Regularization of the self-energy of point vortex dipoles and the increase in the total vorticity during stretching of vortex lines

S. G. Chefranov

Institute of Atmospheric Physics, Academy of Sciences of the USSR
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The balance equation for the total vorticity (the enstrophy) corresponding to a "strong" singularity, i.e., the explosive increase in the enstrophy in a dynamic interaction of two regularized point vortex dipoles (infinitesimal vortex rings which have been "smeared out"), is derived in the three-dimensional case. The effect of viscous dissipation on the process is evaluated. A comparison is made with the dynamics of two-dimensional vortex dipoles which leave the enstrophy invariant (in the absence of dissipative factors), despite the possible realization of a "weak" singularity for the local properties of the vortex field.

The regularization problem always arises in the study of point vortex entities which ahve an indefinite self-energy.[1] For point vortices in the two-dimensional case (i.e., for rectilinear vortex filaments) an infinite self-energy is simply discarded in the analysis of the interaction of such vortices.[1,2] A more complicated question is whether it is valid to discard the self-energy in treating the interaction of point vortex dipoles in either two or three dimensions.[1)] The complexity stems from the fact that the vortices have an infinite self-induction velocity along the symmetry axis.[3,6] The mathematical difficulties which arise here (and which, incidentally, also arise for point vortices in a plane) have the same source as in a quantum field theory based on the concept of a point interaction described by the product of δ-function operators taken at the same spatial point.[7] In quantum field theory this problem of the regularization of expresssions containing products of δ-functions has been approached by various formal paths which nevertheless lead to excellent agreement with experimental data.[8] Dyson regularization, for example, can be associated with a subtraction of the self-energy of point vortex entities from the total invariant kinetic energy of the system, as mentioned above. A somewhat less formal approach to quantum field theory was derived by Landau,[7] who suggested treating the point interaction"...as the limit of some 'smeared' interaction of finite range as this range decreases to zero." In other words, the idea is to abandon the use of δ-functions to describe a point interaction. This idea of Landau's underlies the procedure developed in Sec. 1 of the present paper for regularizing the self-energy of point vortex dipoles, T_s, in which T_s does not affect the Hamiltonian dynamics of the relative motion in a system of such vortex entities. A system of this sort was studied in Sec. 2 in connection with an analysis of the problem of spontaneous singularities in three-dimensional turbulence.[6,9]

1. REGULARIZATION OF THE VORTEX SELF-ENERGY

1. In the three-dimensional case, the total kinetic energy corresponding to the vorticity distribution $\boldsymbol{\omega}(\mathbf{x})$ in an unbounded space is[1]

$$T=\frac{\rho_0}{8\pi}\int d^3x'\int d^3x\,\frac{\omega_i(\mathbf{x}')\,\omega_i(\mathbf{x})}{|\mathbf{x}-\mathbf{x}'|},\qquad (1)$$

where ρ_0 is the constant density of the liquid. The self-energy of a point vortex dipole, T_s, is found from (1) for the vorticity[1,6]

$$\omega_i=\varepsilon_{ijl}\gamma_l\frac{\partial\delta(\mathbf{x})}{\partial x_j},\qquad (2)$$

which corresponds to a point vortex dipole at the origin of coordinates, with a Lamb momentum γ. In (2), ε_{ijl} is the Levi-Civita pseudotensor, and a repeated index means a summation from 1 to 3. The value of T_s is undetermined because of the product of δ-functions, taken at the same point, in (1) [for $\boldsymbol{\omega}$ from (2)].

We will regularize the energy T_s by Landau's approach: We replace the δ-function in (2) by a finite "smeared" modification $\tilde{\delta}(\mathbf{x})$, where the regular function $\tilde{\delta}(\mathbf{x})$ satisfies the parity and normalization conditions

$$\tilde{\delta}(\mathbf{x})=\tilde{\delta}(-\mathbf{x}),\qquad \int d^3x\,\tilde{\delta}(\mathbf{x})=1.\qquad (3)$$

The distribution of the solenoidal velocity field corresponding to (2) takes the following form as a result of the replacement $\delta\to\tilde{\delta}$:

$$\tilde{u}_i(\mathbf{x})=\gamma_i\tilde{\delta}(\mathbf{x})+\frac{\partial\Phi}{\partial x_i},\qquad (4)$$

where

$$\Phi=\Phi_3=\gamma_l\frac{\partial}{\partial x_l}f(\mathbf{x}),\qquad f(\mathbf{x})=\frac{1}{4\pi}\int d^3x'\,\frac{\tilde{\delta}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|},$$

$\tilde{\mathbf{u}}(\mathbf{x})\approx O(|\mathbf{x}|^{-3})$ as $|\mathbf{x}|\to\infty$, and we have $\operatorname{div}\tilde{\mathbf{u}}=0$, regardless of the nature of $\tilde{\delta}(\mathbf{x})$, since we have $\Delta f=-\tilde{\delta}(\mathbf{x})$, where Δ is the Laplacian. For $\tilde{\delta}(\mathbf{x})\to 0$ as $|\mathbf{x}|\to 0$, the velocity field (4) of the "smeared" vortex dipole no longer has a singularity at the point $\mathbf{x}=0$, since we have $\tilde{\mathbf{u}}(\mathbf{x})\to 0$ as $|\mathbf{x}|\to 0$.

In general, we can write a regular finite function $\tilde{\delta}(\mathbf{x})$ which satisfies conditions (3) as an infinite series in spherical harmonics. For simplicity we restrict the analysis to the representation (at $t=0$)

$$\tilde{\delta}(\mathbf{x})=a_0\delta_0(r)\,[1+yP_2(\cos\theta)],\qquad (5)$$

where α_0 is found from the normaliztion condition

$$a_0=\frac{1}{4\pi}\left[\int_0^b dr\,r^2\delta_0(r)\right]^{-1},$$

$P_2=(3\cos^2\theta-1/2)$ is the Legendre polynominal, θ is the spherical angle measured from the direction specified by the momentum vector γ, $r\equiv|\mathbf{x}|$, b is the scale of the smearing of

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the δ-function, and y is an arbitrary constant factor.

