# Distribution of local density of states and NMR line shape in a one-dimensional disordered conductor

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(Submitted 14 June 1988)
Zh. Eksp. Teor. Fiz. **95**, 348–370 (January 1989)

The Berezinskiĭ diagram technique is used to study the fluctuation moments of the local density of electron states in a one-dimensional conductor with weak disorder. It is possible to reconstruct from these moments the distribution functions of this density. It is found that the distribution of the density of states depends substantially on the conditions on the sample boundary. If the electron is able to leave the sample, its local density of states has a logarithmically normal distribution. One can speak of the distribution of the density of states in a closed sample only after regularizing the density, i.e., after broadening the exact energy levels. The distribution turns out then to depend not only on the broadening scale, but also on the type of regularization. The fluctuation moments of the density of states are determined for arbitrary regularization. Also calculated are the moments of the participation ratio. The distribution of the Knight shifts is determined for an arbitrary ratio of the temperature and the Zeeman splitting.

## 1. INTRODUCTION

Considerable advances were made recently in theoretical investigations of statistical fluctuations in disordered conductors. We refer here to the spread of the parameters in an ensemble of macroscopically equivalent samples (mesoscopic fluctuations) or to fluctuations, from point to point, of local electronic properties of a disordered conductor.

The final aim of the theory of these fluctuations is the determination of their distribution functions. This requires knowledge of not only the variance but of all the higher fluctuation moments. The first work in this direction was that of Wegner.[1] He investigated the local density of states of noninteracting electrons in a random Gaussian potential in a space of dimensionality $d$ larger than two. It turned out that at energies approaching from above the threshold of the mobility (averaged over a random potential of degree $n$) the $n$th fluctuation moment of the local density of states increases very rapidly with $n$, like $\exp(n^2)$. This growth of the moments is a characteristic attribute of a logarithmically normal distribution of a random quantity. This means that the logarithm of the local density of states has a Gaussian normal distribution, while the decrease of the probability of a large deviation of the density of states itself from a typical value is much slower than exponential.

Logarithmically normal asymptotics were later observed in the distribution of mesoscopic fluctuations of the conductivity and of the total density of states in an open sample of finite size.[2] These results, as well as those of Ref. 1, were obtained by using the formalism of the effective-field theory (the nonlinear $\sigma$ model) in the one-loop approximation. The latter means that formally these results are valid only if the conductance (the reciprocal of the total resistance) of the sample is large compared with $e^2/\hbar$.

Wegner[3] has recently advanced arguments favoring the premise that when higher loops are taken into account the growth of the fluctuation moment with increase of its number $n$ slows down at the largest $n$. This would mean that the probability of the largest deviations of the local density of states from its mean value decreases exponentially and not according to a logarithmic normal law.

On the other hand, a logarithmically normal distribution was obtained even earlier[4,5] in an exact solution of the problem of the statistics of the resistances of disrodered one-dimensional conductors of finite size. In this case the electrons are localized and the average resistance $R$ is much larger than $\hbar/e^2$, but the resistance distribution function is very similar to that obtained in the single-loop approximation (which is formally valid for $R \ll \hbar/e^2$) for the distribution function of the conductances.

The present paper is devoted to an exact determination of the distribution function of the local density of states of noninteracting electrons in a one-dimensional disordered conductor of finite size $L$. It has turned out that in an open sample this distribution has indeed a logarithmically normal form that agrees surprisingly well with the results of single-loop calculations in the region of good metallic conductivity. As to the local density of states in a closed sample, from which the electrons cannot depart, there exist several methods of determining this quantity, each with its own distribution function. At the very largest values of the argument all these distribution functions decrease exponentially.

In the next section we describe the distributions of the local density of states in various cases and present a qualitative interpretation of these results.

Inhomogeneity in the local density of states leads to an inhomogeneous Knight shift, i.e., to an inhomogeneous nuclear magnetic resonance (NMR) line broadening. If this broadening exceeds appreciably the homogeneous one due to elastic scattering, the NMR line shape is directly connected with the distribution function of the local density of states. These questions were discussed earlier in Ref. 6 for a three-dimensional Anderson dielectric, and in Ref. 7 for the one-dimensional case, on the basis of model premises concerning the form of the localized-state wave function. In Sec. 3 we discuss the results of a direct calculation of the NMR line shape in a one-dimensional conductor with weak disorder.

In Sec. 4 we derive, using the Berzinskiĭ diagram technique,[8] the basic equations satisfied by the higher fluctuation moments. In Secs. 5 and 6 we derive the distribution func-

 0038-5646/89/010198-12$04.00  

tions, described in Sec. 2, of the local density of the states in closed and open samples, respectively.

Some of the results of the present paper were discussed briefly in our earlier paper.[9]

## 2. DISTRIBUTION OF LOCAL DENSITY OF STATES

The density of electronic states with energy $\varepsilon$ at a point $x$ can be defined as

$$\rho(\varepsilon,x)=\sum_{\nu}|\Psi_{\nu}(x)|^{2}\frac{\delta(\varepsilon-\varepsilon_{\nu})}{N(\varepsilon)}=\frac{1}{N(\varepsilon)}\langle x|\varepsilon-\hat{H}|x\rangle. \quad (1)$$

The subscript $\nu$ numbers here the exact eigenstates of a Hamiltonian $\hat{H}$ with energy $\varepsilon_{\nu}$ and with wave function $\Psi_{\nu}(x)$. For convenience, the local density of states at a certain point $x$ of a given sample will be given in units of $N(\varepsilon)$, which is the density, averaged over a random potential, of states of energy $\varepsilon$. In the one-dimensional case $N(\varepsilon)=(\pi v_{\varepsilon})^{-1}$, where $v_{\varepsilon}$ is the velocity of electrons of energy $\varepsilon$.

If the sample is finite and closed (i.e., its boundaries are impermeable to electrons), we deal with a discrete spectrum of exact electron states. It is then meaningless to speak of a distribution function $\rho(\varepsilon,x)$, since it can take on only two values; infinity if $\varepsilon$ is exactly equal to any one of the eigenenergies $\varepsilon_{\nu}$, or zero if $\varepsilon\neq\varepsilon_{\nu}$.

The natural way to regularize the $\delta$-functions in (1) is to replace them by Lorentzians:

$$\rho_{1}(\eta,\varepsilon,x)=\frac{1}{\pi N(\varepsilon)}\sum_{\nu}|\Psi_{\nu}(x)|^{2}\frac{\eta}{(\varepsilon-\varepsilon_{\nu})^{2}+\eta^{2}}. \quad (2)$$

This means that all the energy levels are of equal width $\eta$ (due, e.g., to inelastic processes), or that the physical situation corresponds to averaging of the density of states over the energy with a Lorentzian weight:

$$\rho_{1}(\eta,\varepsilon,x)=\int_{-\infty}^{\infty}\rho(\varepsilon',x)f_{\eta}(\varepsilon-\varepsilon')\,d\varepsilon',$$

$$f_{\eta}(\varepsilon-\varepsilon')=\frac{\eta}{(\varepsilon-\varepsilon')^{2}+\eta^{2}}\frac{1}{\pi}. \quad (3)$$

This regularization is necessary, in particular, to permit a transition to the thermodynamic limit in an infinite sample. The gist of this transition is that $\eta\to 0$ only after the sample size $L$ is increased to infinity. In a metal, when the wave functions of the electrons are delocalized, this limiting transition is meaningful, since it makes the local density of states a smooth function of the energy. If, on the other hand, we deal with an Anderson dielectric, such a regularization ($L\to\infty$, and then $\eta\to 0$) is ineffective. The reason is the exponential decrease of the localized wave functions, the result of which is that even at $L=\infty$ a substantial contribution to $\rho_{1}$ is made at $\eta\neq 0$ only by a finite number of levels. Therefore as $\eta\to 0$ the local density of states can be either zero or infinity.

We know[8,10] that in a one-dimensional conductor all the electronic states are localized even in the case of weak disorder. At the same time, the fluctuation moments and the distribution function of $\rho_{1}(\eta,\varepsilon,x)$ for finite $\eta$ can be determined exactly. The calculations are described in detail in Secs. 4 and 5, while here we present and discuss only the final results.

No regularization is needed to determine the local density of states $\langle\rho_{1}\rangle$ averaged over the realizations of a random impurity potential, since $\langle\rho_{1}\rangle=1$ for all $\eta$. All the remaining fluctuation moments of $\rho_{1}(\eta)$ depend substantially on $\eta$ and diverge as $\eta\to 0$. From these moments, exact expressions for which are given in Sec. 5 [Eqs. (68) and (71)], we can reconstruct the distribution function. In an infinite sample it takes the form of the so-called inverse Gaussian distribution[11]:

$$W(\rho_{1})=(4\eta\tau/\pi\rho_{1}^{3})^{1/2}\exp\left[-4\frac{(\rho_{1}-1)^{2}}{\rho_{1}}\eta\tau\right], \quad (4)$$

where $\tau=\tau_{2}/2$ is the free path time of an electron moving in a random potential, and $\tau_{2}$ is the free path time relative to backward scattering (here and hereafter $\hbar=1$).

The quantity $\tau^{-1}$ has also the meaning of the characteristic energy gap between two levels adjacent in energy and localized near a given point, i.e., at a distance from the point smaller than or of the order of the localization radius $\xi$. On the average, $\xi$ is equal to $4l$, where $l=v_{\varepsilon}\tau=v_{\varepsilon}\tau_{2}/2=l_{2}/2$ is the electron mean free path. When a typical wave function $\Psi_{\nu}$ moves away from the localization center $x_{\nu}$ it decreases like

$$\Psi_{\nu}(x)\propto\exp(-|x-x_{\nu}|/\xi).$$

The theory of the inverse Gaussian distribution (4) is described in detail in Jorgensen's book.[11] We shall note here only its main properties. $W(\rho_{1})$ decreases exponentially as a function of $\rho_{1}$ as $\rho_{1}\to\infty$, and as a function of $\rho_{1}^{-1}$ as $\rho_{1}\to 0$. If $4\eta\tau\gg 1$, contributions to $\rho_{1}(x)$ are made by many levels localized near the point $x$. $W(\rho_{1})$ has then a sharp maximum when $\rho_{1}$ is almost equal to unity, i.e., to the mean value $\langle\rho_{1}\rangle$. The distribution function $W(\rho_{1})$ has a form close to normal if $|\rho_{1}^{-1}|\ll 1$.

