Classification of finite particle motion in the Kerr metric

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We examine possible finite motions of particles in the Kerr metric. We determine the topology of the various sets of constants of the motion of the particle, and identify invariants that are independent of the rotation parameter.

In a pair of previous papers,[1,2] the present author considered various types of particle motion in the Kerr metric that are unbounded in the radial direction, and established the topology of sets of constants characterizing the various types of motion. Finite motions in the equatorial plane were examined in Refs. 3, 4 (de Felice[3] studied finite motion without making use of the first integrals, which were derived by Carter,[5] also in 1968), and they were studied in a fair amount of detail by a number of other authors,[6–9] who considered the continuous transition from stable to unstable equatorial orbits. Wilkins[10] studied a number of properties of orbits not lying in the equatorial plane, including those typical only of extreme Kerr-Newman geometry. The latter subject was treated by the present author,[11] who also described certain orbital properties peculiar to the extreme rotation of a black hole. Detailed studies of equatorial orbits have also been carried out by Chandrasekhar,[12] Novikov and Frolov,[13] and Shapiro and Teukolsky,[9] and a rather more comprehensive survey of the field is to be found in Ref. 14.

Thus far, however, nowhere has the topology of the various types of radial motion in the space of constants defining that motion been considered for finite orbits. Also left unresolved is the problem of finding a continuous transformation that takes one through a continuum of stable (or unstable) orbits from stable (or unstable) prograde equatorial orbits (in which the particle angular momentum with respect to the black hole rotation axis has the same sign as the black hole angular momentum) to stable (or unstable) retrograde equatorial orbits (in which the particle angular momentum is oppositely directed), for a test particle of given energy.

1. TYPES OF FINITE RADIAL MOTION

The motion of a test particle in the Kerr metric is described by the Carter equations (the first integrals of the geodesic equations),[5] and the corresponding equation for the rate of change of the coordinate r may then be written in the form[1,2]

$$(d\hat{r}/d\sigma)^2=R(\hat{r}), \tag{1}$$

where the polynomial $R(\hat{r})$ is given by

$$R(\hat{r})=(\hat{E}^2-1)\hat{r}^4+2\hat{r}^3+[\hat{a}^2(\hat{E}^2-1)-\hat{L}_z{}^2-\hat{Q}]\hat{r}^2$$
$$+2[(\hat{a}\hat{E}-\hat{L}_z)^2+\hat{Q}]\hat{r}-\hat{a}^2\hat{Q},$$
$$\hat{E}=E/\mu,\quad \hat{L}_z=L_z/M\mu,\quad \hat{Q}=Q/M^2\mu^2,\quad \hat{r}=r/M,\quad \hat{a}=a/M, \tag{2}$$

and the constants M and a are specific to the black hole: M is its mass, and a is its rotation parameter (angular momentum per unit mass), i.e., $a = S/M$, where S is the angular momentum of the black hole. In Eq. (2), E, L_z, Q, and μ relate to the particle: E is the particle energy, L_z is its z-component of angular momentum, μ is the particle mass, and Q is Carter's separation constant:

$$Q=p_\theta{}^2+\cos^2\theta[a^2(\mu^2-E^2)+\sin^{-2}\theta L_z{}^2].$$

Below we omit the cap from all variables.

Here we shall consider only finite orbits, for which the particle energy satisfies $E^2<1$, since the case $E^2\geqslant 1$ was thoroughly investigated in our previous papers.[1,2] We already know that the only possible finite orbits with $E^2\geqslant 1$ have constant r, and such orbits are unstable. In the present paper, we classify the various types of finite motion of test particles in the Kerr metric by investigating the roots of $R(r)$, as Synge did for the Schwarzschild metric,[15] and as the present author did for unbounded particle motion in the Kerr metric.[1,2]

We thus plan to classify particle motion having $E^2<1$. The polynomial $R(r)$ will clearly then have at least one root with $r\geqslant r_+ = 1+(1-a^2)^{1/2}$, since $R(r_+)\geqslant 0$, and for large enough r, $R(r)<0$. Since for $r>r_+$ the polynomial can have no more than three distinct roots when multiplicity is taken into account,[10] possible types of motion are as follows:

1) the polynomial $R(r)$ has one nonmultiple root at $r\geqslant r_+$. The particle will fall into the black hole in a finite proper time;

2) $R(r)$ has three nonmultiple roots ($r_+\leqslant r_1<r_2<r_3$). There is then a range of r-values ($r_2\leqslant r\leqslant r_3$) for which finite motion is possible over an infinitely long particle time. If $r_1<r<r_2$ or $r>r_3$, no motion is possible. If $r\leqslant r_1$, the particle will fall into the black hole in a finite proper time;

3) $R(r)$ has two distinct roots r_1 and r_2, with $r_+\leqslant r_1<r_2$, where r_1 is a nonmultiple root and r_2 is double ($R(r_1)=R(r_2)=R'(r_2)=0$). Particle motion is impossible for $r>r_2$ or $r_1<r<r_2$. There is a stable orbit at $r=r_2$, since $R''(r_2)<0$;

4) $R(r)$ has two distinct roots r_1 and r_2, with $r_+\leqslant r_1<r_2$, where r_1 is a double root and r_2 is a single root. Motion is impossible for $r>r_2$, and an unstable orbit exists for which $r=r_1$;

5) $R(r)$ has one triple root $r=r_1$ ($r_1\geqslant r_+$, $R(r_1)=R'(r_1)=R''(r_1)=0$).

Types 3 and 4 are clearly manifolds of codimension 1, and type 5 is a manifold of codimension 2 in the space of constants defining the motion, namely, E, L_z, and Q. We shall refer to orbits for which particle motion is confined to a surface of constant r as being spherical. This terminology is not entirely accurate, but it is quite widespread.[16]

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Note that if the black hole is undergoing extreme rotation ($a = 1$), the possible types of motion are as follows (with $L_z = 2E$ and $E^2 > 1/2$):

1) type 4 for $0 \leqslant Q \leqslant 3(E^2 - 1/3)$;

2) for $Q = 3(E^2 - 1/3)$, there is a triple root ($r = 1$) and a root at $r_2 > 1$. Thus, there is an unstable spherical orbit (type 6 motion) when $r = 1$;

3) for $3(E^2 - 1/3) < Q < E^4/(1 - E^2)$, we have a double root at $r = 1$, as well as roots with $1 < r_1 < r_2$, so a region $r_1 \leqslant r \leqslant r_2$ exists in which finite motion is possible, and at $r = 1$ we have a stable spherical orbit (type 7 motion);

4) for $Q = E^4/(1 - E^2)$, we have two double roots, $r = 1$ and $r = E^2/(1 - E^2)$, and there are thus two stable spherical orbits (type 8 motion);

5) when $Q > E^4/(1 - E^2)$ we have one double root ($r = 1$), and thereby one stable spherical orbit (type 9 motion).

It should also be pointed out that just as for the corresponding orbits in the case of infinite particle motion, orbit types 6–9 disappear when the rotation parameter no longer has an extreme value (structural instability[11]); this also happens to the properties examined by Dymnikova.[17,18] The case $1/3 \leqslant E^2 \leqslant 1/2$ may be treated similarly.

2. TYPES OF FINITE RADIAL MOTION IN THE SCHWARZSCHILD METRIC

For a moving particle in the Schwarzschild metric ($a = 0$), radial motion is governed by the equation

$$(dr/d\sigma)^2 = R(r), \tag{3}$$

where $R(r) = (E^2 - 1)r^4 + 2r^3 - L^2r^2 + 2L^2r$ (E is the particle energy and L is its angular momentum). With the notation $l = L^2/(E^2 - 1)$, $\alpha = (E^2 - 1)^{-1}$, we may write out the condition for a multiple root of the polynomial $R(r)$ by setting the discriminant to zero (as in Ref. 19):

$$l^2 + l(\alpha^2 - 18\alpha - 27) - 16\alpha^3 = 0, \tag{4}$$

which yields

$$l_\pm = [-(\alpha^2 - 18\alpha - 27) \pm D^{1/2}(\alpha)]/2, \tag{5}$$

where $D(\alpha) = \alpha^4 + 28\alpha^3 + 270\alpha^2 + 972\alpha + 729$. It is straightforward to show that $D(\alpha) = (\alpha + 9)^3(\alpha + 1)$. Clearly, if $\alpha \leqslant -9$, we have $D(\alpha) \geqslant 0$. We thus find that for a given particle energy, the angular momentum corresponding to a stable or unstable circular orbit (L^s or L^u) is given by

$$(L^s)^2 = \frac{-(\alpha^2 - 18\alpha - 27) - (\alpha + 9)[(\alpha + 9)(\alpha + 1)]^{1/2}}{2\alpha}, \tag{6}$$

$$(L^u)^2 = \frac{-(\alpha^2 - 18\alpha - 27) + (\alpha + 9)[(\alpha + 9)(\alpha + 1)]^{1/2}}{2\alpha}. \tag{7}$$