Representation (5) corresponds to expression (A1) for $f(r,\theta)$, while for T_s we find the following results from (1) and (2), respectively, when we make the replacement $\delta \to \tilde{\delta}$:

$$T_s = \frac{\rho_0 \gamma^2}{2} \int d^3x\, \tilde{\delta}(\mathbf{x}) \left[\tilde{\delta}(\mathbf{x}) + \frac{\gamma_i \gamma_k}{\gamma^2} \frac{\partial^2}{\partial x_i\, \partial x_k} f(r,\theta) \right]. \quad (6)$$

Expression (6) can also be derived in an elementary way from the definition

$$T_s = \frac{\rho_0}{2} \int d^3x\, \mathbf{u}^2$$

for $\tilde{\mathbf{u}}$ from (4), since in an unbounded space we have

$$\int d^3x \frac{\partial \Phi}{\partial x_i} \tilde{u}_i = 0$$

by virtue of the condition div $\tilde{\mathbf{u}} = 0$.

2. We first consider the possibility of separating from T_s in (6) that part of the energy (T_{s0}) which is most responsible for the existence of a self-induced motion of the vortex dipole in the direction specified by the vector $\boldsymbol{\gamma}$, i.e., along the axis $\theta = 0$. Here we have $T_s = T_{s0} + T_{s1}$, where T_{s0} is given by (6) when we make the replacement $f(r,\theta) \to f(r,\theta = 0)$ [See (A1)]. Note also that the θ dependence of $f(r,\theta)$ is determined by the θ dependence of $\tilde{\delta}$ in (5) even in the limit $b \to 0$, in which we have, for $\tilde{\delta} \to \delta$, $f(r,\theta) \to 1/4\pi r$. In the Appendix we derive an expression for T_{s0}, specifically, expression (A2), which corresponds to an initial (at $t = 0$) smearing:

$$\delta_0(r) = r^{\alpha}(b-r)^2 \tilde{\theta}(b-r), \quad (7)$$

where $\alpha > -3/2$, and

$$\tilde{\theta}(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

is the unit step function. It follows from (A2) that for real values of the parameter y the quantity T_{s0} can vanish only for large values of the structure parameter $\alpha \geqslant \alpha_{cr} = 137$ (we are rounding to integers). For example, we have $T_{s0} = 0$ for $\alpha = 137$ and $y \approx 3$. A vortex structure of this sort, corresponding to (2), (5), and (7), is topologically equivalent to a vortex ring of radius b which is smeared out toward the center of the ring, since we have $\omega \neq 0$ at $r \neq 0$ for any $0 < r < b$ and $\boldsymbol{\omega} = 0$ at $r = 0$ and $r \geqslant b$. A large value of α here corresponds to the greatest concentration of the vorticity which is nevertheless near the periphery of the vortex region, i.e., at $r \approx b$. The total energy T_s of course increases without bound in the limit $b \to 0$, although T_{s0} no longer depends on this limit, remaining zero for arbitrary b (for the values of y and α stated above). As a result, there is a cancellation of the energy

$$\frac{\rho_0 \gamma^2}{2} \int d^3x\, \tilde{\delta}^2(\mathbf{x}),$$

associated with the anisotropic induced motion of the vortex dipole.

3. To evaluate the possibility that the total self-energy T_s vanishes, we must also consider complex values of the parameter y. We assume a factor $y = iy_1$, where $i^2 = -1$, and y_1 is a real random quantity with a zero mean, $\langle y_1 \rangle = 0$ (the angle brackets mean a statistical average). Only the values of the vorticity field and the velocity which are averaged over y_1, and which are real in this case, are physically meaningful, of course. The same comment applies to the global characteristics of the vortex field: the energy, the integral vorticity (enstrophy), etc. Despite the formal nature of this statistical approach to the problem of the regularization of the self-energy T_s, it may be justified by the circumstance that in practice only average properties of vortex fields, not the fields themselves, are measurable and, correspondingly, predictable.[10,11] Furthermore, complex vortex structures also find applications in elementary particle theory.[12]

For $\langle T_s \rangle$ in this case we find the following expression from (6) and (A1), after we average over y_1:

$$\langle T_s \rangle = \frac{4\pi}{3} \rho_0 \gamma^2 a_0^2 \int_0^b dr\, r^2 \delta_0^2(r) \left(1 - \frac{\langle y_1^2 \rangle}{7} \right), \quad (7a)$$

where we should have $\langle y_1^2 \rangle = 7$ according to physical considerations. From the Cauchy-Bunyakovskiĭ inequality we find the following estimate of the coefficient in front of the parentheses in (7a):

$$\frac{4\pi}{3} \rho_0 \gamma^2 a_0^2 \int_0^b dr\, r^2 \delta_0^2(r) \geqslant \rho_0 \frac{\gamma^2}{4\pi b^3}.$$

This estimate does not depend on the form of $\delta_0(r)$ at $0 \leqslant r \leqslant b$. In the case $\langle y_1^2 \rangle = 7$, however, we have $\langle T_s \rangle = 0$, and this energy no longer depends on the last limit ($b \to 0$).

We wish to stress that the smearing in (5) and (7) does not by itself determine the steady-state solution of the Helmholtz equations (Ref. 13, for example), but it can serve as an initial condition for a time-varying vortex structure (which is localized in the limit $b \to 0$) for which the self-energy should again be zero, $\langle T_s \rangle = 0$, by virtue of energy conservation. It is shown in Sec. 2 that a vortex structure of this sort, corresponding to (5) and (7), is nevertheless a steady-state structure on the average.

We also note that the results of the regularization found above agree with the conclusion reached by Goman *et al.*[5] that it is possible to eliminate the self-induction velocity for a localized time-varying vortex region modeled by a set of thin vortex rings. Furthermore, Finkel'shteĭn[14] used arguments similar to those presented by Goman *et al.*[5] as a basis for equating the self-energy for vortex filaments in type II superconductors to zero. In quantum field theory also, regularization brings the self-energy of a photon to zero.[7]

In the Appendix we present some procedures for regularizing the self-energy of point vortex dipoles in the two-dimensional case. Those procedures are similar to the ones discussed in Subsections 2 and 3.

