Theory of the dynamic stability of plasma systems

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Internal instabilities of the plasma of a diffuse pinch result from the acceleration of the plasma in the course of its compression and the expansion of the current channel. The spectra of the growth rates $\sigma_{m,k}$ of the hydromagnetic instabilities responsible for the disruption of the initial cylindrical symmetry during compression are calculated. For a Z-pinch with a Gaussian density profile, the major instabilities in the course of the compression are the small-scale sausage and kink instabilities with $kR \gg 1$ (R is a typical radius of the pinch). Superimposed on these small-scale instabilities is a filamentation instability with $m \gg 1$, which develops more slowly. If the density instead has a power-law profile, the filamentation instabilities will develop more rapidly than the sausage and kink instabilities. Dynamic stabilization of a pinch by a longitudinal magnetic field makes it possible to maintain symmetry up to radial compressions of the plasma significantly higher than in the absence of a field. A numerical solution of the problem of the instability evolution shows that the error of the method used is at most 5%. The method can be applied to a long list of problems involving the stability of fast plasma motions.

1. INTRODUCTION

Research on the hydrodynamics of fast plasma motions, primarily implosions ("cumulative processes"), is important for a long list of applications, including problems in inertial fusion, in the development of sources of intense optical and neutron radiation, and for studying pinch and liner systems. Research on the stability of the plasma motion plays a key role here, determining in particular the maximum compression which can be achieved while the original spherical or cylindrical symmetry is preserved, since it is this stage of the motion which is the most important for most applications. The theory of the stability of equilibrium plasma configurations which has been developed[1,2] cannot be applied directly to these problems since the stage in which the plasma undergoes acceleration makes a significant, usually dominant, contribution to the evolution of instabilities. Because of this circumstance, the final equilibrium state of the plasma which is the object of the stability estimates made by the conventional methods may not be reached at all.

The theory of the dynamic stability of plasmas has been developed to a far lesser extent. Several exact analytic solutions are available for problems concerning the stability of a self-similar motion. These solutions have been derived for certain special cases in which it is possible to separate variables in the perturbation equations.[3–5] Most of the other results concern idealized models of a plane slab[6–8] or an infinitely thin shell[9]; alternatively, they come from numerical calculations on the evolution of initial perturbations for specific situations.[10,11]

In the present paper we propose, and derive a theory for, a general apprach to research on the dynamic stability of fast plasma motions in the linear stage of instabilities. We illustrate it in the particular example of the compression stability in a diffuse Z-pinch with a longitudinal external magnetic field. We derive explicit analytic expressions for the spectra of the hydromagnetic instabilities of diffuse pinches. These expressions yield interpretations of some effects seen experimentally: the plasma filamentation in compressing pinches and the dynamic stabilization of the current channel of a Z-pinch by a comparatively weak longitudinal magnetic field.

2. DISPERSION RELATION AND ENERGY PRINCIPLE FOR AN ACCELERATED PLASMA

The plasma implosion process in a pulsed system is always unstable in the sense that small perturbations of hydrodynamic variables grow during the compression. Instabilities develop in the course of the acceleration, and often during the deceleraton, of the plasma, because of a force which is exerted on the plasma by the "lighter fluid." This situation is seen in a long list of problems: in the compression of shells by the pressure of a plasma corona in laser fusion, in the compression of a low-density fuel by a shell, in the compression of a current channel in a pinch by the pressure of the self-magnetic field of the current, in the deceleration of a hollow cylindrical liner during magnetic flux compression,[12] etc. If this force is concentrated on a certain boundary, the stage is clearly set for the onset of a Rayleigh-Taylor instability; in the more general case of a distributed force, the stage is set for a convective (interchange) instability.

We will be discussing pulsed systems for which the instability rise time is limited by, for example, the plasma compression time τ. If the instabilities grow at a pace slow enough that their typical rise rate satisfies $\sigma \lesssim \tau^{-1}$, the disruption of the symmetry of the flow by the instability will not exceed a certain permissible level, and the flow may be called "dynamically stable."

We will restrict the discussion to the fastest instabilities, the convective hydrodynamic or MHD instabilities of a moving plasma, which in many cases determine the limits on symmetric compression of the system. To describe the unperturbed motion of the plasma we use the equations of ideal MHD. We wish to stress that the method which we are proposing here for studying dynamic stability has a wider range of applicability; it can be generalized directly to incorporate radiative loss, dissipation, etc.

We start from the equations of ideal MHD in the form

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$$\frac{\partial \rho}{\partial t}+\nabla(\rho \mathbf{u})=0, \quad (1)$$

$$\frac{\partial \mathbf{B}}{\partial t}-[\nabla[\mathbf{uB}]]=0, \quad (2)$$

$$\frac{\partial p}{\partial t}+(\mathbf{u}\nabla)p+\gamma p(\nabla \mathbf{u})=0, \quad (3)$$

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}\right]+\nabla p-\frac{1}{4\pi}[[\nabla \mathbf{B}]\mathbf{B}]=0, \quad (4)$$

where ρ, p, $\mathbf{u}$, and γ are the density, pressure, mass velocity, and adiabtic index, respectively, of the plasma, and $\mathbf{B}$ is the magnetic field. So that we can illustrate the results in the examples of plasma compression in pinches, we will assume that the unperturbed motion is cylindrically symmetric. Only the radial component of the velocity, u_r, is then nonzero, and ρ, p, u_r, B_z, and B_φ depend on r and t alone.

The equations for the small perturbations ρ', p', $\mathbf{u}'$, and $\mathbf{B}'$ (of the density, pressure, velocity, and magnetic field, respectively) are

$$\frac{\partial \rho'}{\partial t}+\nabla(\rho'\mathbf{u})=-\nabla(\rho\mathbf{u}'), \quad (5)$$

$$\frac{\partial \mathbf{B}'}{\partial t}-[\nabla[\mathbf{uB}']]=[\nabla[\mathbf{u}'\mathbf{B}]], \quad (6)$$

$$\frac{\partial p'}{\partial t}+(\mathbf{u}\nabla)p'+\gamma p'(\nabla\mathbf{u})=-(\mathbf{u}'\nabla)p-\gamma p(\nabla\mathbf{u}'), \quad (7)$$

$$\rho\left[\frac{\partial \mathbf{u}'}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}'+(\mathbf{u}'\nabla)\mathbf{u}\right]=-\rho'\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}\right]$$
$$-\nabla p'+\frac{1}{4\pi}[[\nabla\mathbf{B}]\mathbf{B}']+\frac{1}{4\pi}[[\nabla\mathbf{B}']\mathbf{B}]. \quad (8)$$

In Sec. 5 below we will examine solutions of the complete system of equations, (5)–(8). At this point we note that by virtue of our comments above, dynamic stability can be disrupted only by perturbations for which the growth rate σ is large in comparison with $\tau^{-1}\sim u/R$, where R is a characteristic radius of the compressing plasma, and u is a characteristic radial velocity. For such perturbations we can ignore the spatial derivatives in comparison with the time derivatives on the left sides of Eqs. (5)–(8). In the case of the compression of shells whose thickness δ is much less than the radius R, so that we have $\nabla\sim 1/\delta$, this approximation is justified in, for example, regimes of uniform compression.[3–5] This simplification allows us to integrate Eqs. (5)–(7) by introducing the vector ξ, which describes a displacement of a plasma particle with respect to the unperturbed motion, so we have $\mathbf{u}'=\partial\xi/\partial t$. As a result of an integration of (5)–(7), the perturbations ρ', $\mathbf{B}'$, and p' can be expressed explicitly in terms of ξ:

$$\rho'=-\nabla(\rho\xi), \qquad \mathbf{B}'=[\nabla[\xi\mathbf{B}]],$$
$$p'=-(\xi\nabla)p-\gamma p(\nabla\xi). \quad (9)$$

In our approximation, Eqs. (8) and (9) are formally analogous to the standard formulation of the problem of the stability of an equilibrium plasma configuration in an effective "gravitational" field. The equilibrium takes the following form, when the "gravitational force" or inertial force is taken into account at each instant:

$$\frac{dp}{dr}+\frac{1}{4\pi}\left[B_z\frac{dB_z}{dr}+\frac{B_\varphi}{r}\frac{d}{dr}(rB_\varphi)\right]=\rho g, \quad (10)$$

where

$$g=-\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}\right)$$

is the negative of the local plasma acceleration at the given instant.