If, however, $4\eta\tau\ll 1$ and $\rho_{1}(x)$ is determined by the position of the level closest in energy and in coordinate, the maximum of $W(\rho_{1})$ becomes asymmetric and shifts into the region of very low values of the local density of states. The most probable value $\rho_{1}=8\eta\tau/3$ turns out to be much smaller than $\langle\rho_{1}\rangle=1$. At the same time, the fluctuation moments of $\langle\rho_{1}^{n}\rangle$ for $n\gg 1$ are determined by the region $\rho_{1}\gg 1$.

It is of interest to see how averaging the local density of states over a coordinate influences its distribution function. Consider, for example, the quantity

$$\rho_{1,\Delta}(\eta,x)=\int_{-\Delta/2}^{\Delta/2}\frac{dy}{\Delta}\rho_{1}(\eta,x+y). \quad (5)$$

The quantity $\rho_{1,L}(\eta,L/2)$ is the total density of states of the size $L$. It was proven in Ref. 12 that the total density of states is self-averaging, i.e., that it does not fluctuate as $L\to\infty$. This is a natural consequence of the localization of the electronic states, which suppresses the repulsion of the energy levels. Fluctuations of the total density of states in an Anderson dielectric of finite size are considered in Ref. 13.

As $L\to\infty$ and for finite $\Delta$, the distribution function $\rho_{1,\Delta}(\eta,x)$ depends substantially on $\Delta$. If $\Delta>v_{\varepsilon}/\eta_{\varepsilon}$, the number of levels on the interval $\Delta$ in an energy interval of width $\eta$,

$$n(\Delta,\eta)=\rho_{1,\Delta}(\eta,x)N(\varepsilon)\eta\Delta=\frac{\eta\Delta}{\pi v_{\varepsilon}}\rho_{1,\Delta}(\eta,x), \quad (6)$$

is large. Therefore $n(\Delta,\eta)$, meaning also $\rho_{1,\Delta}(\eta,x)$, has a Gaussian distribution. If, however, $\Delta$ for $\eta\tau\ll 1$ lies in the

interval $v_\varepsilon/\eta > \Delta > -l\ln(\eta\tau)$, the level repulsion is insignificant as before, but $n(\Delta,\eta)$ is small. This quantity has therefore a Poisson distribution:

$$W(n)=\frac{n_0^n}{n!}e^{-n_0},\qquad n_0=\frac{\eta\Delta}{\pi v_\varepsilon}. \tag{7}$$

We shall not consider here the distribution function $n(\Delta,\eta)$ or $\rho_1$ in the region $-l\ln(\eta\tau)<\Delta<l$ in which the Poisson law ceases to hold because of the repulsion of the levels that contribute substantially to $\rho_{1,\Delta}(\eta,x)$. In addition to the purely local density of states (3), which is equal to $\rho_{1,\Delta}$ only if $\Delta$ is small compared with the wavelength $p_\varepsilon^{-1}$ (with the reciprocal momentum) of the electron, we consider in the present paper also the density of states $\tilde\rho_1(\eta,x)$ averaged over a scale intermediate between the electron wavelength and the localization radius (i.e., the mean free path $l$):

$$\tilde\rho_1(\eta,\ x)=\rho_{1,\Delta}(\eta,\ x)\ \text{ for }\ l>\Delta>p_\varepsilon^{-1}. \tag{8}$$

The distributions $W(\rho_1)$ and $W(\tilde\rho_1)$ differ because the fluctuations of $\tilde\rho_1$ are determined only by the smooth envelope of the exact electron wave function which decreases over scales of order $l$, while the oscillations of this wave function over scales on the order of the electron wavelength $p_\varepsilon^{-1}$ are also important for the $\rho_1$ fluctuations.

In Sec. 5 we obtain an expression for the moments $\tilde\rho_1$ at $4\eta\tau\ll 1$ [see (69)], from which it follows that the distribution function $W(\tilde\rho_1)$ can be written in the form

$$W(\tilde\rho_1)=\frac{\eta\tau}{\pi}\int_4^\infty t\,dt\sin(\pi\eta\tau t)\left(\frac{t+4}{t-4}\right)^{\eta\tau t}\exp\left(-\frac{1}{2}\eta\tau\tilde\rho_1 t^2\right). \tag{9}$$

From (9) we can obtain the following asymptotic expressions:

$$W(\tilde\rho_1)=\begin{cases}(\pi\eta\tau/2\tilde\rho_1^3)^{1/2}\exp(-\pi^2\eta\tau/2\tilde\rho_1), & \tilde\rho_1<\eta\tau\\ (\pi\eta\tau/2\tilde\rho_1^3)^{1/2}, & \eta\tau<\tilde\rho_1<1/\eta\tau,\\ (4\eta\tau/\tilde\rho_1)\,e^{-8\eta\tau\tilde\rho_1}, & \tilde\rho_1>1/\eta\tau\end{cases} \tag{10}$$

To derive (10) we must recognize that if $\tilde\rho_1>\eta\tau$ we can expand the sine function under the integral sign, and if $\tilde\rho_1<\eta\tau$ we can omit the power-law factor. Comparison of (10) with (4) shows that the averaging (8) leads, as expected, to a noticeable narrowing of the distribution of the local density of states [the argument of the exponentials in (10) is larger than in (4)]. On the other hand, only the numbers change in the asymptotic expressions, and the dependences on $\tilde\rho_1$ remain the same as in $W(\rho_1)$ [Eq. (4)] in all three regions.

Expressions (4) and (9), as well as (10), for the distribution of the local densities of state are valid not only in an infinite unbounded sample. They are valid also in a bounded system under the following two conditions: 1) the distance $x$ from the boundary to the point at which the local density of states is investigated is large compared with the localization length, i.e., with $l$, and 2) the sample is closed, i.e., the electrons are reflected from the boundary and do not leave the sample.

The presence of an open boundary can lead, as shown in Sec. 6, to a qualitative change of the distribution of the fluctuations of the local density of states even for $x\gg l$. In this case there is no need to introduce a regularization of type (2); the ability of the electron to leave the sample leads by itself to a broadening of the exact energy levels (it is assumed that after passing through the boundary the sample will either never return, or its phase coherence will be completely lost by the time it does return). The calculations described in Sec. 6 show that the fluctuations of the local densities of states (1), averaged over the atomic scales (8), have the normal logarithmic distribution:

$$W(\tilde\rho)=\left(\frac{l}{2\pi x}\right)^{1/2}\frac{1}{\tilde\rho}\exp\left\{-\frac{(\ln(1/\tilde\rho)-x/2l)^2}{2x/l}\right\}. \tag{11}$$

This distribution function has a maximum at $\tilde\rho=\exp(-3x/l)$ with a sharp edge on the side of small $\tilde\rho$. The width of the maximum at half height is $\exp(-x/l)$. The probabilities of anomalously large and anomalously small values of $\tilde\rho$ decrease much more slowly than exponentially.

It can be noted that the distribution (11), just as in (4), has a symmetry that leads to the equality $\langle\tilde\rho^n\rangle=\langle\tilde\rho^{-n+1}\rangle$.

The distribution of the purely local density of states $\rho$ in the presence of an open boundary differs somewhat from (11), but has exactly the same logarithmically normal asymptotes as in (11). It was already noted in the Introduction that these asymptotes agree very well with the distribution function of the density of states of a metallic system if the dimensionality exceeds two. We shall discuss this question in greater detail in the Conclusion.

The distribution function $\tilde\rho_1$ for finite $\eta\tau\ll 1$ retains the form (11) only in the region $\eta\tau<\tilde\rho_1<1/\eta\tau$. Outside this region, the presence of an open sample boundary is immaterial and $W(\tilde\rho_1)$ decreases exponentially in accordance with (10).

According to (11), $W(\tilde\rho_1)$ has a logarithmically normal behavior and decreases at $|\ln\tilde\rho_1|>x/l$ much more slowly than exponentially. This behavior, however, takes place only on the interval

$$|\ln\eta\tau|>|\ln\tilde\rho_1|>x/l.$$

Therefore the distribution function for $\eta\tau>\exp(-x/l)$ is described by expressions (10) for all $\tilde\rho_1$ [the distribution function (11), just as (10), is proportional in the region $|\ln\tilde\rho|<x/l$ to $\tilde\rho^{-1/2}$].

The qualitative behavior of the described distributions of the densities of state (4), (10), and (11) lends itself to a natural physical interpretation. We consider first the region of small $\rho_1(\varepsilon,x)$ or $\tilde\rho_1(\varepsilon,x)$. For the local density of states to be anomalously small the spectrum of the states localized near a point $x$ must have near this point a gap of width $\delta\varepsilon$ much larger than $\tau^{-1}$. We can obtain from (2) a relation that permits an estimate of the necessary width of the gap:

$$\rho_1\approx\frac{\eta}{4\tau}\sum_{|\varepsilon_\nu-\varepsilon|>\delta\varepsilon}\frac{1}{(\varepsilon_\nu-\varepsilon)^2+\eta^2}\propto\frac{\eta}{\delta\varepsilon}. \tag{12}$$

We have taken it into account that the characteristic value of $|\psi_\nu(x)|^2$ is of the order of the reciprocal localization region $\xi^{-1}=(4l)^{-1}$, and have assumed that the levels are more or less uniformly distributed outside the gap. The exponential decrease [Eqs. (4) and (10)] of $W$ as a function of $\rho_1^{-1}$ attests to a Poisson distribution of the levels, at which the probability of formation of a gap of width $\delta\varepsilon$ is proportional to $\exp[-N(\varepsilon)\xi\delta\varepsilon]$, since $N(\varepsilon)\xi\approx 4\tau/\pi$ is the characteristic inverse of the distance between levels. Substitution of

(12) leads directly to the exponential asymptotes (4) and (10).