4. In the three-dimensional case, for a system of N point vortex dipoles with a vorticity distribution

$$\omega_i(\mathbf{x}, t) = \varepsilon_{ijl} \sum_{m=1}^{N} \gamma_l^m(t) \frac{\partial \delta}{\partial x_j}(\mathbf{x} - \mathbf{x}^m(t)), \quad (8)$$

a Hamiltonian formulation with a Hamiltonian

$$H = \sum_{m=1}^{N} \gamma_i^m v_i(\mathbf{x}^m)$$

and with canonical variables γ_i^m, x_i^m ($m = 1,2,...,N$), satisfying the system of equations

$$\frac{d\gamma_i^m}{dt} = -\frac{\partial H}{\partial x_i^m} = -\gamma_l^m \frac{\partial v_l(\mathbf{x}^m)}{\partial x_i^m}, \qquad (9)$$

$$\frac{dx_i^m}{dt} = \frac{\partial H}{\partial \gamma_i^m} = v_i(\mathbf{x}^m), \qquad (9')$$

where

$$v_i = \frac{\partial^2}{\partial x_i^m \partial x_l^m} \sum_{n=1}^{N} \frac{\gamma_l^n}{|\mathbf{x}^m - \mathbf{x}^n|}$$

[i.e., $v_i = \partial\Phi_3/\partial x_i$ as $\tilde{\delta} \to \delta(\mathbf{x})$ in (4)], was derived in Ref. 6. An expression for H was found in Ref. 6 from (1) and (8) by considering only the interaction energy of the point vortex dipoles, W (so that the expression $W = \rho_0 H/2$ resulted). It was assumed that the invariance of the total energy T of the system of vortex dipoles (8) leads to a regular energy W, found from T by subtracting the singular self-energy T_s of all of the dipoles. The regularization procedure described above, $T_s \to \langle T_s \rangle$, which corresponds to the possible vanishing of this quantity, confirms that this suggestion regarding the invariance of W and thus H is feasible. It thus becomes possible to write down the Hamiltonian dynamic system (9), (9'). Furthermore, the possibility of eliminating the singular self-induction velocity for point vortex dipoles (with $T_{s0} = 0$; see also Refs. 5 and 6) makes it possible to also derive system (9), (9') with an invariant Hamiltonian H and (W) as a weak solution of the three-dimensional Helmholtz equation for the vortex field (8) (see the Appendix). The possible existence of such a weak solution was pointed out in Ref. 15 and also in Ref. 3. In Ref. 3, however, a wrong sign on the right side of (9) led to the incorrect conclusion that the invariance of the total angular momentum $\mathbf{M}$ was violated for a system of point vortex dipoles (8).[2]

5. In the two-dimensional case the Hamiltonian dynamics of point vortex dipoles can be introduced in a similar way. The total kinetic energy corresponding to the vorticity distribution in the plane, $\omega(\mathbf{x})$, is[1]

$$T = -\frac{\rho_0}{8\pi} \int d^2x \int d^2x' \omega(\mathbf{x}) \omega(\mathbf{x}') \ln|\mathbf{x} - \mathbf{x}'|^2. \qquad (10)$$

For a system of N point vortex dipoles we would have

$$\omega(\mathbf{x}, t) = \sum_{m=1}^{N} \varepsilon_{\alpha\beta} \gamma_\beta^m(t) \frac{\partial}{\partial x_\alpha} \delta[\mathbf{x} - \mathbf{x}^m(t)], \qquad (11)$$

where $\varepsilon_{\alpha\beta}$ is the Levi-Civita density, and a repeated index means a summation from 1 to 2. From (10) and (11) we find the interaction energy of the dipoles, W_2, and the corresponding Hamiltonian H_2, which is given by ($W_2 = \rho_0 H_2/2$)

$$H_2 = \sum_{m=1}^{N} \gamma_\alpha^m \tilde{v}_\alpha(\mathbf{x}^m),$$

where

$$\tilde{v}_\alpha(\mathbf{x}^m) = -\frac{1}{2\pi} \frac{\partial^2}{\partial x_\alpha^m \partial x_\beta^m} \sum_{n=1}^{N} \gamma_\beta^n \ln|\mathbf{x}^m - \mathbf{x}^n|.$$

The canonical variables γ_m^α, x_α^m ($\alpha = 1,2, m = 1,2,...,N$) are described by the system of equations (which corresponds to H_2)

$$\frac{dx_\alpha^m}{dt} = \tilde{v}_\alpha(\mathbf{x}^m), \quad \frac{d\gamma_\alpha^m}{dt} = -\gamma_\beta^m \frac{\partial \tilde{v}_\beta(\mathbf{x}^m)}{\partial x_\alpha^m}, \qquad (12)$$

which can also be derived as a weak solution of the two-dimensional Helmholtz equations for the vorticity field (11) by eliminating the corresponding singular self-induced velocity.

Equations (12) leave invariant the Hamiltonian H_2, the total momentum

$$P_\alpha = \sum_{m=1}^{N} \gamma_\alpha^m,$$

and the angular momentum

$$M = \sum_{m=1}^{N} \varepsilon_{\alpha\beta} x_\alpha^m \gamma_\beta^m.$$

As in the three-dimensional case,[6] we can use these invariants to derive (without any difficulty) an exact solution of Eqs. (12) for two vortex dipoles. In particular, we consider the case $P_\alpha = 0$, i.e., the case with $\gamma^1 = -\gamma^2 = \gamma$. We assume $\mathbf{x}_1 - \mathbf{x}_2 = 1$; then we have

$$M = \varepsilon_{\alpha\beta} l_\alpha \gamma_\beta, \quad H_2 = \frac{\gamma^2}{\pi l^2}\left(1 - \frac{2(\gamma\mathbf{l})^2}{\gamma^2 l^2}\right).$$

From (12) we then find the system of equations

$$\frac{d}{dt}(\gamma\mathbf{l}) = 2H_2 = \text{const}, \quad \frac{dl}{dt} = -\frac{(\gamma\mathbf{l})}{\pi l^3},$$

$$\frac{d\gamma}{dt} = \frac{\gamma(\gamma\mathbf{l})}{\pi l^4}\left[3 - \frac{4(\gamma\mathbf{l})^2}{\gamma^2 l^2}\right], \quad \frac{d\varphi}{dt} = -\frac{M}{\pi l^4}, \qquad (13)$$

where φ and φ_1 are the polar angles of the vectors $\mathbf{1}$ and γ, which lie in a common plane, so that we have $\sin(\varphi_1 - \varphi) = M/\gamma l$. Equations (13) can be integrated easily; their solution is $\gamma_0 \equiv \gamma(t=0)$, $\mathbf{l}_0 \equiv \mathbf{l}(t=0)$

$$l(t) = l(0)\left[1 - \frac{4t}{\pi(l(0))^4}((\gamma_0\mathbf{l}_0) + H_2 t)\right]^{1/4}. \qquad (14)$$

$$\gamma(t) = [M^2 + (2H_2 t + (\gamma_0\mathbf{l}_0))^2]^{1/2}/l(t). \qquad (15)$$