We can define the concept of an instantaneous growth rate by writing the plasma particle displacement as a function of the unperturbed "equilibrium" motion at each instant, in accordance with

$$\xi(\mathbf{r},t)=\xi(r)\exp[\sigma t+im\varphi+ikz].$$

By analogy with ordinary stability theory,[13] we would naturally call the motion at a given instant "dynamically σ-stable" if at this instant there exist no perturbations which are growing more rapidly than $\exp(\sigma t)$. The condition for dynamic σ-stability can be formulated as the requirement that a corresponding energy functional be positive definite for small perturbations. Since we are interested in the stability of a diffuse pinch, in which the plasma fills the entire volume, out to the conducting walls, we will retain only the volume integral in the expression for the energy. Using (8) and (9), we can rewrite this integral as

$$W(\xi)=\frac{1}{2}\int dr\Big\{\gamma p(\nabla\xi)^2+(\xi\nabla p)(\nabla\xi)+\frac{1}{4\pi}\{[\nabla[\xi\mathbf{B}]]^2$$
$$+[\nabla\mathbf{B}][\xi[\nabla[\xi\mathbf{B}]]]\}+(\xi g)\nabla(\rho\xi)+\rho\sigma^2\xi^2\Big\}>0. \quad (11)$$

In view of the cylindrical symmetry of the unperturbed state, condition (11) can be reduced to a one-dimensional energy functional which depends on the profile of the radial component of the displacement, $\xi_r(r)$:

$$W(\xi_r)=\int_0^{R_s(t)} dr\left\{K\left[\frac{d}{dr}(r\xi_r)\right]^2+rL\xi_r^2\right\}\geqslant 0, \quad (12)$$

where $R_s(t)$ is the outer boundary of the plasma in the pinch,

$$K=\frac{F^2}{rD}\left[\rho\sigma^2\left(\frac{B^2}{4\pi}+\gamma p\right)+\gamma pf^2\right],$$

$$L=F^2+\frac{B_\varphi}{2\pi}\frac{d}{dr}\left(\frac{B_\varphi}{r}\right)+g\frac{d\rho}{dr}-\frac{1}{D}\left[\rho^2g^2\left(k^2+\frac{m^2}{r^2}\right)F^2\right.$$
$$-\frac{\rho^2\sigma^2 g}{\pi r}kB_\varphi\left(kB_\varphi-\frac{m}{r}B_z\right)+\left.\frac{k^2B_\varphi^2}{\pi r^2}\left(\rho\sigma^2\frac{B^2}{4\pi}+\gamma pf^2\right)\right]$$
$$-r\frac{d}{dr}\left\{\frac{kB_\varphi}{2\pi r^2D}\left(kB_\varphi-\frac{m}{r}B_z\right)\left[\rho\sigma^2\left(\frac{B^2}{4\pi}+\gamma p\right)+\gamma pf^2\right]\right.$$
$$\left.+\frac{\rho^2\sigma^2 gF^2}{rD}\right\}, \quad (13)$$

$$D=\rho^2\sigma^4+\left(k^2+\frac{m^2}{r^2}\right)\left[\rho\sigma^2\left(\frac{B^2}{4\pi}+\gamma p\right)+\gamma pf^2\right],$$

$$f^2=\frac{1}{4\pi}\left(kB_z+\frac{m}{r}B_\varphi\right)^2, \quad F^2=\rho\sigma^2+f^2, \quad B^2=B_\varphi^2+B_z^2.$$

By virtue of the condition $K>0$, a sufficient condition for dynamic σ-stability is $L\geqslant 0$. The minimum value of the parameter σ for which the motion at the given instant is dynamically σ-stable is equal to the largest of the instantaneous growth rates which are found from the following boundary-value problem for the Euler-Lagrange equation corresponding to the functional (12):

$$\frac{d}{dr}\left[K\frac{d}{dr}(r\xi_r)\right]-L\xi_r=0. \quad (14)$$

The boundary conditions

$$r\xi_r=0 \quad \text{at } r=0 \text{ and } r=R_s(t) \quad (15)$$

define a class of convective bulk instabilities and bulk plasma oscillations. If the second of these conditions is imposed as $r\to\infty$, it can be formulated as the weaker requirement that the magnetic field perturbations vanish in the limit $r\to\infty$.

It can be shown that the eigenvalues σ^2 of the boundary-value problem (14), (15) are real. Positive values of σ^2, which correspond to various radial perturbation modes with given m and k, form a discrete spectrum, which becomes closely spaced toward the origin and which is bounded from above. Negative values of σ^2 correspond to eigenfunctions which form a continuous spectrum, as for the spectrum of perturbations of a steady-state quilibrium Z-pinch.[13] The relationship between the number of zeros of an eignfunction and the index of the eigenvalue in order of decreasing σ^2 for the unstable part of the spectrum has the form characteristic of a Sturm-Liouville problem. The maximum growth rate $\sigma_{m,k}$ for any m and k corresponds to an eigenfunction which has no zeros between the axis and the outer boundary of the pinch.

Figure 1 illustrates the situation with the spectra and eigenfunctions of the boundary-value problem (14), (15) as calculated for a steady-state Z-pinch (at the left) and for a dynamic Z-pinch with a Gaussian radial profile of the unperturbed plasma density.

We thus see that for arbitrary wave numbers m and k the solution of the given boundary-value problem reveals the perturbation mode which grows most rapidly and whose growth rate has the instantaneous value $\sigma_{m,k}(t)$. To evaluate the growth of the perturbations over a finite time interval we use the expression

$$\exp\left[\int_0^t \sigma_{m,k}(t')\,dt'\right],$$

which corresponds to the ordinary quasiclassical approximation. Note that since we are interested primarily in the growth rates which influence the dynamic stability of the system, our approximation is justified: Growth rates which are large in comparison with $u/R\sim\tau^{-1}$ are determined relatively accurately, and the error in the calculation of the comparatively small growth rates is inconsequential, since the total contribution of these growth rates to the growth of the perturbations over the time τ is small.