Of course, this reasoning is valid if $N(\varepsilon)\xi\delta\varepsilon > 1$. The distribution function reaches a maximum when the gap width becomes comparable with the characteristic distance $[N(\varepsilon)\xi]^{-1} = \tau^{-1}$ between levels. According to (12), this corresponds at $\eta\tau \ll 1$ to

$$\rho_{1,\,max} \approx \eta\tau,\ W(\rho_{1,\,max}) \approx (\eta\tau)^{-1}.$$

To increase the local density of states further it is necessary that the energy level closest to $\varepsilon$ be located at a distance $\delta\varepsilon$ much shorter than $\tau^{-1}$. In this case the value of $\rho_1(\varepsilon)$ is completely determined by this level:

$$\rho_1 \approx \frac{\eta}{4\tau}\frac{1}{(\delta\varepsilon)^2+\eta^2}. \tag{13}$$

If $\tau^{-1} > \delta\varepsilon > \eta$, i.e., if $\eta\tau < \rho_1 < (\eta\tau)^{-1}$, the uniform distribution of the distances to the nearest level $W(\delta\varepsilon) \approx \tau$ in (13) leads to the distribution

$$W(\rho_1) = W(\delta\varepsilon(\rho_1))\frac{d(\delta\varepsilon)}{d\rho_1} \approx (\eta\tau)^{1/2}\rho_1^{-3/2},$$

which coincides with (4) and (10).

The same dependence, with $\eta$ replaced by $\eta_0 \approx \tau^{-1}\exp(-x/2l)$, follows from (11) for the distribution of the fluctuations of $\tilde{\rho}$ if the sample has an open boundary in the region $|\ln\tilde{\rho}| < x/2l$. This is natural since, as already mentioned, the departure of the electrons from the sample broadens the energy levels, and $\eta_0$ is a typical value of this broadening.

The substantial difference between the distribution functions of the local densities of states in open and closed samples at $\eta = \eta_0$ manifests itself in the region of large values of the argument at $\tilde{\rho} > (\eta_0\tau)^{-1}$. The point is that a level with a typical width, even if its energy $\xi$ is identical, cannot lead to a local density of states much larger than $(\eta_0\tau)^{-1}$. Large values of $\tilde{\rho}$ are therefore connected with anomalously strongly localized levels. If the energy $\varepsilon_\nu$ of this state differs from $\varepsilon$ by not more than its width $\eta_\nu$, we have

$$\tilde{\rho} \approx 1/\xi_\nu\eta_\nu N(\varepsilon), \tag{14}$$

where $\xi_\nu$ is the localization radius of this state.

In an open sample the level width decreases exponentially with decrease of the localization region; $\eta_\nu \propto \exp(-2x/\xi_\nu)$. The natural assumption of a normal distribution of the reciprocal localization radii

$$W\left(\frac{1}{\xi_\nu}\right) = \left(\frac{x}{8\pi l}\right)^{1/2}\exp\left\{-\frac{x}{8l}\left(\frac{4l}{\xi_\nu}-1\right)^2\right\} \tag{15}$$

leads therefore to a logarithmically normal distribution of the widths $\eta_\nu$, meaning also of the fluctuations of $\tilde{\rho}$ [see (14)].

According to the definitions (2) and (3), all the levels in a closed sample have one and the same width $\eta$. An increase of $\rho_1$ is therefore possible if the localization radius is decreased. Substituting in (15) $x \sim \xi_\nu \sim [\rho_1\eta N(\varepsilon)]^{-1}$ in accordance with (14), we obtain $W(\rho_1) \propto \exp(-\rho_1\eta\tau)$, as follows indeed from (4) and (10).

We have thus described and qualitatively explained the distributions of the local density of states in the case when the electron energy levels are Lorentzian. These results became qualitatively understandable by assuming absence of correlation in the positions of the energy levels (Poisson distribution) and a normal distribution of the reciprocal localization lengths. On the other hand, averaging with a Lorentz weight (3) is not the only one possible. We shall show in the next section that the local density of states, which differs from (3) in that the Lorentzian under the integral is replaced by other functions $f_\eta(\varepsilon - \varepsilon')$, can be most directly studied by using NMR. It turns out that the fluctuation moments of the local density of states can be obtained for arbitrary $f_\eta(\varepsilon - \varepsilon')$. These results will also be obtained in the next section.

### 3. INHOMOGENEOUS BROADENING OF NMR LINE

Hyperfine exchange interaction between a nuclear spin at a point $x$ and a conduction-electron spin is known to shift the resonance frequency of a nuclear spin placed in an external magnetic field $H$—the Knight shift[14]:

$$\Delta\omega = Jg\mu_e H\int N(\varepsilon)\rho(\varepsilon, x)f(\varepsilon)\,d\varepsilon \equiv Jg\mu_e HN(\varepsilon_F)\rho_f(x). \tag{16}$$

Here $J$ is the hyperfine-interaction constant, $g\mu_e H$ is the Zeeman splitting of the electron level by spin projections, $\rho(\varepsilon,x)$ is defined in (1), and $f(\varepsilon)$ is connected with the Fermi distribution function $n_F(\varepsilon)$ by the relation

$$f(\varepsilon) = -\frac{n_F(\varepsilon+g\mu_e H)-n_F(\varepsilon-g\mu_e H)}{2g\mu_e H},$$
$$n_F(\varepsilon) = \left[\exp\left(\frac{\varepsilon-\varepsilon_F}{T}\right)+1\right]^{-1} \tag{17}$$

where $\varepsilon_F$ is the Fermi energy.

The fluctuations of the local density of states of the electrons lead thus to fluctuations of the Knight shifts at different points of the sample, i.e., to an inhomogeneous broadening of the NMR line. If the spin-lattice relaxation rate $1/T_2$ is less than the dispersion $\Delta\omega$, the NMR line shape is determined entirely by this inhomogeneous broadening.

Depending on the ratio of the temperature $T$ to the Zeeman splitting $g\mu_e H$, the function $f(\varepsilon)$ is given by

$$f(\varepsilon) = \begin{cases} \dfrac{1}{4T}\operatorname{ch}^{-2}\left(\dfrac{\varepsilon-\varepsilon_F}{2T}\right), & T \gg g\mu_e H \\ \dfrac{1}{2g\mu_e H}\theta[g\mu_e H - |\varepsilon-\varepsilon_F|], & T \ll g\mu_e H \end{cases} \tag{18}$$

where $\theta(t) = 1$ for $t \geqslant 0$ and $\theta(t) = 0$ for $t < 0$. It is meaningful therefore to consider distributions of local densities of state, defined as

$$\rho_2(T, \varepsilon_F, x) = \int \rho(\varepsilon, x)\frac{d\varepsilon}{4T\operatorname{ch}^2((\varepsilon-\varepsilon_F)/2T)}, \tag{19a}$$

$$\rho_3(g\mu_e H, \varepsilon_F, x) = \frac{1}{2g\mu_e H}\int_{\varepsilon_F-g\mu_e H}^{\varepsilon_F+g\mu_e H}\rho(\varepsilon, x)\,d\varepsilon\,. \tag{19b}$$

Since the Knight shift differs from $\rho_2$ or $\rho_3$ by only the factor

$$\frac{\Delta\omega}{Jg\mu_e HN(\varepsilon_F)} = \begin{cases} \rho_2, & g\mu_e H \ll T \\ \rho_3, & g\mu_e H \gg T \end{cases}, \tag{20}$$

the NMR line shift is completely defined by the distribution functions $W(\rho_2)$ and $W(\rho_3)$.

To describe the NMR line shape observed in one-dimensional conductors in Ref. 15, the authors of Ref. 7 calculated $W(\rho_2)$ by starting from the assumption that the elec-

tronic-state wave functions are given by

$$\Psi_\nu(x)=\frac{\xi^{-1/2}}{2}\operatorname{ch}^{-1}\left(\frac{x-x_\nu}{2\xi}\right) \tag{21}$$

with one and the same localization radius $\xi$ and with randomly distributed centers $x_\nu$ and energies $\varepsilon_\nu$. The same considerations were used even earlier[6] to determine $W(\rho_3)$ in the three-dimensional case.

We present here the results of an exact calculation of the distributions $W(\rho_2)$ and $W(\rho_3)$ in the one-dimensional case; no model assumptions are used in the calculations which, in contrast to the results of Ref. 7, are valid for arbitrary $\rho$ and $T$ or $H$.

To find the distribution of the local density of states $\rho_f$ averaged over the energy with arbitrary weight $f(\varepsilon)$ [see (3)] we consider the correlation function $\langle\rho(\varepsilon_1)\rho(\varepsilon_2)...\rho(\varepsilon_n)\rangle$ of the densities of states (1) at different energies. It turns out that this correlator has the following structure:

$$\left\langle\prod_{i=1}^{n}\rho(\varepsilon_i)\right\rangle=1+a_2\sum_{i\neq j}\delta(\varepsilon_i-\varepsilon_j)+a_3\sum_{i\neq j\neq k}\delta(\varepsilon_i-\varepsilon_j)\delta(\varepsilon_i-\varepsilon_k)\ldots+a_n\prod_{i=2}^{n}\delta(\varepsilon_i-\varepsilon_1). \tag{22}$$

It follows from (22) that the quantities $\rho(\varepsilon_i)$ for different $\varepsilon_i$ are not correlated at all. This conclusion seems surprising, since we are dealing with levels localized near the point $x$, and these levels should be repelled. Relation (22) for $n=2$ was obtained by direct calculation in Refs. 13 and 16. Moreover, it turned out that there is no correlation only for the densities of states in one and the same point $x$. The quantity $\langle\rho(\varepsilon_1,x)\rho(\varepsilon_2,y)\rangle$ differs noticeably from unity not only for $|x-y|\gtrsim l$, but also for $|x-y|\approx p_F^{-1}$, where $p_F^{-1}$ is the electron wavelength on the Fermi level. We have no good qualitative explanation for the absence of correlations between $\rho(\varepsilon_i,x)$ The negative correlation connected with the level repulsion apparently offsets the positive correlation of the $|\Psi_\nu(z)|^2$.

