$)

$$C_{fl}=\frac{1}{16\pi}\frac{\alpha^{3/2}}{c^{3/2}\tau^{1/2}}. \tag{2.24}$$

In the vicinity of the TCP under the conditions (2.17) and (2.18) the fluctuational contribution (2.24) is small in comparison with C_0 for the same value of $|\tau|$, while the contribution (2.22) of the droplets is comparable in order of magnitude to C_0 at $\tau\approx\tau_c$ [more precisely, $C(\tau_c)\sim C_0(-\tau_c)$]. It is easily verified that $(\partial C/\partial\tau)_{\tau=\tau_c}<0$, and, therefore, the maximum of the total specific heat C_{tot} lies at a temperature below T_c (see the figure).

2) $D=2$. The condition that the thermal fluctuations be small on scales smaller than ξ has the form

$$Td/\alpha^2c\ll\gamma \tag{2.25}$$

[where $\gamma\ll c/\alpha$; see (1.3)] and is a limiting condition for our results, if the condition (2.25) is not fulfilled it is necessary to consider the scaling behavior near the two-dimensional TCP, and this goes beyond the scope of the present paper. Assuming that (2.25) is fulfilled and proceeding as in the three-dimensional case, we find that a transition of the percolation type occurs for values of b in the range

$$\begin{aligned}
&b\ll\gamma\alpha^2/T, \quad b>0,\\
&-b\ll\alpha(\gamma d/c)^{1/2}, \quad b<0.
\end{aligned} \tag{2.26}$$

The contribution of the droplets to the specific heat is given by the expression

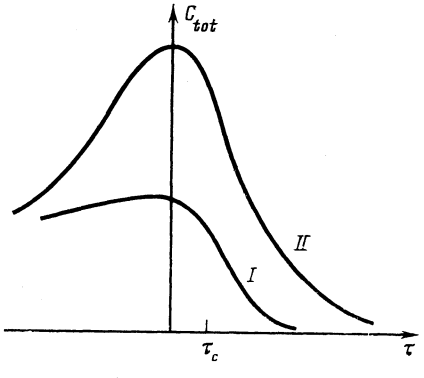

FIG. 1. Qualitative dependence of the specific heat on the temperature near the transition point τ_c; the branches I and II correspond to the cases $b\gg\gamma\alpha^2(d/c^3)^{1/2}$ and $|b|\ll\gamma\alpha^2(d/c^3)^{1/2}$, respectively, for $D=3$, and to analogous conditions for $D=2$ [see (2.27)].

$$C \propto \frac{1}{T} \exp\left(-11 \frac{\tau}{\gamma} \frac{c}{\alpha}\right) \begin{cases} \frac{\alpha^{3/2} c^{1/2}}{b\gamma^{1/2}} \tau^{1/2}, & \alpha \left(\frac{\gamma d}{c}\right)^{1/2} \ll b \ll \gamma\alpha^2 \frac{1}{T} \\ \frac{\alpha^{1/2} c}{a^{1/2}\gamma} \tau^{1/2}, & |b| \ll \alpha \left(\frac{\gamma d}{c}\right)^{1/2}. \end{cases} \tag{2.27}$$

The qualitative conclusions concerning the temperature dependence of the specific heat that were reached for the three-dimensional case are also valid here (see Fig. 1).

At the transition point itself we should not expect any appreciable anomaly in the behavior of the specific heat in either the two-dimensional or the three-dimensional case. Indeed, such an anomaly could arise only on account of the disregarded unbroken bonds between droplets. However, the derivative of the concentration of such bonds with respect to τ does not have a singularity at τ_c, and their energy T_c, by virtue of the conditions (2.21) or (2.26), is considerably smaller than the energy of a droplet. Strictly speaking, the phase transition in the system of interacting droplets is equivalent in symmetry to that in the initial system (with $b_{\text{eff}} \sim (\alpha^2/T)\gamma_{\text{eff}}$), and this should lead to singularities of the dirty-scaling type[1,2]; however, these singularities are scarcely observable, since they contain a small factor related to the proximity to the percolation situation.

