Superconductivity of quasiperiodic layer structures

A. Yu. Kitaev and L. S. Levitov

L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

(Submitted 6 July 1988)
Zh. Eksp. Teor. Fiz. **95**, 311–321 (January 1989)

The critical behavior of the thermodynamic quantities near the superconducting-transition point for a quasiperiodic layer structure is studied. An explanation is given of the unusual power dependence (discovered experimentally by Karkut, Triscone, Ariosa, and Fischer) of the critical magnetic field H_{c2} on $T_c - T$. The critical indices are calculated in the Ginzburg-Landau approximation.

INTRODUCTION

After the discovery of quasicrystals great interest has arisen in various natural and artificial quasiperiodic structures. In Ref. 1 the properties of a quasiperiodic superlattice *SNSNSNS*..., consisting of layers of vanadium (superconductor) and molybdenum (normal metal), were studied. The thicknesses d_n of the layers of normal metal were equal, and the superconducting layers had two different thicknesses, d_{s1} and d_{s2}, which alternated by the Fibonacci rule *ABAABABAABAAB*... (Ref. 2). Near the transition point a nonlinear dependence of H_{c2} on $\tau = (T_c - T)/T_c$ was discovered, and the estimate given in Ref. 1 for the coherence length ξ_0 shows that $\xi_0 \gg d_n$ (strong coupling between the layers). If we plot the results of Ref. 1 on logarithmic scales ($\log \tau$, $\log H_{c2}$), we obtain a straight line with slope 0.74. This critical index lies between 1.0 (for a homogeneous superconductor or periodic system of strongly coupled layers) and 0.5 (for one layer).

Since the size of a nucleus of the superconducting phase in a weak field is much greater than the lattice constant, the reason for the appearance of the anomalous critical index is the quasiperiodicity of the structure. Other thermodynamic quantities (the order parameter ψ, specific heat C, correlation length ξ, London length δ_L, etc.) should also have a power dependence on τ with unusual power exponents.

Some of the results of this work have been published in Ref. 3.

1. ANALYSIS OF THE INITIAL HAMILTONIAN

We shall consider the problem in the Ginzburg-Landau approximation. As is well known, this approximation is applicable if the correlation length ξ_0 of the electrons of a Cooper pair is much shorter than the characteristic length scale of the spatial nonuniformity of the problem. In our case this length scale is determined by the layer thicknesses d_n, d_{s1}, and d_{s2}; in the experiment of Ref. 1 a structure having $d_n = 15$ Å, $d_{s1} = 58$ Å, and $d_{s2} = 30$ Å was studied. According to the estimate of Ref. 1, $\xi_0 \approx 120$ Å, i.e., $\xi_0 > d_{s1}, d_{s2}, d_n$. This implies that the Ginzburg-Landau approximation is quantitatively inapplicable. It is possible, however, to hope that, as is usually the case, it will give a correct qualitative picture.

The free-energy functional has the form

$$\mathcal{F}[\psi]=\int\left[\frac{1}{4m}\left|\left(\nabla-i\frac{2e}{c}\mathbf{A}\right)\psi\right|^2+(U(x)-\tau)|\psi|^2+a|\psi|^4\right]dr, \tag{1.1}$$

in which the function $U(x)$ takes two values: $U(x) = V_n > 0$ in a region of normal metal, and $U(x) = V_s < 0$ in a region of superconducting metal, and τ is proportional to $T_c - T$. In the Landau gauge $A_y = Hx$, $A_x = A_z = 0$ the problem reduces to a one-dimensional problem in the usual way:

$$\psi(r)=\varphi(x)\exp(ipy), \tag{1.2}$$

$$\mathcal{F}[\varphi]=\int\left[\frac{1}{4m}\left(\frac{2eH}{c}\right)^2\varphi^2(x-x_0)^2+\frac{1}{4m}\left|\frac{d\varphi}{dx}\right|^2+(U(x)-\tau)|\varphi|^2+a|\varphi|^4\right]dx,$$

$$x_0=cp/2eH.$$

Near the transition point the function $\varphi(x)$ is a linear combination of eigenfunctions $\varphi_\varepsilon(x)$ of the Hamiltonian

$$\mathcal{H}=-\frac{1}{4m}\frac{d^2}{dx^2}+U(x) \tag{1.3}$$

with energies ε near the lower boundary $\varepsilon_{\min}$ of the spectrum of the operator (1.3). By adding a constant to $U(x)$ we can ensure that $\varepsilon_{\min}$ becomes equal to zero, i.e., that the transition in zero field occurs at $\tau = 0$ [precisely such a choice of $U(x)$ is assumed in (1.1)]. In the case of a superconducting structure the role of ε is played by τ.

The spectrum of such an operator was studied in detail in Ref. 4. We shall give certain definitions and results from Ref. 4 that are necessary for what follows.

Let $\varphi(x)$ be an arbitrary solution of the equation $\mathcal{H}\varphi(x) = \varepsilon\varphi(x)$, where $\mathcal{H}$ is the operator (1.3). The transfer matrix of a layer is the 2×2 matrix that expresses φ and $\varphi' = d\varphi/dx$ on the right side of the layer (the corresponding values are indicated by the subscript r) in terms of the values of these quantities on the left side of the layer (with subscript l):

$$\begin{pmatrix}\varphi_r\\ \varphi_r'\end{pmatrix}=T\begin{pmatrix}\varphi_l\\ \varphi_l'\end{pmatrix}. \tag{1.4}$$

The transfer matrix T depends on ε.