The coefficients $a_n$ in (22) can be written in the form

$$a_n=\frac{1}{N(\varepsilon)^n}\left\langle\sum_\nu|\Psi_\nu(x)|^{2n}\delta(\varepsilon-\varepsilon_\nu)\right\rangle, \tag{23}$$

i.e., they are the irreducible fluctuation moments of the quantity known as participation ratio. Using (22) we can express, for arbitrary $f(\varepsilon)$, the fluctuation moments ("cumulants") of the quantities $\rho_f$ in terms of $a_n$:

$$\langle\rho_f^n\rangle_c=a_n\int d\varepsilon\, f^n(\varepsilon). \tag{24}$$

For the Lorentz weighting function (3) the fluctuation moments $\langle\rho_1^n\rangle_c$ are obtained below in Sec. 5 by direct calculation. Substituting them in (24) we can obtain the coefficients $a_n$:

$$a_n=(\pi/4\tau)^{n-1}\Gamma(n), \tag{25}$$

where $\Gamma(n)$ is a gamma function. Substituting (25) in (24) we get an equation for the irreducible fluctuation moments of $\rho$ for arbitrary $f(\varepsilon)$. It is easily seen, in particular, that

$$\langle\rho_3^n(g\mu_eH,\,\varepsilon_F,\,x)\rangle_c=\Gamma(n)\,(8\pi^{-1}g\mu_eH\tau)^{1-n}, \tag{26}$$

$$\langle\rho_2^n(T,\varepsilon_F,x)\rangle_c=\frac{\Gamma(3/2)\Gamma^2(n)}{\Gamma(n+1/2)}\left(\frac{16T\tau}{\pi}\right)^{1-n}. \tag{27}$$

From (26) we can determine the distribution $W(\rho_3)$. The Laplace transform of this function is directly connected with $\langle\rho_3^n\rangle$ by the known relation

$$\int_0^\infty W(\rho)\exp(-p\rho)\,d\rho=\exp\left\{\sum_{n=1}^{\infty}\frac{(-p)^n}{\Gamma(n+1)}\langle\rho^n\rangle_c\right\}. \tag{28}$$

Substitution of (25) in (28) yields

$$\int_0^\infty W(\rho_3)\exp(-p\rho_3)\,d\rho_3=\left(\frac{s_3}{s_3+p}\right)^{s_3},\qquad s_3\equiv\frac{8}{\pi}g\mu_eH\tau, \tag{29}$$

or

$$W(\rho_3)=\frac{(s_3\rho_3)^{s_3}}{\Gamma(s_3)\rho_3}\exp(-\rho_3s_3). \tag{30}$$

Together with (20), the distribution (30) describes the NMR line shape for $g\mu_eH\gg T$. If $s_3\leqslant1$ then, evidently, the most probable value of $\rho_3$ is zero, i.e., for $T=0$ and $g\mu_eH\leqslant\pi/8\tau$ there is no average Knight shift, and the NMR line is asymmetrically broadened.

The distribution of $\rho_2$ can be found similarly. From (27) and (28) it follows that

$$\int_0^\infty W(\rho_2)\exp(-p\rho_2)\,d\rho_2=\exp\left\{-s_2\ln^2\left[\left(\frac{p}{s_2}\right)^{1/2}+\left(\frac{p}{s_2}+1\right)^{1/2}\right]\right\},\qquad s_2=\frac{16}{\pi}T\tau. \tag{31}$$

The distribution function $W(\rho_2)$ is determined from (31) by taking the inverse Laplace transform. After straightforward albeit cumbersome transformations this function takes the form

$$W(\rho_2)=\frac{s_2}{2\pi}\int_{-\infty}^{\infty}dx\,\operatorname{sh}2x\sin(\pi s_2x)\times\exp\left\{-s_2\left[\rho_2\operatorname{ch}^2x+x^2-\frac{\pi^2}{4}\right]\right\}, \tag{32}$$

If $\rho_2\gg\max\{s_2^{-1},1\}$, the integral (32) is determined by the region of small $x$ and can be calculated by expanding $\sinh 2x$ and $\cosh^2 x$ in powers of $x$. As a result we obtain for $W(\rho_2)$ an asymptotic relation that decreases exponentially with increase of $\rho_2$:

$$W(\rho_2)=(\pi s_2/4\rho_2^3)^{1/2}\exp[-\rho_2s_2+\pi^2\rho_2s_2/4(\rho_2+1)],\qquad \rho_2\gg s_2^{-1},\ \rho_2\gg1. \tag{33}$$

On the other hand, it can be seen from (32) that

$$W(\rho_2=0)=\frac{s_2}{4\pi}\exp\left(\frac{\pi^2s_2}{4}+s_2^{-1}\right)\times\int_{-\infty}^{\infty}dx\sin(\pi s_2x)\left\{e^{-s_2(x-s_2^{-1})^2}-e^{-s_2(x+s_2^{-1})^2}\right\}=0. \tag{34}$$

It can be shown similarly that for $\rho_2 = 0$ all the derivatives $d^n W/d\rho_2^n$ of the distribution function also vanish. Zero is thus an essential singular point of the function $W(\rho_2)$. If $\rho_2 \ll 1$, the main contribution to (32) is made by the region $x \gg 1$. In this case $\sinh x = \cosh x = (1/2)\exp x$ and (32) can be represented in the form

$$W(\rho_2) = \frac{s_2}{8\pi i}\int_{-i\infty}^{i\infty} dy \exp\left\{\frac{s_2}{4}[\rho_2 y - \ln^2 y]\right\}. \quad (35)$$

If furthermore the condition

$$\rho_2 \ll \frac{1}{2s_2}\exp\left(-\frac{1}{s_2}\right) \quad (36)$$

is also met, the integral in (35) can be calculated by the saddle-point method. As a result,

$$W(\rho_2) = \left[\frac{s_2}{4\pi} z(\rho_2)\rho_2^{-2}\right]^{1/2} \exp\left\{-\frac{s_2}{4}[z^2(\rho_2) - 2z(\rho_2)]\right\}, \quad (37)$$

$$z(\rho_2) \gg \frac{1}{s_2},$$

where $z(\rho_2)$ is the solution of the equation

$$z(\rho_2) = \ln\frac{2z(\rho_2)}{\rho_2}, \quad z(\rho_2) = \ln\left\{\frac{2}{\rho_2}\ln\left[\frac{2}{\rho_2}\ln\left(\frac{2}{\rho_2}\ln\ldots\right)\right]\right\}. \quad (38)$$

$W(\rho_2)$ increases with $\rho_2$ in the region in which Eqs. (37) are valid.

It is seen from (27) that for $s_2 \gg 1$ the distribution $W(\rho_2)$ is Gaussian (since $\langle\rho_2^n\rangle_c^2 \ll \langle\rho_2^2\rangle_c^n$), with a center at $\rho_2 \approx 1$ and a width $(2/3s_2)^{1/2}$;

$$W(\rho_2) = \left(\frac{3s_2}{4\pi}\right)^{1/2}\exp\left[-\frac{3s_2}{4}(\rho_2 - 1)^2\right], \quad \rho_2 \sim 1, \ s_2 \gg 1. \quad (39)$$

On the other hand, the asymptotic forms of $W(\rho_2)$ are described by Eqs. (33) and (37).

For $s_2 \ll 1$ the maximum of the function $W(\rho_2)$ becomes very narrow and shifts to the region of small values:

$$\rho_{2\,max} \sim \frac{1}{2s_2}\exp\left(-\frac{1}{s_2}\right), \quad W(\rho_{2\,max}) \sim s_2\exp\left(\frac{1}{s_2}\right). \quad (40)$$

Equation (37) describes $W(\rho_2)$ correctly only for $\rho_2 < \rho_{2\,max}$, while for $\rho_2 > s_2^{-1}$ relation (33) is valid. At intermediate values $\rho_{2\,max} \ll \rho_2 \ll s_2^{-1}$ of the local density of states the integral in (32) is determined by the region $1 < x < (\pi s_2)^{-1}$. It is therefore possible to replace in the integrand $\sin(\pi s_2 x)$ by $\pi s_2 x$ and $\sinh 2$ and $\cosh^2 x$ by $\exp(2x)/2$. The integral can then be easily calculated:

$$W(\rho_2) = \frac{s_2}{2\rho_2}\ln\left(\frac{4}{\gamma s_2\rho_2}\right)\exp\left\{-\frac{s_2}{4}\ln^2\left(\frac{4}{\rho_2 s_2}\right)\right\}, \quad (41)$$

$$\frac{4}{s_2}\exp\left(-\frac{2}{\pi s_2}\right) \ll \rho_2 \ll \frac{1}{s_2},$$

where

$$\gamma = \exp\left\{-\int_0^\infty dx \ln x \exp(-x)\right\} \approx \exp(0.577) \approx 1.781.$$

We note right away that Eq. (41) is in no way related to the logarithmically normal distribution (11) of the local density of states in an open system, discussed in the preceding section. Equation (11) is apparently valid for any sample dimensionality $d$. At the same time it follows from the model (21) used in Refs. 6 and 7 that in the region where the asymptotic relation (41) is valid we have

$$\ln W(\rho_2) \propto s_2 \ \ln^{1+d}(\rho_2 s_2), \quad \ln W(\rho_3) \propto s_3 \ \ln^d(\rho_3 s_3). \quad (42)$$

This is the probability distribution for observing a nearest level at a distance on the order of $\ln\rho_3$ in a $d$-dimensional space, and at a distance of order $\ln\rho_2$ in a $(d+1)$-dimensional space, in which the role of one of the dimensions is assumed by the energy.

The asymptotic values (37) and (41) obtained by us for the function $W(\rho_2)$ for $s_2 \ll 1$ and $\rho_2 \ll s_2^{-1}$ agree qualitatively with the results of Refs. 6 and 7, where the authors started out from model-based premises concerning the wave function of a localized state [see Eq. (21)]. The limitations of these premises are manifest by the substantial quantitative difference between (37) and (41), on the one hand, and the results of Ref. 7, on the other. As to the region of applicability of Eq. (33), $W(\rho_2)$ is determined in this region entirely by the localization-length fluctuations. So large values of $\rho_2$ are therefore never encountered in the approach developed in Ref. 7.