In the ordered phase near the transition point τ_c such quantities as the average order parameter $\bar{\varphi}$ and correlation length L (the size of a bound cluster) have, according to percolation theory, a power-law dependence on the reduced concentration $x_0 = |(N - N_c)/N_c|$:

$$\varphi \propto x_0^{\beta}, \tag{2.28}$$

$$L \propto x_0^{-\nu}. \tag{2.29}$$

The region of this power behavior is in any case bounded by the inequality $x_0 \ll 1$. In our case, N depends on the temperature as in (2.13), and to a range of variation of $N(|\Delta N| \ll N_c)$ there corresponds a range of variation of $\tau(|\Delta\tau| \ll \tau_c)$. Then x in (2.22) and (2.23) can be replaced by $\Delta\tau/\tau_c$. The values of the critical indices for $D = 2, 3$ can be found, e.g., in Ref. 7.

3. DOMAIN-WALL STABILITY AND THE FIRST-ORDER TRANSITION

We turn now to the region of large negative values of b. We shall consider the question of the first-order transition in this region. Such a transition is characterized by the possibility that at the transition point phases with zero and nonzero order parameters coexist and, consequently, there is an interface between them. Thus, we seek the region of values of b for which a domain wall is stable against fluctuations induced by the disorder.

We show that the functional (1.1) can be reduced to the form of the Hamiltonian of a domain wall in the "dirty" Ising model with a random magnetic field.[9] We consider a planar domain wall in a pure system on a first-order transition line determined by the condition $\alpha\tau = 3b^2/16d$. We introduce the coordinates $x_1, \ldots, x_{D-1}$ in the plane of the wall, and z in the perpendicular direction. The order parameter φ_0 in this case depends only on z; minimizing (1.1) (for $\delta\tau \equiv 0$), we obtain an expression from which the order parameter can be determined:

$$-c\partial^2\varphi/\partial z^2 + \alpha\tau\varphi_0 + b\varphi_0^3 + d\varphi_0^5 = 0, \tag{3.1}$$

where $\varphi_0(z)$ satisfies the boundary conditions $\varphi_0(-\infty) = 0$ and $\varphi_0^2(+\infty) = 3|b|/4d$.

Multiplying (3.1) by $\partial\varphi/\partial z$, integrating, and substituting the result into (1.1) (where now the integration is performed over z), we obtain an expression for the surface tension of such a wall:

$$\sigma = \int c(\partial\varphi_0/\partial z)^2 dz = Ab^2(c/d^3)^{1/2}, \quad A \sim 1. \tag{3.2}$$

The estimate (3.2) has been obtained with allowance for the fact that the wall thickness satisfies $\xi \sim (dc)^{1/2}/|b|$.

We turn to a dirty system, in which $\delta\tau(\mathbf{x}, z) \not\equiv 0$. Now the boundary is deformed and deviates from a plane; however, if we assume that the curvature of the boundary at each point is not large, the order parameter can be represented in the form

$$\varphi = \varphi_0\left(\frac{z - \zeta(\mathbf{x})}{[1 + (\nabla_{\mathbf{x}}\zeta)^2]^{1/2}}\right), \tag{3.3}$$

where $\zeta(\mathbf{x})$ is the deviation of the boundary from a plane.

Substituting (3.3) into (1.1), we obtain

$$\mathcal{F} = \mathcal{F}_0 + \sigma \int [1 + (\nabla_{\mathbf{x}}\zeta(\mathbf{x}))^2]^{1/2} d^{D-1}x + \int {}^1/_2 \alpha\delta\tau(\mathbf{x}, z)\varphi^2 d^{D-1}x\, dz. \tag{3.4}$$

The second term in (3.4) is a random function of the variable $\zeta(\mathbf{x})$ determining the shape of the domain wall. When studying deformations of the wall on scales greater than its thickness ξ we can rewrite (3.4) approximately in the form

$$\mathcal{F} = \mathcal{F}_0 + \int [1 + (\nabla_{\mathbf{x}}\zeta(\mathbf{x}))^2]^{1/2} d^{D-1}x + \int \theta(z - \zeta(\mathbf{x}))h(x, z)\, d^{D-1}x\, dz, \tag{3.5}$$

where $\theta(y)$ is the Heaviside function and

$$h(\mathbf{x}, z) = \alpha\varphi^2(\infty)\delta\tau(\mathbf{x}, z), \tag{3.6}$$

$$\langle h(\mathbf{x}, z)h(\mathbf{x}', z')\rangle = \Delta\delta(\mathbf{x} - \mathbf{x}')\delta(z - z'), \quad \Delta \sim \gamma\alpha^2 b^2/d^2. \tag{3.7}$$