It is not hard to see that for $\alpha \leqslant -9$, the right-hand sides of Eqs. (6) and (7) are greater than zero. There are thus two values of the angular momentum corresponding to stable circular orbits and two corresponding to unstable circular orbits, for a given particle energy:

$$L^s = \pm\left\{\frac{-(\alpha^2 - 18\alpha - 27) - (\alpha + 9)[(\alpha + 9)(\alpha + 1)]^{1/2}}{2\alpha}\right\}^{1/2}, \tag{8}$$

$$L^u = \pm\left\{\frac{-(\alpha^2 - 18\alpha - 27) + (\alpha + 9)[(\alpha + 9)(\alpha + 1)]^{1/2}}{2\alpha}\right\}^{1/2}. \tag{9}$$

Orbits for which the angular momenta in (8) and (9) have opposite signs clearly differ only in the direction of particle motion in those circular orbits. We can also easily write out the expression for the radius of a stable or unstable circular orbit.

$$r^u = [{}^8/_{27}\alpha^3 + l + (\alpha/3 + 1)]^{1/3} - {}^2/_3\alpha, \tag{10}$$

$$r^s = [{}^8/_{27}\alpha^3 + l - (\alpha/3 + 1)]^{1/3} - {}^2/_3\alpha, \tag{11}$$

where r^u is the radius of the unstable orbit, and r^s is the radius of the stable orbit. If the particle energy is such that circular orbits exist (and $E^2 < 1$) and $\alpha \leqslant -9$ ($E^2 \geqslant 8/9$), then both r^u and r^s will be greater than zero, since $l_\pm < 0$ and $\alpha/3 + 1 < 0$. It follows from the form of the polynomial $R(r)$ that when $0 < r < 2$ there are no roots, so $r_u > 2$ and $r_s > 2$. A more thorough analysis[9] shows that $r^s \geqslant 6$, $r^u \geqslant 3$.

It is not difficult to verify that when the particle energy E is fixed and $|L| > |L^s|$, type 1 motion takes place; for $|L| = |L^s|$ we have type 3 motion, for $|L^u| < |L| < |L^s|$ we have type 2, for $|L| = |L^u|$ it is type 4, and for $|L| < |L^u|$ the motion is of type 1. Finally, $|L| = |L^u| = |L^s|$ results in type 5 motion (this accurately summarizes the well-known fact[7,9] that the maximum of the polynomial $R(r)$ merges with the minimum at $\alpha = -9$, $E^2 = 8/9$.

3. TYPES OF RADIAL MOTION IN THE KERR METRIC

The radial motion of a particle in the Kerr metric is determined by three constants, E, L_z, and Q (Refs. 1, 2). We now investigate those sets of constants of the motion of a particle that correspond to different types of motion, given the value of the black hole rotation parameter (angular momentum per unit mass). As before,[1,2] we pass a plane of constant E through E, L_z, Q space and examine the sets of constants that correspond to the various types of motion. Clearly, the boundary of the set of constants corresponding to motion of the first and second types is a set of constants of the third and fourth types. According to our previous arguments,[1,2] the motion will be of type 3 or 4 (or type 5 in the degenerate case) if

$$R(r) = 0, \quad R'(r) = 0 \tag{12}$$

when $Q \geqslant 0$ and $r \geqslant r_+$.