This solution agrees qualitatively with that found in Ref. 6 for three dimensions, and it corresponds to Eqs. (9), (9') with $N = 2$. Furthermore, despite the difference between the exponents in (14) and (15), on the one hand, and those in the three-dimensional case, on the other [see the expressions for γt) and $l(t)$ in Ref. 6, and see also expressions (21) and (22) below], the exponent of the explosive increase in the square of the local gradient in a neighborhood $|\mathbf{x} - \mathbf{B}| < l(t)$ of the invariant "center of gravity" $\mathbf{B} = (\mathbf{x}^1 + \mathbf{x}^2)/2$ [in the limit $l(t) \to 0$ for $t \to t_2^+$ in (14)], i.e.,

$$\Omega = (\partial u_i/\partial x_j)^2 \approx O(\gamma^2/l^6) \approx O[1/(t_2^+ - t)^q],$$

is exactly the same as the three-dimensional value $q = 2$ [in Ref. 6, $\Omega \approx O(\gamma^2/l^2) \approx O(1/(t^1 - t)^2)$]. Here $\mathbf{u}(\mathbf{x})$ is the velocity field which would be set up at point $\mathbf{x}$ by a pair of vortex dipoles which are collapsing (closing on each other).

On the other hand, a fundamental difference between three-dimensional vortex dynamics and two-dimensional dynamics, which stems from the circumstance that only three-dimensional vortex lies can be stretched, is demonstrated by the analysis in Sec. 2 of the evolution of the enstrophy, for which there is a "strong" singularity as $t \to t^+$. This strong singularity is not found for the invariant enstrophy in the two-dimensional case, despite the existence of a "weak" singularity in the localization of the quantity Ω for $t \to t_2^+$.

2. INTENSIFICATION OF THE INTEGRAL VORTICITY (ENSTROPHY)

1. The regularization of the vortex field $\tilde{\omega}(\mathbf{x})$ which was introduced in Sec. 1 through the replacement of a δ-function by a regular finite function $\tilde{\delta}(\mathbf{x})$ (with infinitesimal support $b \to 0$) also makes it possible to introduce, in a correct way, the square of the regularized vorticity, $\omega^2(\mathbf{x})$, and corresponding integrals of the enstrophy, $I = \int d^3x \tilde{\omega}^2$ and $I_2 = \int d^2x \tilde{\omega}^2$, for point vortex dipoles in the three-dimensional case, (8), and the two-dimensional case, (11). In contrast with the integral $\int d^3\tilde{\omega}$ (and $\int d^2x\tilde{\omega}$), the enstrophy I (and I_2) does not vanish for the vortex dipoles (8) [or (11)] and is thus a convenient measure of the integral vorticity throughout the volume of the liquid.[1] In an unbounded three-dimensional space, the balance equation for the enstrophy is

$$\frac{1}{2}\frac{dI}{dt} = \int d^3x\, \tilde{\omega}_i \tilde{\omega}_l \frac{\partial \tilde{u}_i}{\partial x_l} - \nu \int d^3x \left(\frac{\partial \tilde{\omega}_i}{\partial x_j}\right)^2, \qquad (16)$$

where $\tilde{\omega}_i$ is the regularized vorticity (8), $\tilde{u}_i$ is the corresponding velocity field, and ν is a kinematic viscosity coefficient. Everywhere below, we will consider finite smearing of the δ-functions in the form of $\tilde{\delta}(\mathbf{x})$ [as in (5) and (7)] and the effects of viscous dissipation only in the enstrophy balance equation (16); we will be assuming that we have already taken the inverse limit $\tilde{\delta} \to \delta$ (i.e., $b \to 0$) in the Helmholtz equation (A6). That limit, in contrast with (16) in (A6), is permissible [and leads to a weak solution described by system (9), (9')]. At large Reynolds numbers we can again use the system (9), (9'), which follows from (A6), to describe the dynamic interaction of vortex dipoles and the corresponding evolution in the enstrophy I in (16). Since the smearing of $\tilde{\delta}(\mathbf{x})$ which we discussed in Sec. 1 corresponds to only the initial time, $t = 0$, we must introduce some new notation: $\tilde{b}(t)$, which determines some "average" scale of the smearing at $t > 0$. Specifically, it determines the scale for the enstrophy balance equation[3)] which we are considering here, (16), from which we also find the quantity $\tilde{b}(t)$ [and, correspondingly, $I(t)$] under the initial condition $\tilde{b}(t) = 0) = b$ [for b from (5) and (7), for example] and for the solutions of Eqs. (9), (9') which were found in Ref. 6.

In particular, for a single vortex dipole (in a cylindrical coordinate system with z axis in the $\gamma = \text{const}$ direction[1]), we have

$$\tilde{\omega}_\rho = \tilde{\omega}_z = 0, \quad \tilde{\omega}_\rho = -\gamma \frac{\partial \tilde{\delta}}{\delta \rho}, \quad \tilde{u}_\varphi = 0,$$

$$\tilde{u}_\rho = \frac{\gamma}{4\pi}\frac{\partial^2 f}{\partial z\, \partial \rho}, \quad \tilde{u}_z = \gamma \tilde{\delta}(\rho) + \frac{\gamma}{4\pi}\frac{\partial^2 f}{\partial z^2},$$

$$f(z,\rho) = f(-z,\rho) = \int d^3x \frac{\tilde{\delta}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.$$

The enstrophy balance equation

$$\frac{d}{dt}\left(\frac{1}{2}\int d^3x\, \omega_\varphi^2\right) \equiv \frac{dI}{dt} = \int d^3x \frac{\tilde{\omega}_\varphi^2 \tilde{u}_\rho}{\rho} = 0$$

[since $\tilde{u}_p(z,p) = -\tilde{u}_p(-z,p)$] determines, on the average, a steady-state solution $\tilde{b} = b = \text{const}$ for a smearing of the type (5), (7), since we have $I \approx O(\gamma^2/b^5) = \text{const}$.

2. We now consider the balance equation (16) for the case of two interacting vortex dipoles which satisfy the dynamic equations (9), (9') in the limit $\nu \to 0$. In system (9), (9'), conserved along with the energy H are the momentum

$$P_i = \sum_{m=1}^{N} \gamma_i^m$$

and the angular momentum[6]

$$M_i = \varepsilon_{ijl} \sum_{m=1}^{N} \gamma_j^m x_l^m.$$

are conserved.