Thus, the NMR line shape in a disordered one-dimensional conductor at $T \gg g\mu_e H$ is determined by the distribu-

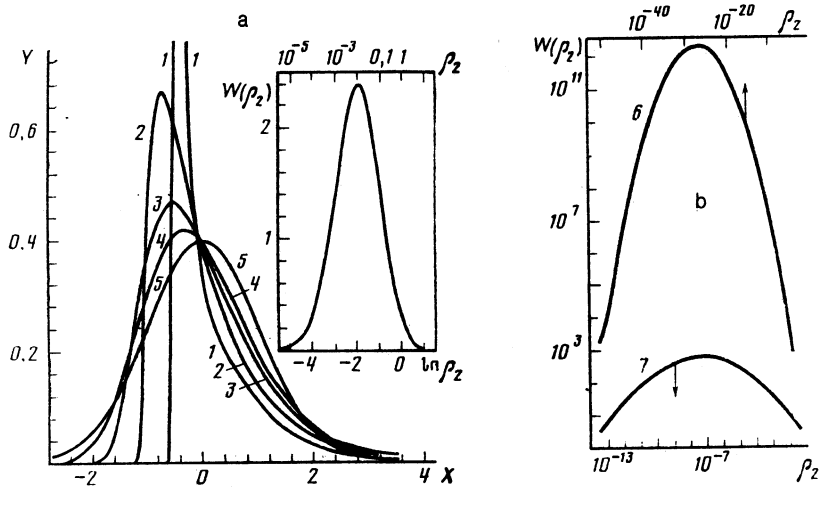


FIG. 1. Distribution functions $W(\rho_2)$ for different $s_2$. a) $W(\rho_2)$ dependence in the relative coordinates $X = (3s_2/2)^{1/2}(\rho_2 - 1)$, $Y = (2/3s_2)^{1/2}W(\rho_2)$: *1*—$s_2 = 0.3$; *2*—$s_2 = 1$; *3*—$s_2 = 3$; *4*—$s_2 = 10$; *5*—plot of the function $Y = (2\pi)^{-1/2}\exp(-X^2/2)$, corresponding according to (39) to the $W(\rho_2)$ dependence as $s_2 \to \infty$. Inset—$W(\rho_2)$ dependence for $s_2 = 0.3$ in semilog scale. b) Plots of $W(\rho_2)$ for $s_2 = 0.1$ (6) and $s_2 = 0.03$ (7) in log-log scale.

tion function (32). For $T \gg \pi/16\tau$ the asymptotes of this function are given by Eqs. (29), (33), and (37). In the opposite limiting case $T \ll \pi/16\tau$ the asymptotic behavior of $W(\rho_2)$ is described by Eqs. (33), (37), and (40). In contrast to the case $T=0$, the Knight shift differs from zero, but is many times smaller than the one produced at $16T\tau \gg \pi$ and corresponds to $\rho_2 \approx 1$, i.e., to a homogeneous density of the electronic states. Highly characteristic at low temperatures is the exponential dependence of the position and width of the NMR line on the reciprocal temperature (40). The distribution functions $W(\rho_2)$ for different $s_2$ are shown in Fig. 1.

To conclude this section, we note that we have calculated in passing the moments of the participation ratio [see (25)]. Such a moment increases factorially when the number is increased, thus pointing to an exponential decrease of the probability of large values of $|\Psi_\nu(x)|^2$. This, however, is a much slower growth than the one observed by Wegner for the participation ratio moments in a metal at $d>2$.

## 4. DERIVATION OF BASIC EQUATIONS

We proceed now to derive our main results described in the preceding sections. The local density of states (2) can be expressed in terms of the electronic Green's functions (GF):

$$\rho_1(\eta, \varepsilon, x) = \frac{1}{2\pi i}[G_-(\varepsilon; x, x) - G_+(\varepsilon; x, x)], \tag{43}$$

where

$$G_\pm(\varepsilon; x, y) = \sum_\nu \frac{\Psi_\nu^*(x)\Psi_\nu(y)}{\varepsilon - \varepsilon_\nu \pm i\eta} \tag{44}$$

is the exact retarded (advanced) GF. The moments $\rho_1(\eta,\varepsilon,x)$ are

$$\langle \rho_1^n(\eta, \varepsilon, x)\rangle = (2\pi i)^{-n} \sum_{k=0}^{n} \frac{(-1)^k \Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \times \langle G_+^k(\varepsilon; x, x) G_-^{n-k}(\varepsilon; x, x)\rangle. \tag{45}$$

The correlator $\langle G_+^k G_-^{n-k}\rangle$ can be calculated by the Berezinskiĭ diagram technique[8] (see also the reviews in Refs. 18 and 19). This calls for summation of the perturbation-theory series over the random impurity potential for an object constituting $n$ closed loops, $k$ of which correspond to the retarded GF $G_+$ and the remainder to the advanced $G_-$. The coordinates of each start and end of the GF in each loop coincide and are equal to $x$.

Following Berezinskiĭ, we use diagrams ordered with respect to the coordinates and break each up into a product of two blocks of the right ($R$) and left ($L$) parts. Thus, $R$ is the sum of the diagrams located to the right of the point $x$ and having at this point $m_1$ pairs of lines from the first "advanced" loop, $m_2$ from the second, $\bar{m}_1$ from the first "retarded" loop, etc. (see Fig. 2). In the general case $R$ depends on each of the variables $m_i$ and $\bar{m}_i$. The equation for $R(\{m_i\}, \{\bar{m}_i\})$ can be derived by following, e.g., Refs. 8 and 19. It is obvious from Fig. 2, however, that in our case, when the energy variables of all the advanced GF (single lines in Fig. 2) are equal to $\varepsilon - i\eta$, while the energy variables of all the retarded GF (double lines in Fig. 2) are equal to $\varepsilon + i\eta$, the

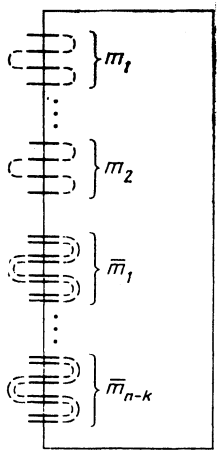


FIG. 2. Right-hand Berezinskiĭ block $R(m_1,m_2,...,m_k; \bar{m}_1,\bar{m}_2,...,\bar{m}_{n-k};x,\eta)$ for multiloop diagrams. Single and double lines—advanced ($G^0_-$) and retarded ($G^0_+$) Green's functions of the free electron.

right-hand side depends only on the summary variable $m = m_1 + m_2 + ... = \bar{m}_1 + \bar{m}_2 + ...$ :

$$R(\{m_i\}, \{\bar{m}_i\}) = R_m(\eta, x)\,\delta_{m,\, m_1+m_2+...+m_{n-k}}\,\delta_{m,\, \bar{m}_1+\bar{m}_2+...+\bar{m}_k}, \tag{46}$$

where $R_m$ is the right-hand side introduced by Berezinskiĭ and satisfying the equation

$$\left(4m\eta\tau - l\frac{d}{dx}\right) R_m = \frac{m^2}{2}[R_{m-1} + R_{m+1} - 2R_m]. \tag{47}$$

Similar equations hold for the left-hand side (only the sign of $d/dx$ is reversed):

$$L_{m'}(\{m_i'\}, \{\bar{m}_i'\}) = \{L_{m'}(\eta, x)\,\delta_{m',\, m_1'+m_2'+...+m'_{n-k}}\,\delta_{m',\, \bar{m}_1'+\bar{m}_2'+...+\bar{m}'_k}. \tag{48}$$

The correlation function of interest to us can be represented in the form

$$\langle G_+^k(x,x) G_-^{n-k}(x,x)\rangle = (-1)^k \nu_\varepsilon^{-n} \sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} L_{m'}(\eta, x) R_m(\eta, x)\,\varphi_{m,m'}(k)\,\varphi_{m,m'}(n-k), \tag{49}$$

where $\varphi_{m,m'}(k)$ are mixing coefficients. To determine them it must be recognized that there are four possible ways of attaching to the point $x$ the electron lines shown for the first loop in Fig. 3. For cases a and d the number of line pairs in

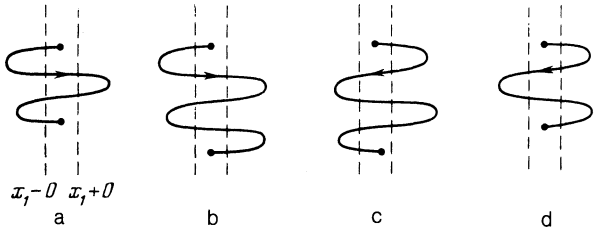


FIG. 3. Possible methods of joining electron lines to an outer vertex. Cases a and d correspond to the oscillating factor $\exp(\pm 2ip_\varepsilon x)$.

the sections $x+0(m)$ and $x-0(m')$ differ by unity, whereas $m=m'$ in cases b and c. Therefore

$$\varphi_{m,m'}(k)=\sum_{\{m_i\}}\delta_{m,\,m_1+m_2+\ldots+m_k}\sum_{\{m_i'\}}\delta_{m',\,m_1'+m_2'+\ldots+m_k'}\times\prod_{j=1}^{k}\left\{\delta_{m_j,\,m_{j'-1}}+\delta_{m_j,\,m_{j'+1}}+2\delta_{m_j,\,m_{j'}}\left[1-\frac{1}{2}\delta_{m_j,0}\right]\right\}. \tag{50}$$

The factor in the square brackets is the result of the fact that b and c do not differ when $m=m'=0$.