We shall consider finite segments A, AB, ABA, $ABAAB$, *ABAABABA*,..., with lengths 1, 2, 3, 5, 8,... (Fibonacci numbers). Each succeeding segment is the combination of the two preceding segments, with their positions exchanged. For the transfer matrix one obtains the recursion formula

$$T_{k+2}=T_kT_{k+1}, \qquad T_0=B(\varepsilon), \qquad T_1=A(\varepsilon), \tag{1.5}$$

where $A(\varepsilon)$ and $B(\varepsilon)$ are the transfer matrices of the elements from which the superlattice is constructed. If we introduce the notation $B_k = T_k$, $A_k = T_{k+1}$, the formula

 0038-5646/89/010176-06$04.00

(1.5) reduces to the following transformation R for the matrix pairs (A_k, B_k):

$$R: \quad (A_k, B_k) \to (A_{k+1}, B_{k+1}), \quad A_{k+1}=B_kA_k, \quad B_{k+1}=A_k. \tag{1.6}$$

The quantity $4J-2=\mathrm{Tr}(ABA^{-1}B^{-1})$ is an invariant of this transformation.[5] The scaling properties of the spectrum and wave functions depend on this quantity (see Ref. 4). In the case under consideration the transfer matrices of the normal and superconducting layers at $\tau=0$ can be written in the form

$$T_n=\begin{pmatrix} \mathrm{ch}(\varkappa_n d) & \varkappa_n^{-1}\mathrm{sh}(\varkappa_n d) \\ \varkappa_n \mathrm{sh}(\varkappa_n d) & \mathrm{ch}(\varkappa_n d) \end{pmatrix}, \qquad T_s=\begin{pmatrix} \cos(\varkappa_s d) & \varkappa_s^{-1}\sin(\varkappa_s d) \\ -\varkappa_s \sin(\varkappa_s d) & \cos(\varkappa_s d) \end{pmatrix}, \tag{1.7}$$

where $\varkappa_n=(4mV_n)^{1/2}$, $\varkappa_s=(-4mV_s)^{1/2}$, and d is the layer thickness.

The superlattice consists of alternating elements NS_1 and NS_2, where S_1 and S_2 are superconducting layers with thicknesses d_{s1} and d_{s2}. The transfer matrices of these elements can be expressed as follows:

$$B=T_{s1}T_n, \quad A=T_{s2}T_n. \tag{1.8}$$

Hence we find the value of the invariant J:

$$J=1+\left[\frac{\varkappa_n^2+\varkappa_s^2}{2\varkappa_n\varkappa_s}\mathrm{sh}(\varkappa_n d_n)\sin(\varkappa_s(d_{s1}-d_{s2}))\right]^2. \tag{1.9}$$

The quantity J can vary from 1 to $+\infty$. For a periodic superlattice ($d_{s1}=d_{s2}$) J is equal to unity.

The transformation R of pairs of matrices can be reduced to the following transformation M for the triplets (x,y,z) of their traces ($x=\frac{1}{2}\mathrm{Tr}AB$, $y=\frac{1}{2}\mathrm{Tr}A$, $z=\frac{1}{2}\mathrm{Tr}B$):

$$M: \quad x\to 2xy-z, \quad y\to x, \quad z\to y. \tag{1.10}$$

The invariant J can be expressed in terms of x, y, and z as follows:

$$J=x^2+y^2+z^2-2xyz. \tag{1.11}$$

As shown in Ref. 2, all boundaries of the allowed bands of the Hamiltonian (1.3), including the lower boundary of the spectrum, correspond to two fixed points Q and Q' of the transformation M^2:

$$\begin{aligned} &Q: \quad x=\tfrac{1}{2}(1+t), \quad y=\tfrac{1}{2}(1+t^{-1}), \quad z=\tfrac{1}{2}(1+t); \\ &Q': \quad x=\tfrac{1}{2}(1+t^{-1}), \quad y=\tfrac{1}{2}(1+t), \quad z=\tfrac{1}{2}(1+t^{-1}); \\ &\qquad M(Q)=Q', \quad M(Q')=Q. \end{aligned} \tag{1.12}$$

Here

$$t=\tfrac{1}{2}[p-2+((p-4)p)^{1/2}], \quad p=\tfrac{1}{2}[3+(16J+9)^{1/2}]. \tag{1.13}$$

We shall make this statement more precise. Let the initial matrices A_0 and B_0 correspond to a band edge. Then under repeated application of the mapping M to the point (x_0,y_0,z_0) the sequence of points (x_n,y_n,z_n) converges to the sequence $Q,Q',Q,Q',Q,\ldots$.