These invariants were used in Ref. 6 with $N = 2$ and $P = 0$ to derive an exact solution for the evolution of the vectors $\mathbf{l}(t) \equiv x^1 - x^2$, $\gamma(t) \equiv \gamma^1 = -\gamma^2$, which lie in the (x,y) plane, with $M_x = M_y = 0$ and $M_z = M = \text{const}$. In the coordinate system at the invariant "center of gravity" of two point vortex dipoles [i.e., at the point $\mathbf{B} = (\mathbf{x}^1 + \mathbf{x}^2)/2$], the regularized vorticity is given by

$$\tilde{\omega}_i = \varepsilon_{ijl}\gamma_l \frac{\partial}{\partial x_j}\left(\tilde{\delta}\left(\mathbf{x} - \frac{1}{2}\right) - \tilde{\delta}\left(\mathbf{x} + \frac{1}{2}\right)\right). \qquad (17)$$

The corresponding enstrophy is

$$I = \int d^3x\, \tilde{\omega}^2 \approx 2\gamma^2 \int d^3x \left[\left(\frac{\partial \tilde{\delta}}{\partial x_i}\right)^2 - \frac{1}{\gamma^2}\left(\gamma_i \frac{\partial \tilde{\delta}}{\partial x_i}\right)^2\right]. \qquad (18)$$

Expression (18) incorporates the assumption that the smeared vortex dipoles are separated from each other by a distance large in comparison with the smearing $\tilde{b}$ (i.e., $l \gg \tilde{b} \to 0$).

When viscous forces are taken into account, the enstrophy balance equation (16) takes the following form, to within, the omitted terms $O(\tilde{b}^2/l^2)$:

$$\frac{1}{2}\frac{dI}{dt} = -\frac{3\gamma^2(\gamma\mathbf{l})}{2\pi l^5}\int d^3x \left[\frac{5}{\gamma^2 l^2}(\mathbf{M}\nabla\tilde{\delta})^2 - (\nabla\tilde{\delta})^2 + \frac{1}{\gamma^2}(\gamma\nabla\delta)^2\right]$$
$$- 2\nu\gamma^2 \int d^3x \left[(\Delta\tilde{\delta})^2 - \frac{1}{\gamma^2}((\gamma\nabla)\nabla\tilde{\delta})^2\right]. \qquad (19)$$

We ignore the viscous forces only in the balance equation (19); everywhere below we assume that we are dealing with large Reynolds number, $\mathrm{Re} = v(t)l(t)/\nu \gg 1$ [for two vortex dipoles, the relative velocity is $v(t) \approx \gamma(t)/4\pi l^3(t)$], in which case we can ignore the effect of viscous dissipation on the relative Hamiltonian dynamics of the vortex dipoles described by Eqs. (9), (9').

In (19) we are using a rather smooth representation of the function $\tilde{\delta}(\mathbf{x})$, which corresponds to, for example, (5) and (7) for $\alpha \geqslant 137$ and for real values of the parameter y [corresponding to $T_{s0} = 0$ at $t = 0$; see (A2)]. From (19) we find, in the limit $\alpha \gg 1$,

$$du/d\tau = A(\tau)u - \beta(\tau)u^{7/5}, \qquad (20)$$

where

$$u=\frac{I(t)}{I(0)},\quad I(t)\approx\frac{\gamma^2(t)\alpha^3}{48\pi\tilde{b}^5(t)}\left(1+\frac{y}{5}+\frac{8}{35}y^2+O\left(\frac{1}{\alpha^2}\right)\right),$$

$$\tau=t/t_0,\quad t_0=\pi l_0^4\sqrt{3}/5\gamma_0,$$

t_0 is the time at which the two vortex dipoles collapse (at which they come together at the same point,[6]

$$l_0\equiv|\mathbf{l}(t=0)|,\quad \gamma_0\equiv|\boldsymbol{\gamma}(t=0)|,\quad \beta(\tau)=\beta_0(\gamma_0/\gamma(\tau))^{4/5},$$

$$\beta_0=6\nu t_0\alpha^2/b_0^2\equiv6\sqrt{3}l_0^2/20\pi b_0^2\,\mathrm{Re}_0,$$

$$\mathrm{Re}_0=v_0l_0/\nu\gg1,\quad v_0\equiv v(t=0),\quad b_0\equiv\tilde{b}(t=0),$$

$$A(\tau)=\frac{3\sqrt{3}}{20}\frac{(\boldsymbol{\gamma}\mathbf{l})l_0^4}{l^5\gamma_0}\left[\frac{M^2}{\gamma^2l^2}\left(5+4y+\frac{11}{7}y^2\right)\Big/\left(1+\frac{y}{5}+\frac{8}{35}y^2\right)-2\right].$$

According to Ref. 6, we have the following expressions for $l(t)$ and $\gamma(t)$:

$$l(t)=l_0\left\{1-\frac{5t}{\pi l_0^5}\left[(\boldsymbol{\gamma}_0\mathbf{l}_0)+\frac{5t}{2}H\right]\right\}^{1/5}$$

$$=\left(\frac{25H}{2\pi}\right)^{1/5}(t^{(+)}-t)^{1/5}(t-t^{(-)})^{1/5},\tag{21}$$

$$\gamma^2(t)=4\pi Hl^3+3[5Ht+(\boldsymbol{\gamma}_0\mathbf{l}_0)]^2/l^2,\tag{22}$$

where

$$t^{(\pm)}=4\pi l_0^4\left(-\cos\psi_0\pm\frac{1}{\sqrt{2}}|\sin\psi_0|\right)/5\gamma_0(1-3\cos^2\psi_0),$$

$$\psi_0=(\varphi-\varphi_1)|_{t=0}\equiv\psi(t=0),$$

φ and φ_1 are the polar angles of the vectors $\mathbf{l}$ and $\boldsymbol{\gamma}$ in the (x, y) plane,

$$\psi_0\equiv\Psi(t=0)=(\varphi-\varphi_1)|_{t=0}$$

$$(\boldsymbol{\gamma}\mathbf{l})=5Ht+(\boldsymbol{\gamma}_0\mathbf{l}_0),\qquad H=\frac{\gamma^2}{4\pi l^3}(1-3\cos^2\psi)=\mathrm{const},$$

and $\gamma l\sin\psi=M=$ const. Since the functions $A(\tau)$ and $\beta(\tau)$ in (20) are given explicitly by expressions (21) and (22), a solution of (20) can be found easily under the initial condition $u(0)=1$:

$$u(\tau)=\exp\left(\int_0^\tau d\tau_1A(\tau_1)\right)\Big/\left[1+\frac{2}{5}\int_0^\tau d\tau_1\beta(\tau_1)\exp\left(\frac{2}{5}\int_0^{\tau_1}d\tau_2A(\tau_2)\right)\right]^{5/2}.\tag{23}$$