3. SPECTRA OF INSTABILITY GROWTH RATES FOR SELF-SIMILAR COMPRESSION OF A *Z*-PINCH

The results of Sec. 2 for arbitrary solutions of Eqs. (1)–(4), describing a cylindrically symmetric compression of a Z-pinch or θ-pinch, can be used to find the maximum instantaneous value of the growth rate $\sigma_{m,k}$ for any perturbation

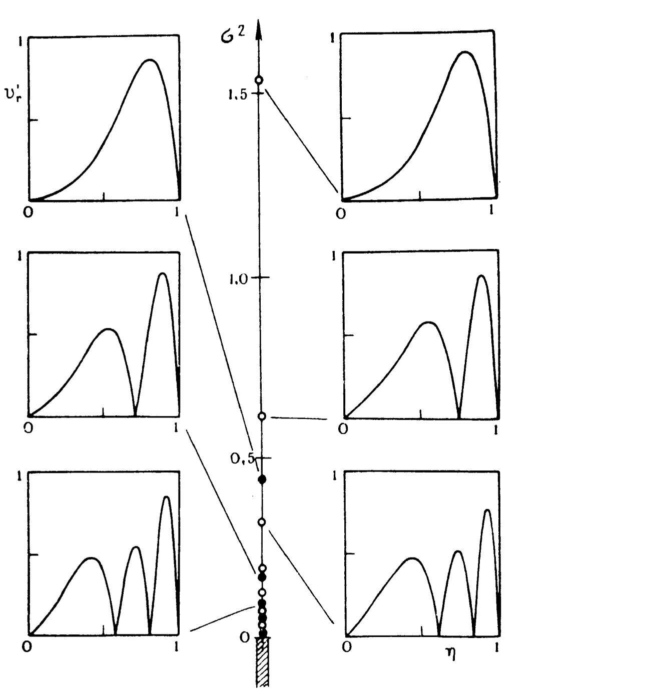

FIG. 1. Absolute values of the eigenfunctions, and spectrum of eigenvalues, for a steady-state Z-pinch (at the left) and for a dynamic Z-pinch (at the right), according to calculations for a Gaussian radial density distribution with $m=0$, $kR_0=5$, $\beta=0.1$, and $b=0$. The continuum at $\sigma^2<0$ is indicated by the hatched band. Here and below, the time is expressed in units of t_0, and the growth rates in units of t_0^{-1}.

component (m,k) at each instant. In particular, this method can be used to study the dynamic stability of one-dimensional solutions found through numerical simulation. Below we will illustrate the method in the example of flows describable by self-similar solutions of Eqs. (1)–(4), for which an analytic expression is known for the profiles of the hydrodynamic variables which figure in (13) for each instant. It thus becomes possible to derive some useful analytic estimates. The solutions which we are using here, like those studied in Refs. 3 and 5, fall in the category of self-similar solutions with a uniform deformation. In the present case, however, it is not possible to separate variables in the equations for the perturbations.

For such self-similar solutions,[14,15] the hydrodynamic variables depend in the following way on the self-similar coordinate $\eta=r/R(t)$ and on the time:

$$u(r,t)=R_0\dot\alpha(t)\eta, \quad (16)$$
$$\rho(r,t)=\rho_0\alpha(t)^{-2}N(\eta),\quad p(r,t)=p_0\alpha(t)^{-2\gamma}P(\eta),$$
$$B_\varphi(r,t)=B_{\varphi0}\alpha(t)^{-1}H_\varphi(\eta),\quad B_z(r,t)=B_{z0}\alpha(t)^{-2}H_z(\eta),$$

where $R(t)$ is a time-dependent characteristic radius of the pinch; $\alpha(t)=R(t)/R_0$; $R_0\equiv R(t=0)$; ρ_0, p_0, $B_{\varphi0}$ and B_{z0} are normalization constants; and the dimensionless functions (representatives) $N(\eta)$, $P(\eta)$, $H_\varphi(\eta)$ and $H_z(\eta)$ describe self-similar profiles which remain constant over time. The time dependence $\alpha(t)$ is determined by the equation of motion

$$t_0^2\frac{d^2\alpha}{dt^2}+\alpha^{-1}-\beta\alpha^{1-2\gamma}-b\alpha^{-3}=0, \quad (17)$$

where the parameters $\beta=4\pi p_0/B_{\varphi0}^2$ and $b=B_{z0}^2/B_{\varphi0}^2$ characterize the relative role played by the kinetic pressure of the plasma and the axial magnetic field. The unit of time here is $t_0 = (4\pi p_0)^{1/2} R_0 / B_{\varphi 0}$; the initial conditions are $\alpha(0) = 1$ and $\dot{\alpha}(0) = 0$.

In the case $\beta + b = 1$, Eq. (17) describes an equilibrium of the plasma: $\alpha(t) = 1 = \text{const}$. For $0 < \beta + b < 1$, we find periodic solutions which describe radial oscillations of the pinch around an equilibrium position with a period on the order of t_0 (Ref. 14). For $\beta + b \ll 1$, the compression of the plasma in the current channel of the pinch in the course of these oscillations is significant. Finally, the case $\beta = b = 0$ corresponds to an unbounded compression of the pinch in the absence of a counterpressure (a collapse) over a finite time $\tau = (\pi/2)^{1/2} t_0$.

As examples we consider the unperturbed flows describable by self-similar solutions with a Gaussian decay of the plasma density with distance from the axis in an isothermal Z-pinch,[14]

$$N(\eta) = P(\eta) = \exp(-\eta^2/2), \tag{18}$$

$$H_\varphi(\eta) = (2/\eta)\left[1 - \exp(-\eta^2/2)(\eta^2/2 + 1)\right]^{1/2},$$

$$H_z(\eta) = \sqrt{2}\exp(-\eta^2/4),$$

and self-similar solutions with power-law decays of the density and the pressure,

$$N(\eta) = (1+\eta)^{-s}, \quad P(\eta) = \frac{(s-1)\eta + 1}{(s-1)(s-2)(1+\eta)^{s-1}}, \tag{19}$$

$$H_\varphi(\eta) = \frac{1}{\eta}\left[2\int_0^\eta d\eta'\, \eta'^3 N(\eta')\right]^{1/2}, \quad H_z(\eta) = [2P(\eta)]^{1/2}.$$

Since the current flowing through the pinch and the mass and thermal energy per unit length of the pinch are finite, the plasma density must fall off more rapidly than r^{-4} at infinity; i.e., we need $s > 4$ in (19).

In order to compare our results with existing results in the theory of the stability of equilibrium plasma configurations, we recall that a steady-state diffuse Z-pinch without a longitudinal magnetic field is stable against perturbations with $m \geqslant 2$ (instabilities with $m \geqslant 2$ can develop only at the pinch boundary), a Z-pinch is also stable against perturbations with $m = 0$ (the sausage instability) provided that the plasma pressure falls off more slowly than $r^{-2\gamma}$ with distance from the axis.[1,2]

In a time-dependent problem these results no longer hold: In the course of the compression, all perturbation modes of the accelerated plasma corresponding to a wide range of m grow, regardless of the profiles of the plasma density and the pressure [see Figs. 2 and 3, which show spectra of the growth rate $\sigma_{m,k}$ for the compression of pinches with Gaussian and power-law profiles at the times $t = t_0$ ($b \neq 0$) and $t = 1.3 t_0$ ($b = 0$)]. We note in particular that $\sigma_{0,k}$ is nonzero in this case at all values of k for the compression of a pinch with a power-law pressure profile which falls off as r^{-3} in the limit $r \to \infty$, i.e., more slowly than $r^{-2\gamma}$, for the value $\gamma = 5/3$, which we have assumed here in all the calculations.