Near the lower boundary $\varepsilon=\varepsilon_{\min}=0$ the spectrum has the form

$$\varepsilon(n)=n^{\beta}F(\lg_{\Phi} n), \tag{1.14}$$

where $\Phi=\zeta^2$ [$\zeta=(5^{1/2}+1)/2$ is the golden section], and F is a certain periodic function with period 1. For the parameter n in the expression (1.14) one takes the number of states per unit length with energy less than ε:

$$n(\varepsilon)=\lim_{L\to\infty}[N_L(\varepsilon)/L]. \tag{1.15}$$

Here $N_L(\varepsilon)$ is the number of states with energy less than ε in a finite system consisting of L lattice elements. The power β in the expression (1.14) is obtained from the linear part of the mapping M^2 at the points q, q':

$$\beta=\lg_{\Phi}\lambda, \tag{1.16}$$

where $\lambda=\{[(2p+1)^{1/2}+(2p-3)^{1/2}]/2\}^2$ is the largest eigenvalue of the mapping M^2 in the neighborhood of the point Q.

2. THE $H_{c2}(T)$ DEPENDENCE

We shall use the results described above to calculate the critical index γ of the field H_{c2}. Corrections to scaling that are periodic in the logarithm of the length scale (like the function F in the expression (1.14) for the spectrum) will be neglected. Constant (i.e., τ-independent) factors in the formulas will be omitted.

Let the characteristic size of the wave function $\varphi(x)$ of a superconducting nucleus in a weak magnetic field be equal to $L\gg l$ (length is measured as the number of NS layers). Then the fraction of states making a contribution to φ is equal to $n\sim L^{-1}$, which corresponds to $N_L\approx 1$ in the formula (1.15). The corresponding energy is equal to

$$\varepsilon(L)\sim n^{\beta}\sim L^{-\beta}. \tag{2.1}$$

If we neglect the term $a|\varphi|^4$, the free energy of a nucleus is equal to the sum of the "kinetic" term and the diamagnetic term:

$$\mathcal{F}\sim(\varepsilon(L)+H^2L^2-\tau)\int|\varphi|^2\,dx. \tag{2.2}$$

Minimizing the expression $L^{-\beta}+H^2L^2$ with respect to L gives the optimum size L_H of a nucleus:

$$L_H\sim H^{-2/(2+\beta)}, \quad \varepsilon(L_H)\sim H^2L^2\sim H^{2\beta/(2+\beta)}. \tag{2.3}$$

The transition temperature in a magnetic field is determined from the condition

$$\varepsilon(L_H)+H^2L_H^2-\tau=0. \tag{2.4}$$

Hence we can express τ:

$$\tau\sim H^{2\beta/(2+\beta)}. \tag{2.5}$$

We obtain

$$H\sim\tau^{\gamma}, \quad \gamma=\tfrac{1}{2}+\beta^{-1}. \tag{2.6}$$

The index γ varies from 1 to 1/2 when J varies from 1 to $+\infty$. The case $J=1$ corresponds to a periodic structure. In this situation the system behaves near T_c as a homogeneous, albeit anisotropic, superconductor. The order parameter varies over a single period of the structure, and on length scales greater than one layer thickness but smaller than L_H. The function $\varphi(x)$ is constant. The case $J=+\infty$ corresponds to two-dimensional behavior. The superconductivity is nucleated in one layer and penetrates weakly into the neighboring layers (J is large when the coupling between the layers is small in comparison with the difference of the values of T_c in different layers).

For intermediate values of J the dependence of the order parameter on the coordinates includes oscillations on all length scales (see Fig. 1). It is this which gives rise to behav-

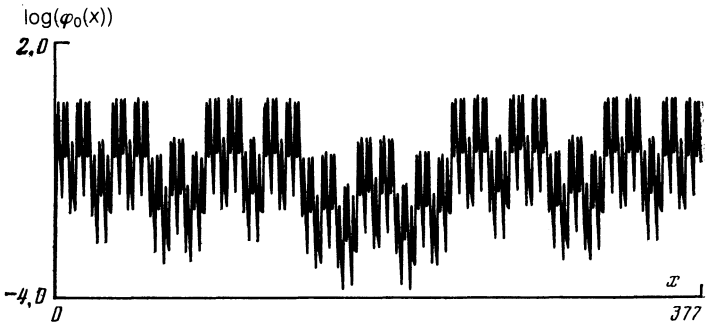

FIG. 1. The function $\varphi_0(x)$—the ground-state eigenfunction of the Hamiltonian (1.3) ($J = 10$).

ior intermediate between two-dimensional and three-dimensional.

The scaling in the case under consideration differs from the usual scaling, since it admits only a discrete group of scale transformations

$$L \to L\Phi^k \quad (k - \text{integer}). \tag{2.7}$$

Under such transformations, accompanied by simultaneous renormalization of the energy

$$\varepsilon \to \varepsilon\lambda^{-k}, \tag{2.8}$$

the spectrum (1.14) remains unchanged. The power dependence of $H(\tau)$ should have corrections periodic in the logarithm of the size L_H of the nucleus:

$$H = \tau^{\gamma} G(\lg_{\Phi} L_H), \tag{2.9}$$

where $L_H(\tau) = \tau^{-1/\beta}$ and G is a periodic function with period equal to unity.

We have considered the case when the field H is parallel to the layers. We shall now determine at what angle θ_c between the direction of the magnetic field and the plane of the layers the dependence $\tau(H)$ changes substantially. For the case of a periodic superlattice such a problem has already been considered in Ref. 6. The explanations accompanying the calculations will be given in terms of the motion of one particle in a potential $U(x)$ and a magnetic field H, since the equations in this case are the same.