Using the relation

$$\delta_{m,k}=(2\pi i)^{-1}\oint dz\,z^{k-m-1}, \tag{51}$$

where the integral is taken counterclockwise over a circle of radius $|z|<1$, we can rewrite (50) in the form

$$\varphi_{m,m'}(k)=\oint_{|z|<1}\frac{dz}{2\pi i}\oint_{|y|<1}\frac{dy}{2\pi i}z^{m'-m-k-1}y^{-m'-1}\left[\frac{(z+y)(1+z)}{1-y}\right]^k. \tag{52}$$

The moments of the density of states (8) averaged over atomic scales can be determined in the same manner. This averaging leads only to vanishing of the diagrams containing attached electron lines of type a and d in Fig. 3 (each such attachment corresponds to a rapidly oscillating factor $\exp(\pm 2ip_0x)$; when $\langle p_1^n\rangle$ is calculated these factor are cancelled out by analogous factors in the retarded loops). What are left ultimately are only diagrams with $m=m'$:

$$\langle\tilde\rho_1{}^n\rangle=(2\pi i\nu_e)^{-n}\sum_{k=0}^{n}\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}\times\sum_{m=0}^{\infty}L_m(\eta,x)R_m(\eta,x)\tilde\varphi_m(k)\tilde\varphi_m(n-k), \tag{53}$$

$$\tilde\varphi_m(k)=\sum_{\{m_i\}}\delta_{m,\Sigma m_i}\prod_{j=1}^{k}(2-\delta_{m_j,0})=\oint_{|z|<1}\frac{dz}{2\pi iz^{m+1}}\left(\frac{1+z}{1-z}\right)^k=\frac{(-1)^{k-1}}{\Gamma(k)}\frac{d^{k-1}}{dz^{k-1}}\left[\frac{(1+z)^k}{z^{m+1}}\right].$$

The aggregate of Eqs. (45), (47)–(49), (52) and (53) make possible a final determination of $\langle\rho_1^n\rangle$ and $\langle\tilde\rho_1^n\rangle$. The calculations become much simpler if $s_1\equiv 8\eta\tau\ll 1$. In this case the significant terms in the sums (39) and (53) over $m$ are those with $m\gg 1$, and

$$\tilde\varphi_m(k)\xrightarrow[m\to\infty]{}2^k m^{k-1}\Gamma^{-1}(k),\quad \varphi_{m,m'}(k)\xrightarrow[m,\,m'\to\infty]{}\to\frac{m^{k-1}\Gamma(2k+1)}{\Gamma(k)\Gamma(m-m'+k+1)\Gamma(m'-m+k+1)}. \tag{54}$$

Substituting (54) in (53) and (49) in (45) we get

$$\langle\tilde\rho_1{}^n\rangle=\frac{n-1}{2n-1}\frac{\Gamma(2n)}{\Gamma^3(n)}I_n(\eta,x),\qquad \langle\rho_1{}^n\rangle=\frac{2n-2}{n(2n-1)}\frac{\Gamma^2(2n)}{\Gamma^5(n)}2^{-n}I_n(\eta,x), \tag{55}$$

where

$$I_n(\eta,x)=\sum_{m=0}^{\infty}m^{n-2}L_m(\eta,x)R_m(\eta,x). \tag{56}$$

Equations (55) are not valid for $n=0$ ($\langle\rho_1\rangle=\langle\tilde\rho_1\rangle=1$ by definition), since expressions (54) for $\varphi(k)$ and $\tilde\varphi(k)$ are not valid when $k=0$. We shall show below that (55) describes more readily irreducible moments (cumulants or semi-invariants $\langle\rho_1^n\rangle_c$ and $\langle\rho_1^n\rangle_c$) than ordinary moments.

The values of $I_n(\eta,x)$ under different boundary conditions will be calculated in the next two sections.

## 5. DENSITY-OF-STATES MOMENTS IN A CLOSED SYSTEM

We consider first a semi-infinite sample located on the $x>0$ axis. If the electron is only elastically scattered from the boundary, the boundary condition for the left-hand side $L_m$ takes the form

$$L_m(x=0)=1. \tag{57}$$

According to (47), $L_m(x)$ for $m\gg 1$ is described by the equation

$$\left(ms_1+\frac{d}{dy}\right)L_m=m^2\frac{d^2L_m}{dm^2},\qquad s_1=8\eta\tau,\qquad y=\frac{x}{2l}. \tag{58}$$

After a Laplace transform with respect to the coordinate $y$ and a change of variable

$$z^2=4ms_1,\qquad \Phi_\lambda(z)=z^{-1}[L_m(\lambda)-\lambda^{-1}]=\int_0^\infty\frac{dy}{z}e^{-\lambda y}(L_m(y)-1) \tag{59}$$

Equation (58) with boundary condition (57) acquires the form of the Bessel equation

$$\frac{d^2\Phi_\lambda(z)}{dz^2}+\frac{1}{z}\frac{d\Phi_\lambda(z)}{dz}-\left(1+\frac{1+4\lambda}{z^2}\right)\Phi_\lambda(z)-\frac{1}{\lambda z}=0, \tag{60}$$

which can be solved with the aid of the Lebedev-Kontorovich transformation[20]:

$$\tilde\Phi_\lambda(\gamma)=\int_0^\infty\Phi_\lambda(z)K_{i\gamma}(z)\,dz,\qquad \Phi_\lambda(z)=\int_{-\infty}^{\infty}\frac{d\gamma}{\pi^2z}\gamma\,\mathrm{sh}(\pi\gamma)\tilde\Phi_\lambda(\gamma)K_{i\gamma}(z), \tag{61}$$

where $K_\mu(z)$ is a modified Bessel function of the second kind. As a result we obtain from (60)

$$\Phi_\lambda(z)=-\int_{-\infty}^{\infty}\frac{d\gamma}{\lambda\pi^2}\gamma\,\mathrm{sh}\,\pi\gamma\frac{K_{i\gamma}(z)}{1+4\lambda+\gamma^2}\int_0^\infty d\zeta\,K_{i\gamma}(\zeta). \tag{62}$$

Substituting (62) in (59), taking the inverse Laplace transform with respect to $y$, and using the identity

$$zK_1(z)=1-\int_{-\infty}^{\infty}\frac{\gamma\,d\gamma}{\pi(1+\gamma^2)}\mathrm{sh}\left(\frac{\pi\gamma}{2}\right)zK_{i\gamma}(z), \tag{63}$$

we obtain

$$L_m(s_1,y)=(4ms_1)^{1/2}\Big\{K_1((4ms_1)^{1/2})+\int_{-\infty}^{\infty}d\gamma\frac{\gamma\,\mathrm{sh}(\pi\gamma/2)}{\pi(1+\gamma^2)}K_{i\gamma}((4ms_1)^{1/2})\times\exp\Big[-\frac{1+\gamma^2}{4}y\Big]\Big\}. \quad (64)$$

The second term in the curly brackets of (64) decreases exponentially with increase of $y$ and is significant if $y\lesssim 1$, i.e., if $x\lesssim l$. Only at such distances from it does a closed boundary influence the local density of states. For $x\gg l(y\gg 1)$ we have

$$L_m(s_1,y)=(4ms_1)^{1/2}K_1((4ms_1)^{1/2})+\Big(\frac{18\pi}{y^3}ms_1\Big)^{1/2}K_0((4ms_1)^{1/2})e^{-y/4}. \quad (65)$$

All the results of this section are therefore valid also for a finite sample much longer than $l$. In this case at least one of the boundaries has no effect regardless of $y$. If, however, we are interested in a point located far enough from both boundaries, the sample does not differ from an infinite one and

$$R_m(s_1)=L_m(s_1)=2(ms_1)^{1/2}K_1(2(ms_1)^{1/2}). \quad (66)$$

Substituting (66) in (56) and then (55), we obtain

$$I_n(\eta)=s_1^{1-n}\frac{n}{n-1}\Gamma^4(n)\Gamma^{-1}(2n), \quad (67)$$

$$\langle\rho_1^n\rangle_c=(2s_1)^{1-n}\Gamma(2n-1)\Gamma^{-1}(n), \quad (68)$$

$$\langle\tilde\rho_1^n\rangle_c=s_1^{1-n}(2n-1)^{-1}\Gamma(n+1). \quad (69)$$

The left-hand sides of (68) and (69) contain in lieu of the moments $\langle\rho_1^n\rangle$, $\langle\tilde\rho_1^n\rangle$ the cumulants (semi-invariants) $\langle\rho_1^n\rangle_c$, $\langle\tilde\rho_1^n\rangle_c$, which are the irreducible moments of the local density of states. The point is that in the case $s_1\ll 1$ considered by us these quantities differ little, $\langle\rho_1^n\rangle=\langle\rho_1^n\rangle_c$ and $\langle\tilde\rho_1^n\rangle=\langle\tilde\rho_1^n\rangle_c$, inasmuch as $\langle\rho_1^n\rangle_c\gg\langle\rho_1^{n-1}\rangle_c$ according to (68) and (69). At the same time, as will be shown below, relation (68) for the cumulants is exact at arbitrary $s_1$.

Using the known relations between the moments and the cumulants:

$$\langle\rho^n\rangle=(-1)^n\frac{d^n}{dp^n}e^{\chi(p)}\Big|_{p=1},\quad \chi(p)\equiv\sum_{n=1}^{\infty}(-p)^n\frac{\langle\rho^n\rangle_c}{\Gamma(n+1)}, \quad (70)$$

we obtain from (68)

$$\langle\rho_1^n(\eta)\rangle=\sum_{k=0}^{n-1}\frac{\Gamma(n+k)}{\Gamma(n-k)\Gamma(k+1)}(2s_1)^{-k}=\frac{K_{n-1/2}(s_1)}{K_{-1/2}(s_1)}. \quad (71)$$

Equation (4) for the distribution function $W(\rho_1)$ follows directly from (71). It can be obtained with the aid of the inverse Mellin transform

$$W(\rho)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\langle\rho^n\rangle\frac{dn}{\rho^{n+1}} \quad (72)$$

(all the singularities of $\langle\rho^n\rangle$ are located to the left of the integration contour). It is even easier to verify (71) by starting from the inverse Gaussian distribution (4).