Thus, we have obtained the Hamiltonian of a domain wall in the Ising model with a random magnetic field.[9] In fact, the correlation function of the random quantities $h(\mathbf{x}, z)$ contains not only the right-hand side of (3.6) but also subsequent terms of the expansion, with derivatives of δ-functions, e.g., the term $\Delta_1\delta(\mathbf{x} - \mathbf{x}')\delta''(z - z')$, corresponding to the random potential $U(\mathbf{x}, \zeta(\mathbf{x}))$. In Appendix B it is shown that allowance for these terms does not affect the result, and, therefore, we shall not take them into account in what follows.

It has been implicitly assumed above that the introduction of impurities does not shift the transition from the curve of first-order phase transitions of the pure substance, determined by the parameters α, τ, b, and d; it has also been assumed that the introduction of impurities leaves the thick-

ness of the wall unchanged. For this, in any case, it is necessary that the fluctuation of the free energy of the wall in a volume of order ξ^D be small in comparison with its average value:

$$\alpha|b|\gamma\xi^{-D/2}/d \ll |b|^3/d^2,$$

i.e.,

$$|b| \gg \gamma\alpha^2(d/c^3)^{1/2}, \quad D=3, \tag{3.8a}$$

$$|b| \gg \gamma^{1/2}\alpha(d/c)^{1/2}, \quad D=2. \tag{3.8b}$$

We note that the conditions (3.8) are opposite to the conditions for the existence of the percolation transition investigated in Sec. 2. Below, we shall convince ourselves that the conditions (3.8) are sufficient for stability of the domain boundaries.

We shall make use of the results of Refs. 9–11. In these papers it was shown that in the Ising model in a random field for $D = 3$ ferromagnetic ordering is observed and, consequently, in the presence of antiperiodic boundary conditions a domain wall exists. Here it is required that

$$\langle(\nabla_x\xi(\mathbf{x}))^2\rangle \ll 1, \tag{3.9}$$

which implies that the deformations are slow. The case $D = 2$ will be considered separately below.

For the mean-square wall deformations $w_{T,h}^2(L) = \langle(\zeta(L) - \zeta(0))^2\rangle$ induced by thermal fluctuations and the disorder, respectively, we have[10,11]

$$w_T(L) \approx \frac{1}{2\pi}\left(\frac{T}{\sigma}\right)^{1/2}\begin{cases} L^{(3-D)/2}, & D<3 \\ \ln^{1/2}(L/\xi), & D=3 \end{cases}, \tag{3.10}$$

$$w_h(L) \sim (\Delta/\sigma^2)^{1/3}L^{(5-D)/3}. \tag{3.11}$$

We note that it makes sense to speak of fluctuations of the wall only if these fluctuations exceed the wall thickness ξ. Since $w_{T,h}$ increases with L less rapidly than L, the condition (3.9) is equivalent to the condition

$$\mathscr{L} \gg \max(\xi,\ w_T(\mathscr{L})) \equiv \xi_{eff}, \tag{3.12}$$

where the length $\mathscr{L}$ is determined by the relation

$$w_h(\mathscr{L}) \sim \xi_{eff}. \tag{3.13}$$

We define $\mathscr{L}_{T,h}$ by the condition

$$w_{T,h}(\mathscr{L}_{T,h}) \sim \xi. \tag{3.14}$$

Thus,

$$\mathscr{L}_T = \xi \exp\,[BT^{-1}(d/c^3)^{-1/2}], \quad B \sim 10, \tag{3.15}$$

$$\mathscr{L}_h = [d^{1/2}c^{5/2}/\gamma\alpha^2|b|]^{1/3} = \xi[|b|/\gamma\alpha^2(d/c^3)^{1/2}]^{1/3}. \tag{3.16}$$

It follows from (3.16) that, if we neglect the thermal fluctuations of the wall, the condition for the existence of a first-order transition takes the form $|b| \gg \gamma\alpha^2(d/c^3)^{1/2}$, i.e. [see (2.19) and (2.20)],

$$|b_0| \gg \gamma\alpha^2\left(\frac{d_0}{c^3}\right)^{1/2}\left(1 + \frac{25}{4\pi^2}\frac{T^2 d_0}{c^3}\ln\frac{1}{\tau}\right)^{-1/10}. \tag{3.17}$$

In fact, because of the smallness of the power 1/10, we can always neglect the renormalization factor.