Let the field deviate in the x direction through angle θ. Then, as before, the variable y can be eliminated. Neglecting the term $a|\varphi|^4$, we arrive at the problem of the spectrum of the following Hamiltonian:

$$\mathcal{H}' = \mathcal{H} - \frac{1}{4m}\frac{\partial^2}{\partial z^2} + \frac{e^2H^2}{mc^2}(x - \theta z - x_0)^2, \tag{2.10}$$

where $\mathcal{H}$ is the previous Hamiltonian (1.3).

For $\theta \ll 1$ the motion along the z axis occurs much more slowly than that along the x axis. We can apply the adiabatic approximation, i.e., in considering the motion along the x axis we can regard the variable z as a parameter.

The Hamiltonian

$$\mathcal{H}^* = \mathcal{H} + \frac{e^2H^2}{mc^2}(x - x_1)^2, \quad x_1 = x_0 + \theta z, \tag{2.11}$$

describing the motion along the x axis for $z = \text{const}$ has smallest eigenvalue $\varepsilon(x_1) \approx \varepsilon(L_H) \sim H^{1/\gamma}$. Upon variation of x_1 the function $\varepsilon(x_1)$ changes by an amount of the order of itself, since there is no other characteristic energy scale. The characteristic distance over which this change occurs is equal to the size of the wavefunction of the nucleus:

$$L_H \sim H^{-2/(2+\beta)} = H^{1/(2\tau)-1}. \tag{2.12}$$

Motion along the z axis is described in the adiabatic approximation by the Hamiltonian

$$\mathcal{H}_z = -\frac{1}{4m}\frac{\partial^2}{\partial z^2} + \varepsilon(\theta z + x_0). \tag{2.13}$$

The neighborhood of the absolute minimum of the function $\varepsilon(\theta z + x_0)$ is a potential well with depth of order $\varepsilon(L_H)$ and width of order $L_z = L_H/\theta$. Important changes of the lower part of the spectrum in such a potential occur when the kinetic energy of the motion along the z axis becomes comparable to the depth of the well: $L_{z_c}^{-2} \sim \varepsilon(L_H)$, i.e., $\theta_c^2 H^{2-1/\gamma} \sim H^{1/\gamma}$, or

$$\theta_c \sim H^{(1-\gamma)/\gamma} \sim \tau^{1-\gamma}. \tag{2.14}$$

As $\tau \to 0$ the angle θ_c tends to zero, i.e., the system becomes sensitive to small deviations of the field from the plane of the layers.

3. DEPENDENCE OF THE ORDER PARAMETER ON τ. CRITICAL BEHAVIOR OF THE SPECIFIC HEAT AND LONDON LENGTH

In this section we shall consider the neighborhood of the transition point in zero magnetic field. The behavior of the specific heat and London length is determined in this case by the volume average $\langle|\varphi|^2\rangle$ of the square of the order parameter. This quantity cannot be calculated using the linearized Ginzburg-Landau equation, since $\langle|\varphi|^2\rangle$ is determined by the competition of the negative second-order term and the positive fourth-order term $a|\varphi|^4$ in the functional (1.2).

It is clear that near the transition point, i.e., for $\tau \ll 1$, the dependence of the order parameter $\varphi(x)$ on the coordinates should resemble the coordinate dependence of the nucleus $\varphi_0(x)$ that appears at $\tau = 0$. The function $\varphi_0(x)$, as already noted, is the eigenfunction corresponding to zero eigenvalue of the operator (1.3). This function has strong oscillations on all scales (see Fig. 1), and this leads to nontrivial effects for the order parameter, analogous to those which arise near a second-order phase-transition point as a result of strong thermal fluctuations.[7] For $\tau \neq 0$ the fourth-order term in the Ginzburg-Landau equation cuts off the oscillations of $\varphi(x)$ on all sufficiently long scales, and makes the superconductor uniform on these scales. Thus, the following picture of the superconducting state arises. There exists a characteristic length scale L_τ, analogous to the correlation length in the theory of phase transitions[7] and such that over distances shorter than L_τ the order parameter oscillates

as the function $\varphi_0(x)$ while over distances greater than L_τ oscillations are absent and the superconductor is uniform.

We shall estimate the quantity L_τ. When $\varphi(x)$ deviates from $\varphi_0(x)$ over distances of order L a positive correction to the energy, of order $\varepsilon(L)\int|\varphi(x)|^2dx$, arises. This must be compared with the term $-\tau\int|\varphi(x)|^2dx$ in the Ginzburg-Landau equation. If $\varepsilon(L)>\tau$, which is true for $L\ll\tau^{-1/\beta}$, the difference of τ from zero can be neglected and we return to the case $\tau=0$, when $\varphi(x)=\varphi_0(x)$. On the other hand, for $L\gg\tau^{-1/\beta}$ the gradient term $\varepsilon(L)\int|\varphi(x)|^2dx$ is unimportant in comparison with $-\tau\int|\varphi(x)|^2dx$. Thus, we find