It can be seen from Fig. 3 that the most important instabilities in the compression of a pinch with power-law density and pressure decays are instabilities with rspect to filamentation, i.e., with respect to a breakup of the plasma column into distinct filaments. For example, for the conditions in Fig. 3a

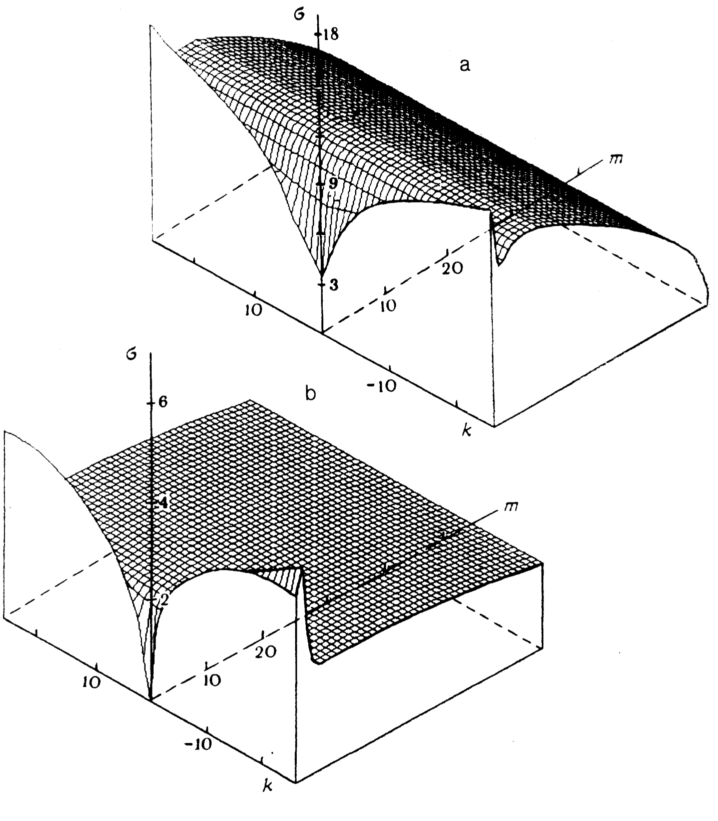

FIG. 2. Spectrum of instability growth rates $\sigma_{m,k}$ for a pinch with a Gaussian density distribution with $\beta = 10^{-4}$ and $\rho_s/\rho_m = 8 \cdot 10^{-4}$. a—$b = 0$, $t = 1.3\, t_0$; b—$b = 0.02$, $t = t_0$.

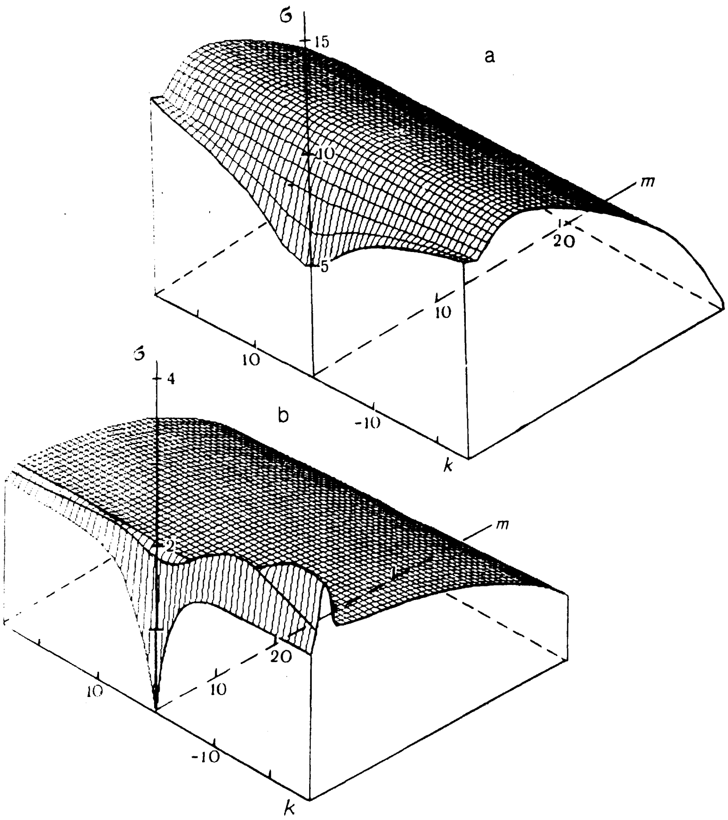

FIG. 3. Spectrum of instability growth rates $\sigma_{m,k}$ for a pinch with a power-law density profile (19) with $s = 5$ and $\beta = 10^{-4}$. a—$b = 0$, $t = 1.3t_0$; b—$b = 0.02$, $t = t_0$.

we see that the maximum growth rate depends weakly on k and is reached at $m \approx 7$–10. A compressing pinch always exhibits a filamentation instability, regardless of the density profile and regardless of whether there is an external magnetic field. Its growth rate is on the order of the growth rate of the fundamental instability mode (Figs. 2 and 3). The onset of all of the instabilities observed in experiments on pinches is usually accompanied by filamentation of the current channel.[16,17]

Pronounced compression of the current channel of the pinch is obviously the case of greatest interest, in which we can expect a high energy density, a high neutron yield, etc. We will thus focus on the dynamics of a pinch with a negligible counterpressure. For an analytic study of the qualitative form of the spectra $\sigma_{m,k}$ in this case we consider the compression of the plasma in a pinch in the limits $\beta \to 0$ and $b \to 0$.

We begin with the $m = 0$ sausage instability, which is the instability observed most frequently in experiments. It can be seen from Figs. 2 and 3 that the growth rate of the sausage instability is either the very highest growth rate or nearly so. Let us consider the limiting case $\beta = b = 0$, $k \to \infty$. In expression (12) for the energy integral we have $K \propto k^{-2} \to 0$ as $k \to \infty$; i.e., the highest instability growth rate is reached specifically at large values of k. From (13) we find in this limit

$$L=\rho\sigma^2\left\{1+\frac{g}{\sigma^2 r}r\frac{d}{dr}\ln\left(\frac{\rho r}{B_\varphi}\right)\right\}. \tag{20}$$

For self-similar solutions with a uniform deformation we have $g/r = -\ddot{a}/a$, and in the limit $\beta = b = 0$ we have $g/r = 1/a^2 t_0^2$. The motion at a given instant is thus σ-stable if $\sigma > \sigma_{0,\infty}$ where

$$\sigma_{0,\infty}=\frac{1}{at_0}\left\{\max_r\left[r\frac{d}{dr}\ln\left(\frac{B_\varphi}{\rho r}\right)\right]\right\}^{1/2}. \tag{21}$$

If the expression in square brackets in (21) is negative, there are no instabilities which grow exponentially against the background of the unperturbed plasma motion. A necessary and sufficient condition for stability is that energy integral (12) be positive definite. In the limit $k \to \infty$ the energy integral is positive if $L \geqslant 0$ for all possible values of r. Specifically, if the relation $L < 0$ holds near some point $r = r_1$, we can choose a displacement $\xi_r(r)$, localized near this point, for which the relation $W < 0$ holds by virtue of the small value of $K \propto k^{-2}$.

It follows from (21) that the boundedness of the growth rate in the limit $k \to \infty$ is determined primarily by the asymptotic density profile. In particular, for an arbitrary power-law profile $\rho \propto r^{-s}$ in the limit $r \to \infty$ the maximum growth rate in the limit $k \to \infty$ is

$$\sigma_{0,\infty}=(s-2)^{1/2}/at_0. \tag{22}$$

For the Gaussian density profile (18), on the other hand, the quantity $-rd(\ln\rho)/dr = [r/R(t)]^2$ is formally unbounded in the limit $r \to \infty$, so the growth rate can be arbitrarily large at sufficiently large values of k, as can be seen from (21). However, it is clear from physical considerations that under actual experimental conditions an exponential decay of the density should give way at a certain level to a slower decay, corresponding to an approach to the background, arbitrarily low value of the plasma density surrounding the pinch. A Gaussian profile must accordingly be cut off at a certain density level.