3. In particular, for $H=0$ (i.e., for $|\cos\psi|=|\cos\psi_0|=1/\sqrt{3}$) we have

$$l(\tau)=l_0(1-\tau)^{1/5},\ \gamma(\tau)=\gamma_0(1-\tau)^{-1/5},\ (\boldsymbol{\gamma}\mathbf{l})=\gamma_0l_0/\sqrt{3},\ M^2/\gamma^2l^2={}^2/_3$$

for $\cos\psi_0=1/\sqrt{3}>0$. In this case we have

$$A(\tau)=A_0/(1-\tau),\ \beta(\tau)=\beta_0(1-\tau)^{4/25},\ A_0=(2+{}^{17}/_5y+{}^{31}/_{35}y^2)/10(1+{}^1/_5y+{}^8/_{35}y^2),$$

and (23) takes the form

$$u(\tau)=1/(1-\tau)^{A_0}\left[1+\frac{\beta_0}{(2.9-A_0)}(1-(1-\tau)^{2/5(2.9-A_0)})\right]^{5/2}.\tag{24}$$

In other words, we have $u(\tau)\approx0(1-\tau)^{-A_0}$ in the limit $\tau\to1$, where $A_0\approx0.55$ for $y\approx3$ and $\alpha=137$. Similar values were found in Refs. 16 through numerical analysis of the evolution of the enstrophy: $A_0\approx0.8\pm0.1$ (Ref. 16a) and $0.5\lesssim A_0\lesssim1$ (Ref. 16b).

We thus see from (24) that at finite values of β_0 the viscous forces are not able to balance the three-dimensional stretching of vortex lines which leads to an unbounded increase in the enstrophy over a finite time as vortex dipoles close on each other in the limit $\tau\to1$. Because of the pronounced increase in the Reynolds number, $\mathrm{Re}\approx\gamma(\tau)/4\pi l^2(\tau)\nu\to\infty$, as $\tau\to1$ the viscous forces furthermore cannot substantially influence the relative Hamiltonian dynamics of point vortex dipoles, described by Eqs. (9), (9′) and the corresponding solutions in (21) and (22) in this regime. In this situation, the intense acoustic emission of the collapsing vortex dipoles can apparently act as a dissipative mechanism which is more effective than the viscosity in limiting the explosive growth of the enstrophy [and in eliminating the corresponding singularity $v(t)\to\infty$ as $\tau\to1$]. A lower limit $(1-\tau)\gg\mathrm{Ma}_0^{5/3}$, was derived in Ref. 9. This limit stemmed from compressibility effects ($\mathrm{Ma}_0=v_0/c$ is the Mach number, and c is the velocity of sound in the slightly compressible medium, with $\mathrm{Ma}_0\ll1$). This limit must be associated with an initial restriction on the scale of the smearing, $l(t)\gg\tilde{b}(t)$. Here, in contrast with the case of an isolated vortex dipole, the smearing $\tilde{b}$ may depend on the time, since we have $u(\tau)\approx O(\gamma^2/\tilde{b}^5)\approx(1-\tau)^{-A_0}$. and $\gamma(\tau)\approx O(1-\tau)^{-1/5}$. Accordingly, for $y\approx3$ and $A_0\approx0.55$, for example, we have $\tilde{b}(\tau)\approx b_0(1-\tau)^{0.03}$ ($b_0\equiv b$) in the limit $\tau\to1$. On the contrary for an isotropic (at $t=0$) smearing (5), (7) with $y=0$ an expansion of the localization region as $\tau\to1$: $\tilde{b}(\tau)\approx b_0(1-\tau)^{-1/25}$, since, at $y=0$ we have[4)] $A_0=0.2$. Since we have $l(\tau)=l_0(1-\tau)^{1/5}$, in the latter case we find from the condition $l(\tau)\gg\tilde{b}(\tau)$ the lower estimate $(1-\tau)\gg(b_0/l_0)^{25/6}$. Comparing this restriction on $1-\tau$ as $\tau\to1$ with the limit which we noted above for the compressibility effects, we reach the conclusion that for $1\gg\mathrm{Ma}_0\gg(b_0/l_0)^{5/2}\to0$ the acoustic effect should smooth out the singularity in $u(\tau)$ as $\tau\to1$. We also note that we have $\beta_0=t_0/t_v$ ($t_v=b_0^2/6\nu\alpha^2$), and we are justified in ignoring the viscous smearing of the vortex-core structure itself, of radius b_0, during the collapse time t_0 only under the condition $t_0<t_v$, i.e., under the condition $\beta_0<1$. In turn, this condition can hold only if the Reynolds numbers are sufficiently large: $\mathrm{Re}_0=v_0/l_0/\nu>(l_0/b_0)^2\gg1$.

4. We now assume $H\neq0$ ($H<0$) and $M=0$ [see the notation preceding (23)]. In this case the vortex dipoles close on each other (if $\psi_0=0$) or move apart (if $\psi_0=\pi$) along a straight line which is collinear with the direction of the momenta of these dipoles, $\boldsymbol{\gamma}=\boldsymbol{\gamma}'=-\boldsymbol{\gamma}^2$. This situation corresponds to the classical problem of the motion of coaxial vortex rings with oppositely directed momenta[1,17] in the limit in which the radii of these rings are arbitrarily small in comparison with the distance between their centers.

We consider vortex dipoles which are moving away from each other under the initial condition $\psi_0 = \pi$. In this case we have $(\gamma) = -\gamma l < 0$ at any time $t \geqslant 0$; this situation corresponds to a positive value $A(\tau) > 0$ and the possibility that the enstrophy will grown in time. In this case we indeed find from (23) [since $l(\tilde{\tau} = l_0(1+\tilde{\tau})^{2/5}$, $\gamma(\tilde{\tau} = \gamma_0(1+\tilde{\tau})^{3/5}$, $A(\tilde{\tau}) = \frac{3}{5}(1+\tilde{\tau})$, $\beta(\tilde{\tau}) = \tilde{\beta}_0/(1+\tau)^{12/25})$]

$$u(\tilde{\tau}) = (1+\tilde{\tau})^{3/5} / [1 + {}^{10}/_{19}\tilde{\beta}_0((1+\tilde{\tau})^{19/25} - 1)]^{5/2}, \tag{25}$$

where

$$\tilde{\tau} = t/\tilde{t}_0,\ \tilde{t}_0 = 2\pi l_0^4/5\gamma_0,\ \tilde{\beta}_0 = 6\nu\tilde{t}_0\alpha^2/b_0^2 = 3l_0^2\alpha^2/5\pi b_0^2\ \mathrm{Re}_0.$$