$$L_\tau\sim\tau^{-1/\beta}. \tag{3.1}$$

Another justification of the validity of this relation is the following. The order parameter can be expanded in eigenfunctions of the operator (1.3):

$$\varphi(x)=\sum_n a_n\varphi_n(x) \tag{3.2}$$

[n is the state label, defined as in (1.15)]. It is clear that if τ is small, only functions $\varphi_n(x)$ such that $\varepsilon(n)<\tau$ will be important in the expansion (3.2). Any function $\varphi_n(x)$ satisfying this condition practically coincides with $\varphi_0(x)$ on any segment of length shorter than L_τ. Consequently, the order parameter $\varphi(x)$ should also possess this property. Over larger distances the behavior of $\varphi(x)$ is influenced by nonlinear effects. This leads to the result that in each piece of size L_τ the equilibrium value of $|\varphi|^2$ is established independently. Consequently, on scales greater than L_τ the quantity $|\varphi|^2$ is essentially unchanging. The length $L_\tau\sim\tau^{-1/\beta}$ is analogous to the correlation length in the fluctuation theory of phase transitions.[7]

We note that although the phenomena described remind one in many respects of the picture of the thermal fluctuations that arise near a second-order phase-transition point, there is an important difference, which is that here $\varphi(x)$ at each point does not fluctuate and the ensemble averaging is replaced by volume averaging.

To determine $\langle|\varphi|^2\rangle$ we should express $\langle|\varphi|^4\rangle$ in terms of $\langle|\varphi|^2\rangle$, i.e., find the dependence on τ of the quantity A in the expression $\langle|\varphi|^4\rangle=A\langle|\varphi|^2\rangle^2$, after which it will be possible to find $\langle|\varphi|^2\rangle$ easily by minimizing the expression

$$-\tau\langle|\varphi|^2\rangle+A(\tau)\langle|\varphi|^2\rangle^2. \tag{3.3}$$

This minimization gives $\langle|\varphi|^2\rangle=\tau/A(\tau)$. Our problem, in essence, is to determine the dependence of A on τ.

We turn to the calculations. In accordance with what was said above, the average value

$$\langle|\varphi|^2\rangle_{a,b}=\frac{1}{b-a}\int_a^b|\varphi|^2dx$$

depends neither on a nor on b if $b-a\gg L_\tau$ (this is also true for $\langle|\varphi|^4\rangle_{a,b}$). Therefore, in order of magnitude the following equalities are true:

$$\langle|\varphi|^2\rangle\approx\frac{1}{L_\tau}\int_0^{L_\tau}|\varphi|^2dx,\qquad\langle|\varphi|^4\rangle\approx\frac{1}{L_\tau}\int_0^{L_\tau}|\varphi|^4dx, \tag{3.4}$$

and it is clear that the values of the integrals in (3.4) do not change in order of magnitude if we replace $\varphi(x)$ by $\varphi_0(x)$ in them.

To find the relation between

$$\langle\varphi_0^2\rangle_L=L^{-1}\int_0^L\varphi_0^2\,dx\text{ and }\langle\varphi_0^4\rangle_L=L^{-1}\int_0^L\varphi_0^4\,dx$$

we make use of the scaling properties of the function $\varphi_0(x)$:

$$\langle\varphi_0^4\rangle_L=L^{1-\delta}\langle\varphi_0^2\rangle_L^2. \tag{3.5}$$

In other words, we introduce a new index δ, which is defined in terms of the ground-state eigenfunction $\varphi_0(x)$ of the Hamiltonian (1.3):

$$\delta=\lim_{L\to\infty}\lg_L\left\{\left(\int_0^L\varphi_0^2\,dx\right)^2\Big/\int_0^L\varphi_0^4\,dx\right\}. \tag{3.6}$$

Its physical meaning is clear. On a segment of length L a particle with wave function $\varphi_0(x)$ can be situated at approximately L^δ sites. When J changes from 1 to $+\infty$ the index δ changes from 1 to $\log_\Phi 2=0.72...$. The latter value follows from the hierarchical structure of the wave function $\varphi(x)$ for $J\gg1$ (see Sec. 6 in Ref. 4, and also Ref. 3). The calculation of the dependence of δ on J is described in the next section.

The value of $\langle|\varphi|^2\rangle$ is determined from the following considerations. The specific free energy F is equal to

$$F\sim-\tau\langle\varphi^2\rangle+\langle\varphi^4\rangle\sim-\tau\langle\varphi^2\rangle+A(\tau)\langle\varphi^2\rangle^2. \tag{3.7}$$

where $A(\tau)\approx(L_\tau)^\delta\approx\tau^{(\delta-1)/\beta}$. Minimizing the expression $-\tau\langle\varphi^2\rangle+\tau^{(\delta-1)/\beta}\langle\varphi^2\rangle^2$, with respect to $\langle\varphi^2\rangle$, we obtain

$$\langle\varphi^2\rangle\sim\tau^{1+(1-\delta)/\beta},\qquad F\sim\tau^{2+(1-\delta)/\beta}. \tag{3.8}$$

The specific heat is calculated from the usual formula

$$C_V=-T\frac{\partial^2F}{\partial T^2}\sim\frac{\partial^2F}{\partial\tau^2}\sim\tau^{(1-\delta)/\beta}. \tag{3.9}$$

Thus,

$$C_V\sim\tau^\alpha,\ \alpha=(1-\delta)/\beta>0. \tag{3.10}$$

There is no discontinuity of the specific heat.