Another cutoff possibility corresponds to, for example, a sharp decrease in the density at the boundry, associated with the skin effect in the part of the pinch current at its outer boundary. We can show that in any case the maximum growth rate of the convective bulk instabilities will depend only weakly on how and where we choose to cut off the Gaussian profile. Denoting by ρ_m and ρ_s the values of the density at the axis and the pinch boundary, respectively, we find from (21)

$$\sigma_{0,\infty}=\frac{\sqrt{2}}{\alpha t_0}\left(\ln\frac{\rho_m}{\rho_s}+1\right)^{1/2}. \qquad (23)$$

This expression for $\sigma_{0,\infty}$, which depends very weakly on the value of ρ_m/ρ_s, does not depend on whether we cut off the density exactly to zero at the sharp pinch boundary or replace a Gaussian profile by some arbitrary power law at a level $\rho\approx\rho_s$. For the conditions in Fig. 2a, for example, we find the estimate ($\sigma_{0,\infty}\approx 16$), because the finite counterpressure plays a stabilizing role.

We now consider the behavior of the maximum growth rate as a function of m in the same limit, in the absence of a counterpressure: β, $b\ll 1$. Assuming $m=kq\to\infty$, we find from (13)

$$K=\frac{q^2}{q^2+r^2}\frac{B_\varphi^2}{4\pi},$$
$$L=k^2q^2\frac{B_\varphi^2}{4\pi\sigma^2r^2}\left(\sigma^2-\frac{g}{r}\frac{4\pi\rho gr}{B_\varphi^2}\right)+\rho\sigma^2$$
$$+\frac{B_\varphi}{2\pi}\frac{d}{dr}\left(\frac{B_\varphi}{r}\right)+\frac{r^2}{q^2+r^2}g\frac{d\rho}{dr}+\frac{B_\varphi^2r^2}{4\pi(q^2+r^2)^2}. \qquad (24)$$

It can be seen from (24) that in the limit $kq\to\infty$ the values of L and W are dominated by the term of order k^2q^2, so the sign of L and W is determined by the factor in parentheses multiplying k^2q^2. Hence

$$\sigma_{\infty,k}=\frac{1}{\alpha t_0}\left(\max_r\frac{4\pi\rho gr}{B_\varphi^2}\right)^{1/2}=\frac{1}{\alpha t_0}\left\{\max_r\left[\frac{1}{B_\varphi}\frac{d}{dr}(rB_\varphi)\right]\right\}^{1/2}. \qquad (25)$$

We see that a filamentation instability, with a growth rate which can be smaller than the growth rate corresponding to the onset of a sausage instability, but comparable in magnitude, should occur regardless of the density profile, at large values of m. For a Gaussian density profile, the numerator in (25) has the value $\sqrt{2}$, so filamentation can develop in the interior superposed on the formation of necks. This circumstance is caused by specifically the plasma acceleration, and as we have already mentioned, it does not occur in a steady-state Z-pinch.

The limit $m\to\infty$ in (24) is possible at a finite value of k and as $q\to\infty$; alternatively, it is possible at a finite value of q as $k\to\infty$. If q remains bounded, the contribution to L from the term containing $d\rho/dr$ can be important at large values of r. For a power-law or truncated Gaussian density profile, however, the first term is again dominant; i.e., expression (25) continues to hold as $k\to\infty$. For large values of m and k, the growth rate is thus nearly independent of k, and it varies slowly with m. For nonzero values $\beta\neq 0$ the asymptotic behavior at large m changes, since L is now dominated by the positive term $f^2=m^2B_\varphi^2/4\pi r^2$; the compression of the Z-pinch is thus stable against perturbations with sufficiently large values of m. The effect of a finite β is determined by the relative magnitude of the terms γpF^2 and $\rho\sigma^2(\gamma p+B_\varphi^2/4\pi)$ in expression (13) for D. The first term, which ensures stability at large values of m, is dominant at $m\gtrsim\beta^{-1/2}$. Since the relation $\sigma_{0,0}<\sigma_{1,0}$ usually holds and since at small values of β the value of $\sigma_{m,0}$ increases with increasing m, the growth rate will have a local maximum at $m>1$ for each value of k. As we see from Figs. 2 and 3, this maximum is sharper for power-law density profiles than for Gaussian profiles.

A longitudinal external magnetic field makes its presence felt primarily through a stabilization of the pinch compression dynamics; it significantly reduces the maximum instability growth rates even at $b=B_{z0}^2/B_\varphi^2\ll 1$. It is easy to see that in the presence of a longitudinal magnetic field the growth rate is no longer an even function of k. Since the stabilizing term f^2 in (13) is smaller when kB_z and mB_φ differ in sign, the growth rates in Fig. 2b and Fig. 3b are larger for negative k.

In the limit $k\to\infty$, for finite b, L is dominated by the term $f^2\approx k^2B_z^2/4\pi$. Taking the limit $k\to\infty$ in (13) with $b\neq 0$ and $\beta=0$, we find that a necessary condition for σ-stability of the motion is $\sigma>\sigma_{m,\infty}$, where

$$\sigma_{m,\infty}=\frac{1}{\alpha t_0}\left\{\max_r\left[\frac{B_z^2}{B^2}\frac{d\ln B_z}{d\ln r}+\frac{B_\varphi^2}{B^2}\left(\frac{d\ln B_\varphi}{d\ln r}+1\right)\right]\right\}^{1/2} \qquad (26)$$

is the limiting value of the growth rate in the limit $k\to\infty$, for all m. The m dependence of the growth rate (for $m>2$) is weaker in the presence of a magnetic field (Figs. 2b and 3b). This circumstance may explain the experimental observation of a large number of filaments in a situation in which filamentation is relatively less pronounced in the presence of a magnetic field.

The asymptotic behavior (26) may not be reached if the longitudinal magnetic field falls off sufficiently rapidly with distance from the axis; i.e., in order to take the limit we need $k\gtrsim B_{\varphi 0}/B_{z0}=b^{-1/2}\gg 1$. In this case, the stabilizing contribution of the thermal counterpressure ($\beta\neq 0$) is more important. It leads to a vanishing of the growth rates at sufficiently large values $k\gtrsim\beta^{-1/2}$. The growth rates are not shown for these large values of k in Figs. 2 and 3.

4. GROWTH OF PERTURBATIONS IN THE COURSE OF THE PINCH COMPRESSION

The maximum growth rate for each perturbation component (m,k) at each instant can be found with the help of the results of Secs. 2 and 3. In principle, we cannot rule out the further possibility that the maximum values of $\sigma_{m,k}$ at different times in the course of the compression will belong to different components (m,k). We will accordingly examine the growth of the individual perturbation components (m,k), although in the most typical cases one particular instability mode will dominate the entire compression process. As a quasiclassical estimate of the growth of the perturbations over a finite time interval we adopt the quantity

$$\Gamma_{m,k}(t)=\exp\left[\int_0^t\sigma_{m,k}(t')\,dt'\right].$$

Figure 4 shows the time evolution of the growth rate

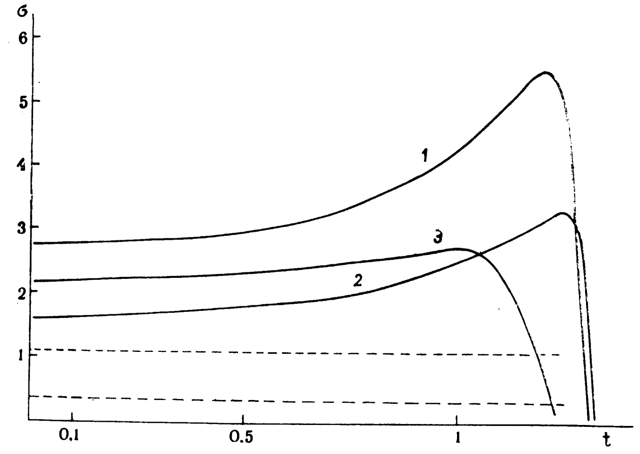

FIG. 4. Time evolution of the perturbation growth rates for a Z-pinch with a Gaussian density profile with $\beta = 0.1$, and $\rho_s/\rho_m = 8\cdot 10^{-4}$. 1—$m = 0, b = 0$; 2—$m = 1, b = 0$; 3—$m = 0, b_+0.22$. The dashed lines show the growth rates found for a steady-state pinch with the same density profile and $\beta = 1$ and $b = 0$. Upper line) $m = 0$; lower line) $m = 1$.