We consider now the distribution of the local density of states $\tilde\rho_1$ averaged in an atomic scale. From (69) we can obtain for the cumulants $\langle\tilde\rho_1^n\rangle_c$ the generating function $\chi(\tilde\rho)$ (70) in terms of which $W(\tilde\rho_1)$ can be directly expressed:

$$\chi(\tilde p_1)=\frac{i}{2}(\tilde p_1 s_1)^{1/2}\ln\frac{1+i(\tilde p_1 s_1^{-1})^{1/2}}{1-i(\tilde p_1 s_1^{-1})^{1/2}}, \quad (73)$$

$$W(\rho_1)=\int_C\frac{d\tilde p_1}{2\pi i}\exp[\chi(\tilde p_1)+\tilde p_1\tilde\rho_1].$$

The integration contour $C$ in (73) is drawn such that all the singularities of $\chi(\tilde\rho_1)$ (root and logarithmic branch points) are located on the left. Following the change of variable $\tilde p_1=s_1t^2$, Eq. (73) takes the form

$$W(\tilde\rho_1)=\frac{s_1}{\pi i}\int_{a-i\infty}^{a+i\infty}t\,dt\exp\Big\{-is_1\frac{t}{2}\ln\frac{1-it}{1+it}+\tilde\rho_1 s_1t^2\Big\}. \quad (74)$$

To obtain Eq. (9) for $W(\tilde\rho_1)$ from (74) it suffices to deform the integration contour in (47) so as to make it pass along the edges $[i\infty,i]$ and $[-i,-i\infty]$ of the cut, and also substitute the definition (58) of the quantity $s_1$.

We have considered so far the case $s_1\ll 1$. We discuss now the situation for arbitrary $s_1$. An expression for $R_m(s_1)$ was obtained in Refs. 8 and 16 for arbitrary $m$ and $s_1$, so that from the set of Eqs. (45), (49), and (52) we can calculate $\langle\rho_1^n\rangle_c$ also for this case. The result, however, is much simpler to obtain. It is necessary for this purpose to consider the inverse limiting case $s_1\gg 1$, in which we can write down a high-frequency expansion in terms of the parameter $s_1^{-1}=(8\eta\tau)^{-1}$, equivalent to the diagram expansion in the impurity digram technique.[8,21]

Diagrams for $\langle\rho_1^2\rangle_c$ in the first two orders of perturbation theory are shown in Fig. 4, where the single and double lines correspond as before to the advanced ($G_-$) and retarded ($G_+$) Green's functions. A wavy line and a cross describe impurity scattering. They correspond to factors $(2\tau)^{-1}$ and $(4\tau)^{-1}$, respectively.[22] Each diagram in Fig. 4 is marked by the number of equal but topologically nonequivalent diagrams.

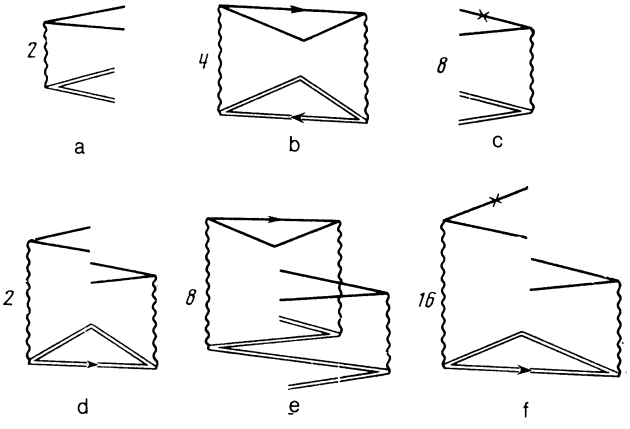


FIG. 4. High-temperature expansion for the cumulants of $\rho_1$: a) diagram for $\langle\rho_1^2\rangle_c$ of first (lower) order in $s_1^{-1}$; b) and c) second-order corrections to $\langle\rho_1^2\rangle_c$; d) diagram of second (lower) order in $s_1^{-1}$ for $\langle\rho_1^3\rangle_c$; e) and f)—corrections to $\langle\rangle\rho_1^3{}_c$ which are proportional to $s_1^{-3}$. A wavy line and a cross describe impurity scattering and correspond to factors $(2\tau)^{-1}$ and $(4\tau)^{-1}$, respectively.[22] The coefficients preceding the diagrams take into account all possible topologically equivalent diagrams.

In lowest-order perturbation theory in $s_1^{-1}$, the cumulant $\langle\rho_1^2\rangle_c$ is described by diagram a of Fig. 4, which yields $\langle\rho_1^2\rangle_c = s_1^{-1}$. This agrees with the result of the lowest order of the high-frequency expansion (68). It can therefore be proposed that the corrections in each of the succeeding orders of the high-frequency expansion cancel out. This can be verified by direct calculation. For example, the contribution of the diagrams of second order in $s_1^{-2}$, Figs. 4b and 4c, actually cancel each other. Relation (68) is therefore exact for all $s_1$.

This statement is valid also for the higher cumulants. Thus, the sum of the diagrams of the two lower perturbation-theory orders for $\langle\rho_1^3\rangle_c$ (diagrams 4d–4f) yields

$$\langle\rho_1^3\rangle_c = \frac{2\cdot 3!}{2^3}\left[\frac{2}{s^2}+\frac{1}{s^3}\left(8-\frac{16}{2}\right)+\ldots\right]=\frac{3}{s^2},$$

which coincides with (68).

Relation (68) for the cumulants $\langle\rho_1^n\rangle_c$ is thus exact, as already mentioned, for all $s_1$ and $n$. Unfortunately, the first-order high-frequency expansion of $\langle\tilde{\rho}_1^n\rangle_c$ is not equal to (69), and the higher-order corrections do not cancel out. Therefore (69) is valid only for $s_1 \ll 1$.

## 6. DENSITY-OF-STATES MOMENTS IN THE PRESENCE OF AN OPEN BOUNDARY

The condition on an open boundary of a sample is

$$L_m(x=0)=\delta_{m0}. \tag{75}$$

We have assumed that an electron departing backwards from a sample will no longer return (e.g., boundary between a disordered sample and an ideal conductor), or that prior to its return the phase coherence of its wave function will be completely lost.

In a semi-infinite sample, an equation for $L_m$ of type (47), after transformation with respect to coordinate and allowance for the boundary condition (75), takes the form

$$\{s_1 m+\lambda\}L_m(\lambda)-\delta_{m0}=m^2[L_{m+1}(\lambda)+L_{m-1}(\lambda)-2L_m(\lambda)]. \tag{76}$$

This equation can be solved as follows: For $m \ll s_1^{-1}$ one can neglect the first term in the curly brackets of (76). A solution of the remaining equation was obtained in Ref. 23, where it was shown that its asymptotic form for $m \gg 1$ is

$$L_m(\lambda)=m^{-q}\Gamma^2(q+1)\,\Gamma(\quad\ \ (2q+2),\ 1\ll m\ll s_1^{-1}, \tag{77}$$

where

$$2q=-1+(1+4\lambda)^{1/2}. \tag{78}$$

On the other hand, if $m \gg 1$ the expression in the square brackets of (76) can be replaced by $d^2L_m/dm^2$. This and the change of variable (59) result in a Bessel equation whose solution, with a correct asymptotic behavior as $m\to\infty$, is equal to

$$L_m(\lambda)=C\cdot 2(ms_1)^{1/2}K_{1+2q}(2(ms_1)^{1/2}). \tag{79}$$

The factor $C$ can be determined by matching the solutions (79) and (77) in the region $1 \ll m \ll s_1^{-1}$. As a result we have

$$C=\frac{s_1^q(q+1)}{8q+4}\Gamma^3(q)\Gamma^{-2}(2q). \tag{80}$$

Substituting (80) in (79) and taking the inverse Laplace transform, we get

$$L_m(y)=2(ms_1)^{1/2}K_1(2(ms_1)^{1/2}) + \frac{2(ms_1)^{1/2}}{2\pi i}\int_{-\infty}^{\infty}\frac{d\lambda}{\lambda+i}\frac{\Gamma^3((1+i\lambda)/2)}{\Gamma^2(i\lambda)} \times e^{-(1+\lambda^2)y/4}s^{(i\lambda-1)/2}K_{i\lambda}(2(ms_1)^{1/2}). \tag{81}$$

This expression can be compared with the analogous expression (64) for a closed sample. Whereas in (64) the presence of the boundary turns out to be inessential for $y>1$, according to (81) an open boundary is effective at distances $y \lesssim \ln s_1^{-1} \gg 1$.

It can be seen from (81) that for $m \gg s_1^{-1}$ the right-hand side decreases exponentially, $L_m \propto \exp(-2(ms_1)^{1/2})$. If $\gamma \equiv y^{-1}\ln s_1^{-1} \gg 1$, there exists a region $\exp(y) \ll m \ll s_1^{-1}$ in which $L_m$ also decreases when $m$ is increased, but more slowly:

$$L_m(y) \propto \exp\left(-\frac{1}{4y}\ln^2 m\right), \quad \gamma \equiv \frac{\ln s_1^{-1}}{y} > \ln\frac{1}{ms_1}. \tag{82}$$

In a semi-infinite sample, or in a finite one having a right-hand boundary located at a distance much larger than $\ln s_1^{-1}$ from the considered point, the right-hand side of $R_m$ is described as before by Eq. (66). Substituting (66) and (81) in (56) we obtain for $I_n(\eta,y)$

$$I_n=\frac{s_1^{1-n}}{\Gamma(2n)}\exp\left[-\frac{y}{4}(1-\gamma)^2\right](2\pi i)^{-1} \times\int_{-\infty+ia}^{\infty+ia}\frac{d\lambda}{i-\lambda}\Gamma^3\left(\frac{1-i\lambda}{2}\right)\Gamma^{-2}(-i\lambda) \times\left|\Gamma\left(n+\frac{i\lambda+1}{2}\right)\Gamma\left(n+\frac{i\lambda-1}{2}\right)\right|^2\exp\left[-\frac{y}{4}(\lambda-i\gamma)^2\right], \tag{83}$$

where $1<a<3/2$. The value of the integral in (83) is determined by the position of the saddle point $\lambda = i\gamma$ relative to the poles. For $\gamma<1$ the main contribution to $I_n$ is made by the pole $\lambda = i$, and this leads to Eq. (67). For the largest [i.e., the smallest $s_1$ and $\eta$, see Eq. (82)], or more accurately for $\gamma > 2n-1$, the value of the integral for $I_n$ in (83) is determined by the pole $\lambda = i(2n-1)$:

$$I_n=(2n-1)(n-1)^{-1}\Gamma^3(n)\Gamma^{-1}(2n)\exp[yn(n-1)]. \tag{84}$$

If $\eta = 0$ then $\gamma = \infty$ and (84) is valid for all $n$. Substitution of (84) in (55) yields

$$\langle\tilde{\rho}_1^n(y)\rangle=\langle\tilde{\rho}_1^n(y)\rangle_c=\exp[yn(n-1)]. \tag{85}$$

Note that for (85) to be valid it is necessary only that the condition $\gamma > 2n-1$ or $s_1 \ll \exp[-y(2n-1)]$ be met. The relation between $y$ and 1 can then be arbitrary.