The dependence of α on γ is depicted in Fig. 2 (the index δ is calculated by the method described below). The index α is equal to zero in the limit of a uniform system ($J=1$, $\gamma=1$) and in the limit of one layer ($J=+\infty$, $\gamma=0.5$).

As already stated, over distances greater than L_τ (along the x axis) the critical phenomena are unimportant. Consequently, over such distances we can define effective macroscopic characteristics of the superconductor—the London length δ_L, the correlation lengths ξ_1 and ξ_2 in the directions parallel and perpendicular to the layers, and also the Ginzburg-Landau parameters K_1 and K_2 in the longitudinal and

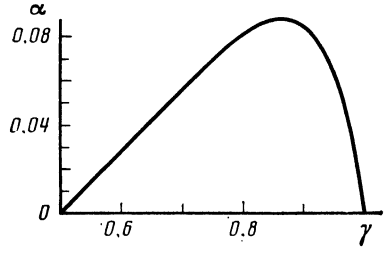

FIG. 2. Relation between the critical indices γ and α, as specified by the function $\alpha(\gamma)$.

transverse directions: $K_1 = \delta_L/\xi_1$, $K_2 = \delta_L/\xi_2$. By making use of the formula (3.8), we find

$$\delta_L^{-2} \approx \langle\varphi^2\rangle \sim \tau^{1+(1-\delta)/\beta} \sim \tau^{1+\alpha}. \tag{3.11}$$

The longitudinal correlation length ξ_1 does not depend on the complicated structure of the order parameter, and ξ_2 has already been calculated (it coincides with L_τ):

$$\begin{aligned} &\xi_1 \sim \tau^{-1/2}, \quad \xi_2 = L_\tau \sim \tau^{-1/\beta}, \\ &K_1 \sim \tau^{-\alpha/2}, \quad K_2 \sim \tau^{-(1/2-1/\beta+\alpha/2)}. \end{aligned} \tag{3.12}$$

The parameters K_1 and K_2 tend to infinity as $\tau \to 0$, and, therefore, near the transition point the superlattice is a type-II superconductor in respect of its macroscopic properties, irrespective of the materials of which the layers are made.

4. METHOD OF CALCULATION OF THE INDEX δ

For study of the properties of the eigenfunction $\varphi_0(x)$ of the Hamiltonian (1.3) the transformation M of the three quantities x,y,z is not sufficient. We need to analyze the transformation (1.6) of the matrix pair (A,B). As shown in Ref. 4, after double application of this transformation the matrices B and A corresponding to the point Q change as follows:

$$B \to BA = S^{-1}BS, \quad A \to ABA = S^{-1}AS. \tag{4.1}$$

Here S is a certain 2×2 matrix. The equations (4.1) are valid inasmuch as the point Q corresponds to a sequence of traces with period 2 [see the discussion preceding the formula (1.14)]. Under repeated application of the transformation (1.6) the following sequence of matrices arises:

$$B,\ A,\ S^{-1}BS,\ S^{-1}AS,\ S^{-2}BS^2,\ S^{-2}AS^2,\ \ldots \tag{4.2}$$

From Eqs. (4.1) we can find the matrices A, B, and S to within a change of basis. In the basis in which S is diagonal, they have the form

$$\begin{aligned} S &= \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}, \quad B = \frac{1}{s-s^{-1}} \begin{pmatrix} ts-s^{-1} & 1-t^{-1} \\ 1 & s-ts^{-1} \end{pmatrix}, \\ A &= \frac{1}{s-s^{-1}} \begin{pmatrix} s(s^2-t-1) & (1-t^{-1})(ts^{-2}-1) \\ ts^2-1 & s^{-1}(t+1-s^{-2}) \end{pmatrix} \end{aligned} \tag{4.3}$$

where $s = \frac{1}{2}[(p+1)^{1/2} + (p-3)^{1/2}]$, and t and p depend on J as in (1.13).

The index δ can be found by direct calculation, starting from the definition (3.6). The ground-state eigenfunction φ is obtained by multiplying a certain initial vector (the vector (φ_l, φ_l') at the left end of the structure) by the matrices A and B in the order specified by the Fibonacci sequence. In the general case it is a linear combination of two functions, one of which increases rapidly. In order to eliminate this rapidly increasing function it is necessary to choose as the initial vector (φ_l, φ_l') an eigenvector of the matrix S with eigenvalue $s^{-1} < 1$. Only with such a choice of the vector (φ_l, φ_l') do we obtain a function that is not increasing but bounded on the entire straight line, and is, in fact the function $\varphi_0(x)$ corresponding to a superconducting nucleus in zero field. An example of a function obtained in this way is shown in Fig. 1.

There also exists another, "almost adiabatic" way of calculating the index δ, in which a numerical calculation is required only for the determination of the largest eigenvalue of a certain tenth-order matrix.