With $\tilde{\tau} \gg 1$ and $\tilde{\beta}_0 \ll 1$ it thus follows from (25) that there can be a significant power-law increase in the enstrophy, since we have $u(\tilde{\tau}) \approx O(\tilde{\tau}^{3/5})$ on the time interval $1 \ll \tilde{\tau} \lesssim (\tilde{\beta}_0)^{-25/19}$. On this time interval we have an expansion of the localization region, $\tilde{b}(\tilde{\tau}) \approx O(\tilde{\tau}^{3/25})$. Furthermore, at $\tilde{\tau}^{19/25}\tilde{\beta}_0 \gg 1$ we find $\tilde{b}(\tilde{\tau}) \approx O(\tilde{\tau}^{1/2})$, since in this limit we have $u(\tilde{\tau}) \approx O(\tilde{\tau}^{-18/10})$ and $\gamma^2(\tilde{\tau}) \approx O(\tilde{\tau}^{6/5})$. There thus exists an intermediate time interval $\tilde{\tau}$ in which we have $\tilde{b}(\tilde{\tau}) \approx O(\tilde{\tau}^{2/5})$; i.e., the increase in the average radius of the smearing of the vortex dipole, $\tilde{b}(\tilde{\tau})$ in time is proportional to the change in the distance between the vortex dipoles, since we have $l(\tilde{\tau}) \approx 0(\tilde{\tau}^{2/5})$ [and the condition $l(\tilde{\tau}) \gg \tilde{b}(\tilde{\tau})$ definitely holds for $l_0 \gg b_0$]. The same tendency toward an increase in the radius of a spherical vortex, proportional to the displacement of the vortex, was noted in Ref. 18, where this process was driven by buoyancy effects.

It thus obviously follows from (25) that the role of viscous forces in the case $M = 0$ $(H < 0)$, is a very important one, and it may differ qualitatively from that in the explosive regime (24) for $H = 0$ and $M \neq 0$), since even in the case of an arbitrarily small viscosity ν the viscous forces will be capable in principle, after a sufficiently long time $\tilde{\tau} \gg (\tilde{\beta}_0)^{-25/19} \gg 1$, of cancelling the effect of the stretching of the vortex lines.[1] On the other hand, it follows from (25) that at sufficiently large Reynolds numbers $\mathrm{Re}_0 \gg 1$ the relative increase in the enstrophy can reach extremely large maximum values before the viscous damping comes into play. Specifically, the function $u(\tilde{\tau})$ in (25) reaches its maximum value

$$u_{max}(\tilde{\tau}_m) \approx \tilde{\beta}_0^{-15/19}(1.14 - \tilde{\beta}_0)^{15/19}(1.57 - \tilde{\beta}_0)^{-5/2}$$

at

$$\tilde{\tau} = \tilde{\tau}_m = \left[\frac{10}{13\tilde{\beta}_0}(1.14 - \tilde{\beta}_0)\right]^{25/19} - 1, \tag{26}$$

where $\tilde{\tau}_m > 0$ only for sufficiently small values $\tilde{\beta} < \tilde{\beta}_0^{\mathrm{cr}} = 1.14$, i.e., only at Reynolds number Re_0 exceeding a critical value,

$$\mathrm{Re}_0 > \mathrm{Re}_0^{\mathrm{кр}} = 3\alpha^2 l_0^2/5.7\pi b_0^2,$$

where $\alpha \gg 1$ [for $\alpha = 137$ we have $I_{\max}/I_0 \approx O(\mathrm{Re}_0^{15/19})$]. Consequently, in this case, $M = 0$ $(H < 0)$, the enstrophy can be significantly enhanced, to a maximum value $I_{\max}/I_0 \approx O(\mathrm{Re}_0^{15/19})$ at $t = t_{\max} \approx \tilde{t}_0 \mathrm{Re}_0^{2/19}$ only for Reynolds numbers above the critical value, $\mathrm{Re}_0 > \mathrm{Re}_0^{\mathrm{cr}}$.

The vorticity bursts of large (but finite) amplitude which are observed in turbulent boundary layers are indeed linked with an interaction of (coaxial) dipole vortices.[19,20] Accordingly, solutions (23)–(25) [especially (25)], derived above, can be of some use in interpreting the corresponding experimental data.[21,22] In fact, the separation of a horseshoe-shaped vortex having a dipole structure[19] from the wall which was observed in Refs. 21 and 22 may be thought of as a result of the interaction of this vortex with its coaxial mirror image (with respect to the plane of the wall) (or with a dipole vortex induced near the wall[19]).

The conclusions reached in Sec. 2 can also be used to model statistically uniform turbulence, if the integral in the definition of I(and I_2) is replaced by a statistical averaging.[23] Furthermore, it would be interesting to study the intensification of vorticity when the effects of buoyancy and a temperature stratification of a medium are taken into account.

We also note that the conclusion, reached in Sec. 2, that a strong singularity of the enstrophy I can be reached in the three-dimensional case (in the limit $\tau \to 1$) is itself independent of the nature of the regularization of the point vortex dipole (cf. the case with $y = 0$ and 3). On the other hand, the index A_0 of the corresponding explosive increase of I nevertheless depends on the nature of the smearing of the δ-function. This circumstance leads in turn to different tendencies in the time evolution of the smearing radius $\tilde{b}$ (in contrast with the case in which there is only an expanding vortex region, studied in Ref. 18). Consequently, and in contrast with the two-dimensional case (in which the enstrophy is invariant, and only a weak singularity of the local characteristics of the velocity field is possible; see Subsection 1.5), in three-dimensional turbulence there can be a special mechanism by which energy is drained off to point vortex singularities. This mechanism would not depend on the presence of viscous dissipation, because of the strong singularity of the enstrophy which was mentioned in Sec. 2. The possibility that a mechanism of this sort would operate to drain off the energy of turbulence (and lead to the establishment of a corresponding universal Kolmogorov-Obukhov regime) because the solutions of the three-dimensional hydrodynamic equations lose their smoothness over a finite time (there is no need to introduce viscous dissipation in the region of small scales) was apparently studied first in the well-known paper by Onsager[24] (see also Refs. 6 and 9 and the bibliographies there).