The possibility of neglecting thermal fluctuations is due to the large magnitude of the exponent in (3.15) [see (2.19)]. A criterion for a first-order transition that practically coincides with (3.17) was obtained in Ref. 12, but there the role of renormalizations and thermal fluctuations was not discussed.

We now consider a two-dimensional system. Although long-range order does not exist in the two-dimensional Ising model in a random field, the correlation length is exponentially large (see, e.g., Ref. 11): $r_c \propto \xi\exp[(\sigma^2/\Delta)^{3/2}]$ [the large magnitude of the exponent is ensured by the inequality (3.8b)]; therefore, we shall also give estimates for $D = 2$.

By analogy with (3.13) we introduce two lengths $\mathscr{L}_1$ and $\mathscr{L}_2$, defined by the relations

$$w_h(\mathscr{L}_1) \sim \xi, \quad w_h(\mathscr{L}_2) \sim w_T(\mathscr{L}_2),$$

or

$$\mathscr{L}_1 \sim [d^{1/2}c^{5/2}/\gamma\alpha^2|b|]^{1/3}, \tag{3.18}$$

$$\mathscr{L}_2 \sim [d^{3/4}T^{3/2}c^{1/4}/\gamma\alpha^2|b|]^{2/3}. \tag{3.19}$$

Two cases are possible.

a) $\mathscr{L}_1 \gg \mathscr{L}_2$. This means that thermal fluctuations can be neglected and the inequality itself can be rewritten as follows:

$$T[d^2/\gamma\alpha^2|b|c^2]^{1/3} \ll 1. \tag{3.20}$$

In this case the criterion (3.12) gives

$$|b| \gg \gamma^{1/2}\alpha(d/c)^{1/2}. \tag{3.21}$$

As in the three-dimensional case, we have obtained a criterion opposite to (2.26). Comparing (3.20) and (3.21) with (2.25), we discover that the condition (3.20) is less stringent than (2.25); on the boundary of the region (3.21) the condition (3.20) goes over into (2.25).

b) $\mathscr{L}_1 \ll \mathscr{L}_2$, or

$$T[d^2/\gamma\alpha^2|b|c^2]^{1/3} \gg 1. \tag{3.20'}$$

Thermal fluctuations of the wall become important, and the effective thickness of the wall is equal to w_T. In this case (3.12) again formally leads to the expression (3.21), but now renormalized quantities appear in it [in the case (a) we neglected the renormalizations, so that (3.20) and (3.21) were written, in fact, for the bare values].

4. PINNING OF DOMAIN BOUNDARIES AND HYSTERESIS NEAR THE FIRST-ORDER TRANSITION

Pinning of the domain wall at impurities should lead to the appearance of long-lived metastable states near the first-order transition point. In the given case there do not exist critical nuclei of the new phase, increase of which would lead to rapid decay of the metastable state. The dynamics of the transition is determined by the motion of the walls for which the pinning force is smaller than the "driving force"—the difference of the densities of the thermodynamic potentials of the two phases. Therefore, hysteresis should be observed in the τ dependence of the thermodynamic quantities in the neighborhood of the transition point.

For the pinning force the following expressions were obtained in Refs. 13 and 14:

$$E_c = \sigma(\Delta/\sigma^2)^{2/(5-D)}\xi^{-(1+D)/(5-D)}, \quad T \ll T_0, \tag{4.1}$$

$$E_c=\Delta/T, \quad T\gg T_0, \tag{4.2}$$

$$T_0=\sigma(\Delta/\sigma^2)^{(3-D)/(5-D)}\xi^{(1+D)/(5-D)}. \tag{4.3}$$

For $D=3$ we have $T_0=(c^3/d)^{1/2}$ (in the renormalized quantities) and we are interested only in the case $T\ll T_0$.