We shall consider a sequence of F_n layers (F_n is the Fibonacci number with index n) and determine the two symmetric second-order and fourth-order forms constructed from the initial vector $\mathbf{x} = (\varphi_l, \varphi_l')$:

$$\Phi_n^{(2)}(\mathbf{x}) = \int_0^{F_n} \varphi^2\, dx, \quad \Phi_n^{(4)}(\mathbf{x}) = \int_0^{F_n} \varphi^4\, dx. \tag{4.4}$$

For the forms $\Phi^{(2)}$ and $\Phi^{(4)}$ we obtain the same recursion relation

$$\Phi_{n+2}(\mathbf{x}) = \Phi_{n+1}(\mathbf{x}) + \Phi_n(T_{n+1}\mathbf{x}). \tag{4.5}$$

Taking into account that $T_{2k} = S^{-k}BS^k$ and $T_{2k+1} = S^{-k}AS^k$, we obtain

$$\begin{aligned} \Phi_{2k+2}(\mathbf{x}) &= \Phi_{2k+1}(\mathbf{x}) + \Phi_{2k}(S^{-k}AS^k\mathbf{x}), \\ \Phi_{2k+3}(\mathbf{x}) &= \Phi_{2k+1}(\mathbf{x}) + \Phi_{2k}(S^{-k}AS^k\mathbf{x}) \\ &\quad + \Phi_{2k+1}\cdot(S^{-k-1}BS^{k+1}\mathbf{x}). \end{aligned} \tag{4.6}$$

We now introduce the forms

$$\Pi_k(x) = \Phi_{2k}(S^{-k}\mathbf{x}), \quad \Sigma_k(x) = \Phi_{2k+1}(S^{-k}\mathbf{x}). \tag{4.7}$$

The recursion relations for the forms Π and Σ do not contain any dependence on k:

$$\begin{aligned} \Pi_{k+1}(\mathbf{x}) &= \Pi_k(AS^{-1}\mathbf{x}) + \Sigma_k(S^{-1}\mathbf{x}), \\ \Sigma_{k+1}(\mathbf{x}) &= \Pi_k(AS^{-1}\mathbf{x}) + \Sigma_k(S^{-1}\mathbf{x}) + \Sigma_k(S^{-1}B\mathbf{x}). \end{aligned} \tag{4.8}$$

These equations specify a linear transformation of the coefficients in the forms Π and Σ. The symmetric second-order and fourth-order forms have three and five independent coefficients, respectively. The problem reduces to the study of linear mappings in a six-dimensional space (for the second-order forms) or in a ten-dimensional space (for the fourth-order forms).

Let η and ρ be the largest eigenvalues of these mappings, all the other eigenvalues being smaller than these in absolute magnitude (this is confirmed by calculations). In the case of a general position,

$$\begin{aligned} &\lim_{k\to\infty} \eta^{-k}\Pi_k^{(2)} = \Pi_*^{(2)}, \quad \lim_{k\to\infty} \rho^{-k}\Pi_k^{(4)} = \Pi_*^{(4)}, \\ &\lim_{k\to\infty} \eta^{-k}\Sigma_k^{(2)} = \Sigma_*^{(2)}, \quad \lim_{k\to\infty} \rho^{-k}\Sigma_k^{(4)} = \Sigma_*^{(4)}, \end{aligned} \tag{4.9}$$

$(\Pi_*^{(2)}, \Sigma_*^{(2)})$ and $(\Pi_*^{(4)}, \Sigma_*^{(4)})$ are the eigenvectors of the mapping (4.8) that correspond to the eigenvalues η and ρ.

Since the initial forms $\Pi_0^{(2)}, \Sigma_0^{(2)}, \Pi_0^{(4)}$, and $\Sigma_0^{(4)}$ are positive-definite, and the non-negativity of Π_k and Σ_k implies the non-negativity of Π_{k+1} and Σ_{k+1}, it follows that Π_* and Σ_* are at least non-negative. We shall prove that $\Pi_*^{(2)}$ and $\Pi_*^{(4)}$ are positive-definite (for the forms $\Sigma_*^{(2)}$ and $\Sigma_*^{(4)}$ the argument is analogous).

We shall assume the contrary: $\Pi_*(\mathbf{x}) = 0$ for a certain $\mathbf{x}$. Then $\Pi_*(AS^{-1}\mathbf{x}) = 0$ and $\Sigma_*(S^{-1}\mathbf{x}) = 0$; consequently, $\Pi_*(AS^{-2}\mathbf{x}) = 0$ [this follows from Eqs. (4.8) and the non-negativity of Π_* and Σ_*]. Continuing these arguments, we arrive at the conclusion that the form Π_* vanishes on the vectors $(AS^{-1})^n\mathbf{x}$ and $(AS^{-2})^n\mathbf{x}$ for any integer $n > 0$. The matrices AS^{-1} and AS^{-2} are nondegenerate and have real eigenvalues. If the vector $\mathbf{x}$ is not an eigenvector for the matrix AS^{-1}, all the vectors $(AS^{-1})^n\mathbf{x}$ are noncollinear (the situation is analogous for the matrix AS^{-2}). A form of

order n can vanish on not more than n noncollinear vectors. Consequently, the vector $\mathbf{x}$ is an eigenvector for both matrices AS^{-1} and AS^{-2}. Hence it follows that the vector $\mathbf{x}$ is also an eigenvector for the matrices A and S. However, it can be seen from the formulas (4.3) that the matrices A and S do not have common eigenvectors. We have arrived at a contradiction.