$\sigma(t)$ of the perturbations with $m = 0$ and $m = 1$ of a Gaussian pinch without an external magnetic field (curves 1 and 2). Shown for comparison in Fig. 4, by the dashed lines, are the instability growth rates of a steady-state pinch with a Gaussian density profile under the same conditions, with $\beta = 1$ ($m = 0$ corresponds to the upper line, and $m = 1$ to the lower line). The imposition of a longtitudinal magnetic field stabilizes the compression, reducing σ, as can be seen by comparing curve 3, plotted for $b = 0.02$, with curve 1. In accordance with the equation of motion (17), the plasma acceleration $g = -(\ddot{a}/a)r$ initially increases and then tends toward infinity as $t \to (\pi/2)^{1/2}t_0$ if there is no counterpressure; alternatively, it vanishes and then changes sign if there is a finite counterpressure, either thermal or magnetic. In the latter case, the conditions for the occurrence of a convective instability will not hold near the time of maximum compression, and the motion will stabilize: All the growth rates will vanish (Fig. 4).

Figures 5 and 6 show the growth of perturbations of a

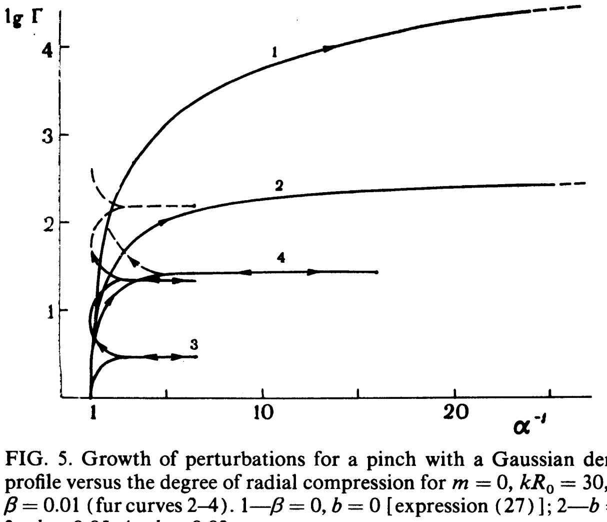

FIG. 5. Growth of perturbations for a pinch with a Gaussian density profile versus the degree of radial compression for $m = 0$, $kR_0 = 30$, and $\beta = 0.01$ (fur curves 2–4). 1—$\beta = 0, b = 0$ [expression (27)]; 2—$b = 0$; 3—$b = 0.05$; 4—$b = 0.02$.

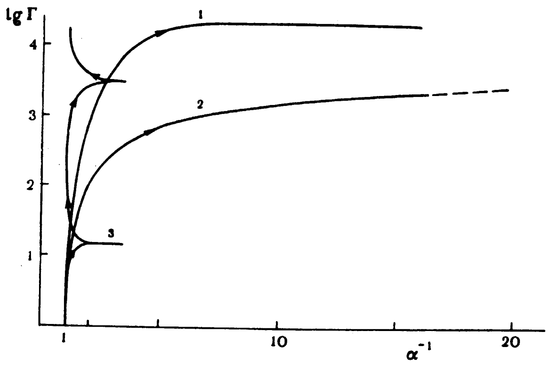

FIG. 6. Growth of perturbations for a pinch with a Gaussian density profile versus the degree of radial compression for $m = 1$, $kR_0 = -30$, $\beta = 0.01$, 1—$b = 0.02$; 2—$b = 0$; 3—$b = 0.24$.

Z-pinch as a function of the degree of compression, $1/\alpha$, for a Gaussian density profile. With $\beta + b \ll 1$ and $k \to \infty$, we find the following functional dependence from (23) for a Gaussian Z-pinch:

$$\Gamma(\alpha) = \exp\left\{2\left[\left(\ln\frac{\rho_m}{\rho_s} + 1\right)\ln\frac{1}{\alpha(t)}\right]^{1/2}\right\}. \qquad (27)$$

This function corresponds to curve 1 in Fig. 5. The corresponding expression for a power-law profile differs by the factor in front of $\ln(1/\alpha)$. Note that in the case $\beta = b = 0$ the growth of the perturbations (like the degree of compression) is not bounded. In this case, $\Gamma(\alpha)$ is dominated by the singularity in $\sigma(t)$ as $t \to (\pi/2)^{1/2}t_0$, which is caused by the unbounded growth of the acceleration near the collapse. As we see from Fig. 4, the counterpressure eliminates this singularity in $\sigma(t)$ and serves as a factor which stabilizes the compression. The asymptotic expression (27) gives us an upper limit on the growth of the perturbations at a given value of ρ_m/ρ_s. Curves 2–4 in Fig. 5 show the corresponding behavior calculated for the finite values $kR_0 = 30$ and $\beta = 0.01$. For a Gaussian Z-pinch, the perturbation modes which grow most rapidly are those with $m = 0$ and $m = 1$; for values of β which are not too small, the sausage instability dominates, while as $\beta \to 0$ the kink instabilities become dominant (Figs. 2 and 5).

By setting some acceptable level of the growth of the initial perturbations (e.g., $\Gamma_{max} = 100$), we can work in the linear theory to determine the maximum plasma compression in the Z-pinch before the time at which the cylindrically symmetric flow is disrupted. It follows from Figs. 5 and 6 that in the case $\beta \ll 1$ we can accept a radial compression of the current channel by a factor of no more than three to five. Even a weak longitudinal magnetic field stabilizes the compression of the pinch plasma, reducing σ noticeably, so that in a sufficiently strong magnetic field it is possible to observe radial oscillations of the plasma in the Z-pinch. A few successive compression-expansion cycles occur while the cylindrical symmetry is preserved, until Γ grows to the limiting permissible value Γ_{max} (curve 3 in Fig. 5), as was observed in some recent experiments.[12] For a given permissible Γ_{max} we can say that there is a certain optimum value of the longitudinal field B_z, which provides the greatest radial compression while retaining the cylindrical symmetry (curve 4 in Fig. 5). This conclusion agrees well with experimental results on the dynamics of a high-current gas-filled

Z-pinch with a longitudinal magnetic field.[12,18]

We recall that the cutoff of the Gaussian density profile which was introduced in the explicit form of the current shell of the Z-pinch involves the appearance of surface wave modes and instabilities of the pinch. The growth rates of the Rayleigh-Taylor instability modes which are localized near the surface are not bounded at large values of k. It is thus necessary to evaluate the relative contributions of the surface instabilities and of the bulk instabilities, with which we are concerned here, to the growth of the perturbations in the volume of the plasma which is being compressed. At large values of k, the Rayleigh-Taylor perturbations mode, with a growth rate $\gamma_{RT} = (gk)^{1/2}$, falls off exponentially with distance into the plasma, with a typical penetration depth of $1/k$. In other words, an estimate of its amplitude at the pinch axis is

$$|\zeta(0)| \propto \exp[(gk)^{1/2}t - kR]. \qquad (28)$$

Corresponding to the maximum amplitude is the value $k = k_m \sim gt^2/R^2$. In other words, the corresponding total contribution to Γ can be estimated to be

$$\int_0^t \gamma_{RT}(k_m, t')\,dt' \sim \int_0^t \frac{gt'}{R(t')}\,dt' \lesssim 1, \qquad (29)$$

since $gt^2/2 \sim R$. A comparison of this expression with (27) shows that the bulk instabilities of a diffuse Z-pinch may dominate even in the presence of a sharp boundary.