Noting that Eqs (12) specify the functions $r(L_z)$ and $Q(L_z)$ implicitly, we obtain

$$(dQ/dL_z)(-\Delta) = 2L_zr^2 - 4(L_z - aE)r,$$

$$(dQ/dL_z)(-2r + 2) + R''\frac{dr}{dL_z} = 4L_zr - 4(L_z - aE) \tag{13}$$

when $r > r_+$, $Q \geqslant 0$ (assuming that $R''(r) \neq 0$, i.e., the motion is not of type 5). A rather lengthy but straightforward calculation then shows that when $Q = (L^u)^2$ and $r = r^u$, where $(L^u)^2$ and r^u are given by (7) and (10), and

$$L_z = -2aE/(r^u - 2), \tag{14}$$

the equations (12) are satisfied identically. Furthermore, it can then readily be shown that $R_{rr}''(r^u) > 0$ (if $E^2 > 8/9$). As was the case previously,[1] one may conclude that these values of the constants of the motion correspond to a maxi-

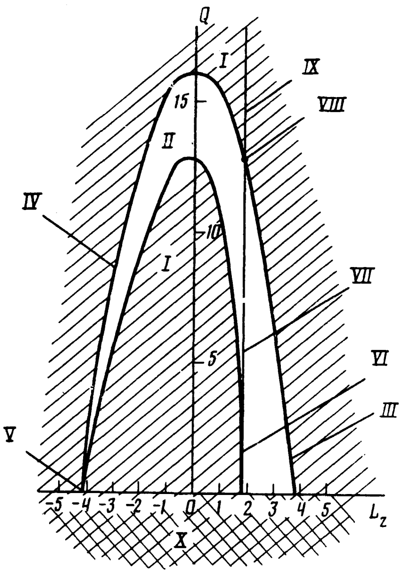

FIG. 1.

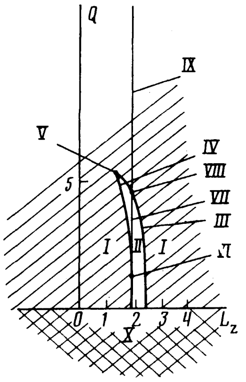

FIG. 3.

mum of $Q(L_z)$—in other words, for a particle of given energy that is initially located in the equatorial plane, the maximum value of p_θ^2 (which corresponds to an orbit of type 4) will be independent of the rotation parameter of the black hole, as will the value of r^u (the radius of the spherical orbit[16]). Similarly, for $Q = (L^s)^2$ and $r = r^s$, where $(L^s)^2$ and

$$L_z = -2aE/(r^s - 2), \tag{15}$$

Eqs. (12) are satisfied identically. Moreover, it then turns out that $R_{rr}''(r^s) < 0$ (for $E^2 > 8/9$). hence, for a particle of given energy that is initially located in the equatorial plane, the maximum value of p_θ^2 (which corresponds to a stable spherical orbit at $r = r^s$) will be independent of the rotation parameter of the black hole, as will the value of r^s. Since an analysis of orbits in the Schwarzschild metric shows $r^s > 6$ and $r^u > 3$ (for $E^2 > 8/9$), the corresponding inequalities will still hold for the orbits under consideration here in the Kerr metric.

Finally, we present some plots of the constants corresponding to the various types of motion, assuming extreme rotation. In Fig. 1, we show the domains of Q and L_z for a particle energy $E = (25/27)^{1/2}$ (resulting in type 5 motion; for higher energy, type 5 motion is not possible). Roman numeral I marks the region corresponding to motion of type 1, and so on up through Roman numeral IX; Roman numeral X identifies the set of constants for which motion is no longer possible. In Fig. 2. we have plotted similar results for a particle energy $E = (8/9)^{1/2}$ (yielding $r^u = r^s$, with the above-mentioned stable and unstable spherical orbits merging into an orbit of type 5). Figure 3 presents our results for $E = 0.9$. The notation in Figs. 2 and 3 is the same as in Fig. 1. Clearly, for particle energy $E < 3^{-1/2}$, only orbits of type 1 are possible.

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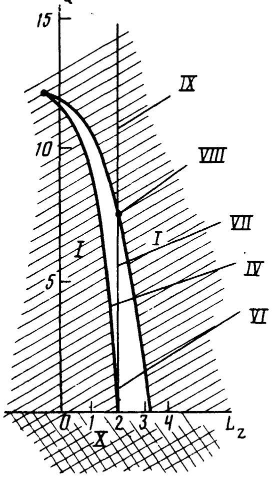

FIG. 2.

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