The positive-definiteness of the forms $\Pi_*^{(2)}$ and $\Pi_*^{(4)}$ implies that, on the set $|\mathbf{x}| = 1$, $\Pi_*^{(2)}(\mathbf{x})$ and $\Pi_*^{(4)}(\mathbf{x})$ have a finite positive maximum and a finite positive minimum. Consequently,

$$\begin{aligned}\Phi_{2k}^{(2)}(\mathbf{x}) &= \Pi_k^{(2)}(S^k\mathbf{x}) \sim \eta^k \Pi_*^{(2)}(S^k\mathbf{x}) \sim \eta^k |\mathbf{x}|^2,\\ \Phi_{2k}^{(4)}(\mathbf{x}) &= \Pi_k^{(4)}(S^k\mathbf{x}) \sim \rho^k \Pi_*^{(4)}(S^k\mathbf{x}) \sim \rho^k |\mathbf{x}|^4.\end{aligned} \tag{4.10}$$

Finally, we find δ:

$$\delta = \lim_{k\to\infty}\left(\lg \frac{(\Phi_{2k}^{(2)})^2}{\Phi_{2k}^{(4)}} \lg^{-1}(F_{2k})\right) = \lg_\Phi \frac{\eta^2}{\rho}. \tag{4.11}$$

For the case of the second-order forms the eigenvalues of the transformation (4.8) can be found analytically. They are equal to 1, s^2, s^{-2}, 1, λ, and λ^{-1}, where s and λ were determined earlier [see the formulas (4.3) and (1.16)]. The largest eigenvalue $\eta = \lambda$. For the case of the fourth-order forms the largest eigenvalue ρ of the corresponding tenth-order matrix is found numerically.

The results obtained by this method coincide with the results of the direct calculation using the formula (3.6).

5. DISCUSSION AND CONCLUSIONS

Near the transition point all quantities in the system under consideration have a power-law dependence on the temperature. We have shown this in the framework of the Ginzburg-Landau approximation, but, apparently, the power dependences are universal. The situation with the critical indices is more complicated. There exist two independent indices γ and α, in terms of which one can express all the other indices (β, δ, the index for the critical tilt angle θ_c of the magnetic field, etc.). These relations between the critical indices were obtained from general physical considerations, without the use of properties of the Fibonacci sequence. The indices γ and α are not universal, but there exists between them a universal (at least in the framework of the Ginzburg-Landau approximation) dependence $\alpha(\gamma)$ (Fig. 2), associated with the order of alternation of the layers in the superlattice. It is evidently of interest to check this dependence experimentally. Unfortunately, the index α is small ($\alpha_{\max} \approx 0.09$).

In the system under consideration the scaling has periodic corrections [see (2.9)] associated with the spatial nonuniformity of the superlattice on all scales. It is curious that the period of the corrections in the logarithm of the correlation length L has turned out to be twice that which follows from purely geometrical considerations (the structure itself admits the transformations $L \to L\Phi^{k/2}$, where k is an arbitrary integer).

In conclusion we shall make one comment concerning the experiment of Ref. 1. One can attempt to use the formulas (2.6), (1.16), and (1.13) to analyze the index value $\gamma = 0.74$ obtained in Ref. 1. We then find $J = 235.2$. If we assume that J is related to the parameters of the lattice by (1.9), it turns out that $\xi_0 \ll d_n, d_{s1}, d_{s2}$, since J is very large. In fact, $\xi_0 > d_{s1}, d_{s2}$ (see the beginning of Sec. 1). This discrepancy conforms with our conclusion that the applicability of the Ginzburg-Landau equation is poor in the given case. For a quantitative analysis of the critical behavior the microscopic theory of superconductivity must be used.

We are grateful to P. A. Kalugin, V. L. Pokrovskiĭ, G. Riener, and G. M. Éliashberg for numerous useful discussions.

[1] M. G. Karkut, J.-M. Triscone, D. Ariosa, and Ø. Fischer, Phys. Rev. B **34**, 4390 (1986).
[2] R. Merlin, K. Bajema, R. Clarke, F.-Y. Juang, and P. K. Bhattacharya, Phys. Rev. Lett. **55**, 1768 (1984).
[3] A. Yu. Kitaev and L. S. Levitov, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 52 (1987) [JETP Lett. **45**, 66 (1987)].
[4] P. A. Kalugin, A. Yu. Kitaev, and L. S. Levitov, Zh. Eksp. Teor. Fiz. **91**, 692 (1986) [Sov. Phys. JETP **64**, 410 (1986)].
[5] M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. **50**, 1870 (1983).
[6] L. I. Glazman, Zh. Eksp. Teor. Fiz. **93**, 1373 (1987) [Sov. Phys. JETP **66**, 780 (1987)].
[7] A. Z. Patashinskiĭ and V. L. Pokrovskiĭ, *Fluctuation Theory of Phase Transitions*, Pergamon Press, Oxford (1979) [Russ. original (2nd ed.), Nauka, Moscow (1982)].

Translated by P. J. Shepherd