5. SOLUTION OF THE COMPLETE TIME-DEPENDENT PROBLEM OF THE DEVELOPMENT OF INSTABILITIES IN A DIFFUSE Z-PINCH

The assumption that the drift (convective) terms of order u/R are small in comparison with the retained terms of order σ—an assumption we used in deriving the equations of the quasiclassical approximation, (14), (15)—is indeed valid in that stage of the compression which is most important for the growth of the perturbations. The self-consistency of this approach is therefore confirmed. It is also interesting to evaluate the contribution to the growth of the perturbations from those stages of the compression in which the acceleration vanishes and changes sign, and the velocity of the plasma particles is high. In this case σ vanishes, so the approximate analytic method developed in Secs. 3 and 4 cannot be used, and we are forced to resort to a solution of the complete system of equations, (5)–(8). The solution of this problem gives us a rigorous quantitative estimate of the accuracy of the quasiclassical approximation.

With an eye toward a direct comparison with the analytic results derived in the preceding sections of the paper, we seek a solution of Eqs. (5)–(8) for an unperturbed motion described by the self-similar solutions (16)–(19), in the form

$$\begin{gathered} \rho'(r,t) = \rho_0\alpha(t)^{-2}N', \quad p'(r,t) = p_0\alpha(t)^{-2\gamma}P', \\ B_r'(r,t) = B_{\varphi 0}\alpha(t)^{-1}H_r', \quad B_\varphi'(r,t) = B_{\varphi 0}\alpha(t)^{-1}H_\varphi', \\ B_z'(r,t) = B_{\varphi 0}\alpha(t)^{-2}H_z', \\ u_r'(r,t) = R_0V_r', \quad u_\varphi'(r,t) = R_0V_\varphi', \\ u_z'(r,t) = R_0\alpha(t)^{-1}V_z', \end{gathered} \qquad (30)$$

where N', P', $\mathbf{H}'$ and $\mathbf{V}'$ are unknown functions of the self-similar variable η and the time t. The dependence of the perturbations on φ and z is determined by the common factor $\exp(im\varphi + ikz)$, which we will not write out any further.

In terms of the variables in (30), the equations for the perturbations are

$$\frac{\partial N'}{\partial t} = -\frac{1}{\alpha}\left\{V_r'\frac{\partial N}{\partial\eta} + N\left[\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta V_r') + \frac{im}{\eta}V_\varphi' + i\varkappa V_z'\right]\right\}, \qquad (31)$$

$$\frac{\partial P'}{\partial t} = -\frac{1}{\alpha}\left\{V_r'\frac{\partial P}{\partial\eta} + \gamma P\left[\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta V_r') + \frac{im}{\eta}V_\varphi' + i\varkappa V_z'\right]\right\}, \qquad (32)$$

$$\frac{\partial H_r'}{\partial t} = \frac{1}{\alpha}\left(\frac{im}{\eta}H_\varphi + i\varkappa\tilde{H}_z\right)V_r', \qquad (33)$$

$$\frac{\partial H_\varphi'}{\partial t} = -\frac{1}{\alpha}\left[\frac{\partial}{\partial\eta}(V_r'H_\varphi)' - i\varkappa(\tilde{H}_zV_\varphi' - H_\varphi V_z')\right], \qquad (34)$$

$$\frac{\partial H_z'}{\partial t} = -\frac{1}{\alpha}\left[\frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta\tilde{H}_zV_r') + \frac{im}{\eta}(\tilde{H}_zV_\varphi' - H_\varphi V_z')\right], \qquad (35)$$

$$\frac{\partial V_r'}{\partial t} = -\frac{\dot\alpha}{\alpha}V_r' - \frac{\ddot\alpha}{\alpha}\eta\frac{N'}{N} - \frac{1}{\alpha^3N}\frac{\partial}{\partial\eta}(\tilde{H}_zH_z') - \frac{1}{\alpha N\eta^2}\frac{\partial}{\partial\eta}(\eta^2H_\varphi H_\varphi') - \frac{\beta}{\alpha^{2\gamma-1}N}\frac{\partial P'}{\partial\eta}, \qquad (36)$$

$$\frac{\partial V_\varphi'}{\partial t} = -\frac{\dot\alpha}{\alpha}V_\varphi' - \frac{im}{\alpha^3\eta N}\tilde{H}_zH_z' - \frac{im\beta}{\alpha^{2\gamma-1}N\eta}P' + \frac{1}{\alpha N}\left\{\left[\frac{1}{\eta}\frac{d}{d\eta}(\eta H_\varphi)\right]H_r' + i\varkappa\tilde{H}_zH_\varphi'\right\}, \qquad (37)$$

$$\frac{\partial V_z'}{\partial t} = -\frac{\dot\alpha}{\alpha}V_z' - \frac{i\varkappa\alpha}{N}H_\varphi H_\varphi' - \frac{i\varkappa\beta}{\alpha^{2\gamma-3}N}P' + \frac{1}{\alpha N}\left[\frac{d\tilde{H}_z}{d\eta}H_r' + \frac{im}{\eta}H_\varphi H_z'\right], \qquad (38)$$

where $\tilde{H}_z = b^{1/2}H_z, \varkappa = kR_0$.

The boundary conditions on system (31)–(38) reduce to the requirement

$$V_r'|_{\eta=1} = 0, \qquad (39)$$

where the normalization of the self-similar coordinate has been chosen in such a way that the value $\eta = 1$ corresponds to the outer boundary of the pinch, $r = R_s(t)$.

There is a formal similarity between the right sides of Eqs. (31)–(35) and the right sides of Eqs. (5)–(8). By virtue of the representation (30), Eqs. (31)–(39) are similar in form to the equations of the quasiclassical approximation, regardless of whether we are dealing with a diffuse Z-pinch or a thin shell (Sec. 2). At each instant, by replacing $\partial/\partial t$ by σ, we can find the eigenvalues σ of the complete system of equations (31)–(39) from the boundary-value problem for an equation of second order in V'_r. The only distinction from the boundary-value problem (14), (15) here stems from the drift terms on the order of $\dot\alpha = 0$ in the equations of motion (36)–(38). At the initial time and at the time of maximum compression, when we have $\dot\alpha = 0$, the boundary-value problem (14), (15) is reproduced exactly; i.e., the values of σ calculated for these times in the quasiclassical approximation are also exact. For $\dot{\alpha}\neq 0$, the eigenvalues σ become complex, and a classification of the corresponding modes as either eigenfunctions or instabilities becomes somewhat arbitrary, justified only asymptotically for large growth rates and frequencies.