The law governing the growth of the cumulants (85) is a characteristic attribute of the logarithmically normal distribution (11). Equation (11) for $W(\tilde{\rho}_1)$ at $\eta = 0$ can also be obtained from (85) directly by using an inverse Mellin transform of (72).

In the intermediate case $1<\gamma<2n-1$ the value of $I_n$ is determined by the contribution of the vicinity of the saddle point $\lambda = i\gamma$ to the integral in (83). Calculating this contribution and substituting in (55), we get

$$\langle\tilde\rho_1^n\rangle_c=\frac{2^n\Gamma^2(n)}{\Gamma(2n)}I_n=\frac{s_1^{1-n}}{2(\pi y)^{1/2}}\frac{\exp[-y(1-\gamma^2)/4]}{[(2n-1)^2-\gamma^2]}\frac{n-1}{2n-1}$$
$$\times\frac{\Gamma^2((\gamma-1)/2)\Gamma^2(n+(1+\gamma)/2)\Gamma^2(n+(1-\gamma)/2)}{\Gamma^2(\gamma-1)}. \tag{86}$$

It can be seen from (86) that once the condition $\gamma>2n-1$ is violated, the growth of the cumulants slows down with increase of the number $n$ in (85) and becomes merely factorial.

According to (11), at a certain value of $\tilde\rho_1$ that differs greatly from the most probable $\exp(-y)$, the distribution function $W(\tilde\rho_1)$ is determined by the cumulants $\langle\rho_1^n\rangle_c$ with numbers $n$ on the order of

$$n=\frac{1}{2y}|\ln\tilde\rho_1|.$$

The condition for the validity of Eq. (85) for cumulants with $2n-1<\gamma$ means that expression (11) for $W(\tilde\rho_1)$ is valid for finite $s_1<\exp(-y)$ only in the region $s_1<\tilde\rho_1<s_1^{-1}$.

If $n\to\infty$, Eq. (8) goes over into (69) apart from a coefficient that depends on $s_1$ and $y$. Consequently, accurate to the same coefficient, the distribution density $W(\tilde\rho_1)$ as $\tilde\rho_1\to\infty$ and as $\tilde\rho\to0$ is described by the asymptotic expressions (10). The determination of $W(\tilde\rho_1)$ in the intermediate region $|\ln\tilde\rho_1|\lesssim|\ln s_1|$ must be based on the exact Eq. (85) for the cumulants. Finally, if $y\gtrsim\ln s_1^{-1}$ the normal distribution law (11) is not valid in any region and is described for all $\tilde\rho_1$ by Eqs. (9) and (10) derived for an infinite sample. Thus, the influence of an open boundary on $W(\tilde\rho_1)$ extends over a region having a size on the order of $|\ln s_1|\gg1$. Recall that a closed boundary is effective only if $y\lesssim1$.

To conclude this section, we consider the distribution of $\tilde\rho_1$ in a finite sample with two open boudnaries at $\eta=0$. Let the dimensionless distances from the considered point to the boundaries by $y_1$ and $y_2$, and let $y_1<y_2$. By a method similar to the one described above, we can obtain from (81) and (55)

$$\langle\tilde\rho_1^n\rangle=\int_{-\infty}^{\infty}\frac{d\gamma}{4\pi^{1/2}}\frac{|\psi((n+i\gamma)/2)|^2}{\psi(n-1/2)}$$
$$\times\exp\left\{-y_1\frac{1+(\gamma+in)^2}{4}-y_2\frac{1+(\gamma-in)^2}{4}\right\}.$$
$$\psi(n)=\frac{\Gamma(n+1/2)\Gamma(n-1/2)}{\Gamma(n)}. \tag{87}$$

It follows hence that if $n<(y_1+y_2)/2y_1$ the equation (85) for the moments is valid, so that for $\tilde\rho_1<\exp(y_2)$ the $\tilde\rho_1$ distribution is given by Eq. (11) with $x=2ly_1$. If, however, $n>(y_1+y_2)/2y_1$, then

$$\langle\tilde\rho_1^n\rangle=\frac{1}{2(y_1+y_2)^{1/2}}\exp\left[\frac{y_1y_2}{y_1+y_2}n^2-\frac{y_1+y_2}{4}\right]$$
$$\times\psi\left(\frac{ny_1}{y_1+y_2}\right)\psi\left(\frac{ny_2}{y_1+y_2}\right)\psi^{-1}\left(n-\frac{1}{2}\right). \tag{88}$$

We have therefore for $\tilde\rho_1>e^{y_2}$

$$W(\tilde\rho_1)=\frac{(\pi y_1y_2)^{-1/2}}{4\tilde\rho_1}\psi\left(\frac{\ln\tilde\rho_1}{2y_1}\right)\psi\left(\frac{\ln\tilde\rho_1}{2y_2}\right)$$
$$\times\psi^{-1}\left(\frac{y_1+y_2}{2y_1y_2}\ln\tilde\rho_1-\frac{1}{2}\right)\cdot$$
$$\times\exp\left\{-\frac{y_1+y_2}{4y_1y_2}(\ln^2\tilde\rho_1+y_1y_2)\right\}. \tag{89}$$

Thus, the far logarithmic normal tail of the function $W(\tilde\rho_1)$ in a finite system is obtained by multiplying of the asymptotic relations of type (11) from each boundary.

## 7. CONCLUSION

We recapitulate briefly the result of the present paper. Using Berezinskiĭ's diagram technique we succeeded in calculating exactly the fluctuation moments of the local density of states in a weak one-dimensional random potential. We used these moments to reconstruct the distribution function of this quantity.

The results were found to depend substantially on the conditions at the sample boundaries. If at least one of the boundaries is open (i.e., an electron exiting through it leaves the sample forever), the distribution of the local density of states has a logarithmic normal form (11) which can be naturally interpreted as a consequence of the normal fluctuations of the localization length. This law agrees splendidly with the results in Refs. 2, where the distribution of the density of states over an ensemble of metallic samples in $d>2$ dimensions was examined. Within the framework of the one-loop approximation of the nonlinear $\sigma$ model it was shown there that the cumulants $N(\varepsilon)$ increase with the number $n$ in accordance with the law

$$\langle N(\varepsilon)^n\rangle_c\propto\exp\left[n^2\ln\frac{\sigma_0}{\sigma}\right], \tag{90}$$

where $\sigma_0=e^2N(\varepsilon)v_Fl/d$ is the classical (Drude) conductivity and $\sigma$ is the observed value of the conductivity renormalized by the quantum effects. In the one-dimensional case, owing to the localization, $\sigma$ decreases exponentially with increase of the sample dimension $x$, and $\sigma_0$ is independent of $x$. Therefore $\langle\ln(\sigma_0/\sigma)\rangle\propto x/l$ and the cumulant-growth laws (89) and (90) agree at least accurate to a numerical factor in the exponential.[1)]

This agreement is quite surprising, since the first of these equations pertains to a $(d>2)$-dimensional metal, and the second to a one-dimensional dielectric, to which the $\sigma$-model formalism is certainly inapplicable. A similar correspondence was observed also for the frequency dependence of the average conductivity.[24,25] All this seems to offer evidence that even in a metal there is some probability of an onset of appreciably localized states, and their localization has a normal distribution.

In the case of a closed sample, the local density of states must be regularized before it is determined, that is to say, the exact energy levels must be broadened in some way or another. The level widths must not tend to zero even in an infinite sample, for even in this case the density of states at each point of an Anderson dielectric (in contrast to a metal) is determined by a finite number of levels. It was found that the distribution function of densities of states depends strongly not only on the width but also on the regularization method. On the other hand, it is possible to determine the fluctuation moments of the density of states for arbitrary regularization [see (24) and (25)]. A common feature of all regularization is an exponential decrease of the distribution function in the region of large values.

A study of the NMR line shape that results from inhomogeneous line broadening permits a direct observation of the distribution function of the local density of states. We have determined this line shape for both high temperatures

(compared with the Zeeman splitting) and low ones [see (30) and (32)].

The main qualitative conclusions are the strong asymmetry of this line, and the exponentially strong temperature dependences of the width and position of the maximum of this line. Consequently, at low temperatures the principal role in the formation of the NMR line should be played by homogeneous broadening. At first glance it might seem that this line should correspond to a Lorentz broadening of the energy levels, and hence to the inverse Gaussian distribution (4). It appears, however, that the fluctuations of the inelastic widths of these turn out to be substantial. One should expect in this case a logarithmically normal NMR line shape of type (11), with $x$ replaced by a certain temperature-dependent length scale (for example, by the characteristic hop length in hopping conduction). This question requires further research.

We determined also the fluctuation moments of the participation ratio. A rapid growth of type (9), typical of a metal,[1] is apparently valid only for an open sample boundary. In a closed sample these moments, just as the moments of the local density of states, increase with the number only factorially. This confirms once more the conclusion that logarithmically normal distributions are consequences of the level widths due to the scatter of the localization lengths only in the presence of an open boundary.

We are deeply grateful to A. G. Aronov, A. M. Vaĭnrub, I. Kh. Zharenkeshchev, V. E. Kravtsov, I. V. Lerner, M. I. Raĭkh, Yu. A. Firsov, and B. I. Shklovskiĭ for valuable discussions.

[1)]Owing to the logarithmically normal distribution of (Refs. 4 and 5) the value of $\ln(\sigma_0/\sigma)$ in the one-dimensional case fluctuates strongly, in contrast to a metal), and $(xn)^{-1}\ln\langle(\sigma_0/\sigma)\rangle$ depends on $n$. This makes uncertain the reconciliation of the numerical factors in (89) and (90).

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Translated by J. G. Adashko