These expressions make it possible to estimate the width $\Delta\tau$ of the hysteresis by comparing E_c and the difference ΔF of the free-energy densities of the two phases. Assuming that the temperature shift $\Delta\tau$ is small, namely, that

$$\alpha\Delta\tau d\ll b^2, \tag{4.4}$$

we obtain for ΔF

$$\Delta F={}^3/_8\alpha|b|\Delta\tau/d \tag{4.5}$$

and from the equality $E_c=\Delta F$ we find $\Delta\tau$:

$$\Delta\tau\approx\gamma\alpha|b|(1/dc^3)^{1/2}. \tag{4.6}$$

It can be seen that (4.4) and (4.6) lead to the condition $|b|\gg\gamma\alpha^2(d/c^3)^{1/2}$, so that the assumption that $\Delta\tau$ is small is fulfilled in the entire region of the first-order transition.

In the case $D=2$ our results are valid in the region in which the renormalization can be neglected, i.e., under the condition (3.20), which again coincides with $T\ll T_0$. Now,

$$\Delta\tau\approx(\alpha\gamma^2b^2/dc^2)^{1/3}. \tag{4.7}$$

We shall return to (4.6). We recall that the discontinuities of such quantities as the compressibility and entropy at a first-order transition are proportional to b. For example, for the entropy we have

$$\Delta S={}^1/_2\alpha\varphi^2/T={}^3/_8\alpha|b|/Td. \tag{4.8}$$

As can be seen from (4.6), in three-dimensional systems a linear relation should be observed between the width $\Delta\tau$ of the hysteresis and the discontinuity of the entropy (and of other quantities experiencing a discontinuity at a first-order transition):

$$\Delta\tau\sim\gamma(T^2d/c^3)^{1/2}\Delta S. \tag{4.9}$$

Strictly speaking, the coefficient of ΔS in (4.9) can have a weak dependence on ΔS because of the renormalization of d [see (2.19), in which it is necessary to use $\tau=3b^2/16\alpha d$], but this effect is unlikely to be observable.

In the two-dimensional case, provided that the thermal fluctuations are small, we obtain

$$\Delta\tau\propto(\Delta S)^{2/3}. \tag{4.10}$$

Once again we stress that we can speak only nominally of a first-order transition in a dirty two-dimensional system, since on exponentially large scales a domain boundary always loses its stability, so that there is a finite thermodynamic "smearing" of the transition. In practice, however, this effect should almost always be unobservable against the background of the hysteresis induced by the pinning of the boundaries.

5. CONCLUSION

We have shown that in the neighborhood of a TCP of a disordered system there is always a region of a percolation-type phase transition associated with the formation and interaction of localized droplets—a transition of a type known previously for superconductors[3] and ferromagnets[4] with strong disorder. We have obtained estimates for the region of existence of such a transition [see (2.21) and (2.26)] in three-dimensional and two-dimensional systems. The contribution of the droplets to the specific heat of the system near the transition point has been found [see (2.22) and (2.27)].

The condition for the occurrence of a first-order phase transition, associated with the stability of the domain boundaries between the ordered and disordered phases, has been found. This condition turns out to be complementary to the condition for a percolation transition, and the intermediate region between these two regimes has not yet been investigated. The width $\Delta\tau$ of the region of hysteresis near the first-order transition, associated with the pinning of domain boundaries at inhomogeneities, has been estimated. In a three-dimensional system $\Delta\tau\propto\Delta S$ while in a two-dimensional system $\Delta\tau\propto(\Delta S)^{2/3}$, where ΔS is the entropy discontinuity in the first-order transition.

Thus, in the two-dimensional case the relative "smearing" of the transition grows as the discontinuity decreases.

APPENDIX A

To determine the energy of interaction of the droplets we shall consider two droplets located on the x_1 axis at equal distances from the coordinate origin; R_{ij} is the distance between the droplets. Two variants are possible depending on whether the order parameters in the droplets have the same or opposite signs:

$$\varphi_\pm=\varphi_i\pm\varphi_j. \tag{A1}$$

Then from (1.1) we obtain

$$-V_{ij}=\mathscr{F}_+-\mathscr{F}_-=2\int_{x_1>0}[2c\nabla\varphi_i\nabla\varphi_j+2\alpha(\tau+\delta\tau(\mathbf{x}))\varphi_i\varphi_j + 2b\varphi_i\varphi_j^3+2d\varphi_i\varphi_j^5+O(\varphi_i)]d^Dx. \tag{A2}$$