To solve Eqs. (31)–(38) we use the collocation method,[19] approximating the time-varying profiles at each instant by a finite sum of Chebyshev polynomials $T_{2j}(\eta)$:

$$
\begin{aligned}
N' &= f_1(\eta)\sum_{j=0}^{l-1} Y_{1j}(t)\,T_{2j}(\eta),\\
&\cdots\cdots\cdots\cdots\\
V_z' &= f_8(\eta)\sum_{j=0}^{l-1} Y_{8j}(t)\,T_{2j}(\eta),
\end{aligned}
\tag{40}
$$

where l is the number of polynomials, which determines the order of the approximation, and the weight functions $f_1(\eta),...,f_8(\eta)$ correspond to the asymptotic behavior of the profiles in the limit $\eta\to 0$. Since we have $T_{2j}(1) = 1$, boundary condition (39) is dealt with by adding the following term to the expansion (40) for V'_r:

$$V_r' = f_6(\eta)\sum_{j=0}^{l} Y_{6j}(t)\,T_{2j}(\eta), \tag{41}$$

where

$$Y_{6l}(t) \equiv -\sum_{j=0}^{l-1} Y_{6j}(t).$$

As the collocation points, whose choice results in the convergence of the polynomial approximation of the solution on the interval (0,1) we adopt the zeros of the Chebyshev polynomials. When this choice is made, the method which we are using to approximate our original system of equations, (31)–(38), is equivalent to the Galerkin method.[19] Substituting the expansion (40), (41) into Eqs. (31)–(38), and equating the left and right sides at the collocation points $\eta = \eta_i$, we find a system of ordinary differential equations for the $n = 8l$ expansion coefficients $Y_{sj}(t)$ $(1\leqslant s\leqslant 8,\ 0\leqslant j\leqslant l-1)$:

$$\frac{d}{dt} Y_{sj} = D_{sq}^{-1} F_{qj}(Y,t), \tag{42}$$

where $D_{ij} = T_{2j}(\eta_i)$ and the linear function of the variables $Y_{sj}(t)$ designated $F_{qj}(Y,t)$ is expressed in terms of the right sides of Eqs. (31)–(38).

The finite-dimensional approximation which we are using is applicable in this problem by virtue of the spectral properties of the problem, which result in the convergence of the method. As was shown in Sec. 2, the spectrum of instability growth rates derived in the quasiclassical approximation, which corresponds asymptotically to the exact spectrum for large growth rates, is discrete and bounded (Fig. 1). Consequently, the growth of the perturbations is determined primarily by a finite number of radial modes for given values of m and k.

Solving system of equations (42), we can follow the linear stage of the evolution of any initial perturbation profile. For our purposes, we are interested primarily in the invariant properties of the evolution of the instabilities—properties which depend on neither the initial profile nor the method used for the numerical approximation. We make use of the circumstance that the self-similar solutions which we are considering are for $\beta + b > 0$, periodic functions with a period of 2τ, where τ is the compression and expansion time. We denote by $\mathbf{x}(t)$ an n-dimensional vector formed by the components of the perturbation $Y_{sj}(t)$. We know from the theory of differential equations[20] that a system of linear, first-order, ordinary differential equations with periodic coefficients

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \tag{43}$$

where $A(t)$ is a periodic $n\times n$ matrix with a period of 2τ, can be reduced through a change of variables to a corresponding system with constant coefficients. In other words the solution of (43) can be written in the form

$$\mathbf{x}(t) = S(t)\exp(Bt)\mathbf{x}(0), \tag{44}$$

where $S(t)$ is a periodic $n\times n$ matrix with a period of 2τ, $S(0) = I$, and I is the unit matrix. Here B is a constant $n\times n$ matrix, and $M = \exp(2B\tau)$ is the monodromy operator of system (43) (Ref. 20). To determine M we should find a solution of the matrix equation $\dot{X} = A(t)X$ for the $n\times n$ matrix $X(t)$ with the initial condition $X(0) = I$; we then have $M = X(2\tau)$. The logarithms of the eigenvalues μ_j $(j = 1,...,n)$ of the operator M, divided by the period of 2τ, are exact analogs of the growth rates and eigenfrequencies for a periodic motion. If

$$\sigma_M = \max_{1\leqslant j\leqslant n} \mathrm{Re}(\ln \mu_j/2\tau), \tag{45}$$

we have the following estimate of the growth of the perturbation at any instant:

$$|\mathbf{x}(t)| \leqslant \mathrm{const}\cdot\exp(\sigma_M t)\,|\mathbf{x}(0)|. \tag{46}$$

The right side of (46) is an exact upper limit on the growth of the perturbations by the time t.

To evaluate the accuracy of the quasiclassical approximation, we carried out numerical calculations for a time-varying diffuse Z-pinch with a Gaussian density distribution. The eigenfunctions and the spectrum of the growth rates of the unstable modes in the quasiclassical approximation for this case are shown at the right in Fig. 1. An estimate of the growth of the perturbations for necks with $m = 0$ and $\varkappa = 5$ has been found on the basis of the quasiclassical approximation:

$$\int_0^{2\tau} \sigma_{0,5}(t)\,dt = 3.6.$$

Under the same conditions we have $2\tau\sigma_M = 3.76$. The eigenvalues of the monodromy operator M converge quite rapidly as the order of the approximation, l, increases. For example, we can achieve a 5% error by retaining only $l = 5$ terms in expansion (40).

The convergence of this procedure with increasing order of the approximation is evidence that the eigenvalues which are found are exact invariants of our original system of equations, (31)–(38). The fact that the estimate σ_M is close to the results of the analytic estimates in Secs. 3 and 4 shows that the quasiclassical representation of an instantaneous growth rate and the method described here for calculating this growth rate lead to good results for the type of problem considered here.

6. CONCLUSION

This new method for studying dynamic stability leads to a simple estimate of the rate of the exponential growth of convective bulk perturbations in the course of the plasma motion. As was shown in Sec. 5, the quasiclassical estimate of the perturbation growth rate is a fairly good approximation of the exact solution of the problem. This new method for studying dynamic stability, which we have illustrated here in the example of the stability of the self-similar compression of a diffuse Z-pinch, has a much wider range of applicability. The instability growth rates determined by the solution of the corresponding boundary-value problem of the form (14), (15) depend on only the profiles of the hydrodynamic variables of the unperturbed flow. Consequently, a method of this type can also be used to calculate the perturbation growth rate in cases in which analytic solutions cannot be found through (for example) numerical solution of the problem at each instant. Furthermore, this method can also be used, under a wide range of conditions, to calculate the perturbation growth rate when dissipative processes play a substantial role in shaping the profiles of the hydrodynamic variables of the unperturbed flow.

Our study of the dynamic stability of the compression of a diffuse Z-pinch shows that the perturbations grow without bound in the course of the plasma compression in the current channel of the pinch, with an unbounded increase in the pinch current. Consequently, radial compression factors substantially greater than three to five cannot be achieved, regardless of the increase in the current in the pinch, if the original cylindrical symmetry is to be retained. Although the stabilization of a Z-pinch by a longitudinal magnetic field $B_z \sim B_\varphi$ is well known from the theory of the stability of a steady-state pinch[1,2] (the Shafranov-Kruskal and Suydam criteria), the dynamic stabilization effect differs substantially: Although the growth rates of the instabilities which are set by the dynamics of the plasma compression are significantly higher than the corresponding values for a steady-state pinch, the condition for dynamic stabilization of the pinch by a longitudinal field is incomparably less stringent. For example, it follows from Figs. 4 and 5 (in accordance with the measurements of Refs. 12 and 21) that an energy of the longitudinal external magnetic field amounting to only 2–3% of the self-field of the current in the pinch, B_φ, is sufficient for dynamic stabilization of the plasma compression in a pinch. It is this circumstance which makes it possible to generate ultrastrong magnetic fields during the compression of the magnetic flux B_z in the plasma of a Z-pinch with a radial compression by a factor of 20–30, which is achievable in practice.[12,18,21] Consequently, pinch systems with an axial magnetic field, e.g., a hollow plasma liner with a trapped azimuthal magnetic flux and with a thread of solid (frozen) DT along the liner axis (a Z–θ configuration), are promising directions for controlled thermonuclear fusion.[22,23]

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