We assume that droplet j is in the half-space $x_1>0$. Here φ_i falls off exponentially:

$$\varphi_i=\begin{cases}\varphi_{i0}\dfrac{\xi}{|\mathbf{r}-\mathbf{r}_i|}\exp\left(-\dfrac{|\mathbf{r}-\mathbf{r}_i|}{\xi}\right), & D=3\\ \varphi_{i0}\left(\dfrac{\xi}{|\mathbf{r}-\mathbf{r}_i|}\right)^{1/2}\exp\left(-\dfrac{|\mathbf{r}-\mathbf{r}_i|}{\xi}\right), & D=2\end{cases} \tag{A3}$$

where $\xi=(c/\alpha\tau)^{1/2}$, $|\mathbf{r}-\mathbf{r}_i|\gg\xi$ [the droplet size $l=\xi(\tau/E)^{1/2}\lesssim\xi$], and $\varphi_{i0}\sim a_i/\xi^{D/2}$ is the value of the order parameter on the boundary of the drop.

Thus, we can write

$$V_{ij}\approx-4\int_{x_1>0}c\,\mathrm{div}(\varphi_i\nabla\varphi_j)d^Dx+4\int_{x_1>0}\varphi_i(-c\nabla^2\varphi_j+\alpha(\tau+\delta\tau(\mathbf{x}))\varphi_j + b\varphi_j^3+d\varphi_j^5)d^Dx=-4c\int_{x_1>0}\mathrm{div}(\varphi_i\nabla\varphi_j)d^Dx. \tag{A4}$$

Finally, we obtain in the three-dimensional case

$$V_{ij}=8\pi c\varphi_{0i}\varphi_{0j}\frac{\xi^2}{R_{ij}}\exp\left(-\frac{R_{ij}}{\xi}\right), \quad D=3, \tag{A5}$$

and in the two-dimensional case, for $R_{ij}\gg\xi$,

$$V_{ij}\approx4\pi^{1/2}c\varphi_{0i}\varphi_{0j}(\xi/R_{ij})^{1/2}\exp(-R_{ij}/\xi). \tag{A6}$$

APPENDIX B

The contribution to the Hamiltonian from the random potential is equal to

$$\int V(x, \zeta(\mathbf{x}))\, d^{D-1}x,$$

where

$$V(x, \zeta(x)) \approx \int_{\zeta(x)-1/2\xi}^{\zeta(x)+1/2\xi} \frac{1}{2}\alpha\delta\tau(\mathbf{x}, z)\varphi^2(\mathbf{x}, z)\, dz,$$

and the correlator Δ_1 can be estimated as

$$\Delta_1 \approx \xi^2\Delta. \tag{B1}$$

For case $D = 3$ the quantity w_V was calculated in Ref. 15:

$$w_V \sim \left(\frac{\Delta\xi}{\sigma^2}\right)^{1/4} L^{1/2}. \tag{B2}$$

It can be seen immediately that w_V increases more slowly with L than does w_h (the same is also true in the two-dimensional case), and on large scales it is certainly possible to consider only the random field. The estimates in Sec. 3 were based on an analysis of fluctuations on small scales; applying the arguments of this section to (B2) and (3.10) we again obtain the condition (3.17) for a first-order transition.

For $D = 2$, following Refs. 16 and 17, we can obtain

$$w_V \sim (\Delta\xi^2/\sigma T)^{1/2} L^{2/3}, \tag{B3}$$

which is valid at sufficiently high temperatures. The region of these temperatures can be estimated using the results of Ref. 13, in which it was shown that at $T_0 \sim (\gamma\alpha^2|b\,|c^2/d^2)^{1/3}$ a change occurs in the dependence of the pinning force on the parameters of the model [as in (4.1)–(4.3)]. Thus, (B3) is valid for $T \gtrsim T_0$. We are interested mainly in the opposite case, when renormalizations can be neglected. In this case it is clear that $w_V \propto L^{2/3}$ as in (B3) (see Ref. 18), but the coefficient should change. Knowing the pinning force in this case,[13] it is not difficult to estimate this coefficient

$$w_V \sim (\Delta^2\xi^3/\sigma^4)^{1/4} L^{1/2}, \quad T < T_0. \tag{B4}$$

Here we have used the fact that the pinning force is determined by fluctuations on the smallest scales; the same estimate is obtained by substituting T_0 for T in (B3). Repeating the arguments of Sec. 3, we again obtain (3.21) as the criterion for a first-order transition.

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