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APPENDIX

1. To derive the expression

$$f(\mathbf{x}) = \frac{1}{4\pi}\int d^3x' \frac{\tilde{\delta}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|},$$

which corresponds to $\tilde{\delta}(\mathbf{x})$ from (5), we use the representation

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{k=0}^{\infty} P_k(\cos\tilde{\psi})\left[\theta(|\mathbf{x}'| - |\mathbf{x}|)\frac{|\mathbf{x}|^k}{|\mathbf{x}'|^{k+1}} + \theta(|\mathbf{x}| - |\mathbf{x}'|)\frac{|\mathbf{x}'|^k}{|\mathbf{x}|^{k+1}}\right],$$

where

$$\cos\tilde{\psi} = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi'),$$

θ and φ are the polar angles of the vector **x**, and θ' and φ' are those of the vector **x**′, over which the integration is carried out. Making use of the orthogonality of the spherical harmonics,[25]

$$\int_0^{\pi} d\theta' \sin\theta' \int_0^{2\pi} d\varphi' P_n(\cos\theta') P_m(\cos\tilde{\psi}) = \begin{cases} 0, & m\neq n \\ \dfrac{4\pi}{2n+1} P_n(\cos\theta), & m=n \end{cases}$$

we find the following expression for $f(\mathbf{x})$:

$$f(r,\theta)=4\pi a_0\Big[\frac{1}{r}\int_0^r dx\, x^2\delta_0(x)+\int_r^{\infty} dx\, x\delta_0(x) + y\frac{P_2}{5}(\cos\theta)\Big(\frac{1}{r^3}\int_0^r dx\, x^4\delta_0(x)+r^2\int_r^{\infty} dx\frac{\delta_0(x)}{x}\Big)\Big]. \quad (A1)$$

To find T_{s0}, we substitute (A1) into (6) and set $\theta=0$ in $l(r,\theta)$ $[P_2(1)=1]$. For $\delta_0(r)$ from (7), we then find

$$T_{s0}=\frac{\rho_0\gamma^2(\alpha+3)^2(\alpha+4)^2(\alpha+5)^2}{80\pi b^3(\alpha+2)(\alpha+3)(2\alpha+3)(2\alpha+5)(2\alpha+7)} \times\Big[y^2\Big(1+\frac{16\alpha^2+143\alpha+327}{(\alpha+5)(\alpha+6)(\alpha+7)}\Big) - y\Big(7-\frac{16\alpha^2+143\alpha+327}{(\alpha+5)(\alpha+6)(\alpha+7)} - \frac{2(16\alpha+59)}{(\alpha+4)(\alpha+5)}\Big)+10\Big], \quad (A2)$$

where $\alpha> -3/2$. With $\alpha=\alpha_{cr}=137$ in (A2) we have $T_{s0}=0$ for $y=3.001\pm0.101$; in the limit $\alpha\to\infty$ we have $T_{s0}=0$ for $y=2$ and 5. With a decrease in the smoothness of $\delta_0(r)$ near $r=b$, the size of the critical structural index α_{cr} decreases (only for $\alpha\geqslant\alpha_{cr}$ do we have $T_{s0}=0$ for real values of y). For example, with $\delta_0(r)=r^\alpha(b-r)\tilde{\theta}(b-r)$ we find $\alpha_{cr}=102$, while for $\delta_0(r)=r^\alpha\tilde{\theta}(b-r)$ we have an even smaller value, $\alpha_{cr}=50$ (we are rounding to integers).

2. We now consider procedures for regularizing the self-energy T_s in the two-dimensional case, for which T_s is given by (10) and (11) with $N=1$ ($\mathbf{x}'=0$, $\gamma\equiv\gamma=$ const). We replace the δ-function by the regular function $\tilde{\delta}(\mathbf{x})=\tilde{\delta}(-\mathbf{x})$ ($\int d^2\tilde{\delta}(\mathbf{x})=1$), which we represent by the series

$$\tilde{\delta}(\mathbf{x})=\sum_{k=0}^{\infty} a_{2k}(r)\cos 2k\varphi,$$

where $r=|\mathbf{x}|$, and φ is the polar angle in the plane, measured from the direction of γ. The corresponding velocity field is of the form in (4), with

$$\Phi=\Phi_2=\gamma_l\frac{\partial f_2}{\partial x_l}, \quad f_2=-\frac{1}{2\pi}\int d^2x'\,\tilde{\delta}(\mathbf{x}')\ln|\mathbf{x}-\mathbf{x}'|.$$

We thus have

$$f_2(r,\varphi)=-\frac{1}{2\pi}\int_0^{\infty} d\rho\rho\sum_{k=0}^{\infty} a_{2k}(\rho)\cos 2k\varphi \begin{cases} -\dfrac{\pi}{2k}\Big(\dfrac{\rho}{r}\Big)^{2k}, & \rho<r, \quad k\neq0 \\ -\dfrac{\pi}{2k}\Big(\dfrac{r}{\rho}\Big)^{2k}, & \rho>r, \quad k\neq0 \\ 2\pi\ln r, & \rho<r, \quad k=0 \\ 2\pi\ln\rho, & \rho>r, \quad k=0 \end{cases} \quad (A3)$$

In particular, with $a_{2h}(r)=a_{2k}\delta_0(r)$ $(k=0,1)$ $(k=0,1)$, $a_{2k}=0$ $(k>1)$, $a_2/a_0=y$ we have the following expression for T_{s0} [$T_s=T_{s0}+T_{s1}$, where T_{s0} is found from T_s as $f_2(r,\varphi)\to f_2(r,\varphi=0)$] in the case $\delta_0(r)=r^\alpha(b-r)\tilde{\theta}(b-r)$:

$$T_{s0}=\frac{\rho_0\gamma^2}{8\pi b^2}\Big[y^2\Big(1+\frac{4(\alpha+3)}{(\alpha+4)(\alpha+5)}\Big) - y\Big(3-\frac{4(\alpha+3)}{(\alpha+4)(\alpha+5)}-\frac{8}{\alpha+3}\Big)+2\Big]. \quad (A4)$$

It follows from (A4) that we have $\alpha_{cr}=98$ (we are rounding to integers), while in the case $\alpha\to\infty$ we have $T_{s0}=0$ for $y=2$ and 1 in (A4) (for $\alpha=98$, we have $T_{s0}=0$ for $y=1.388\pm0.009$).

For the case of complex y,

$$\tilde{\delta}(\mathbf{x})=a_0\delta_0(r)(1+iy_1\cos 2\varphi),$$

where $\langle y_1\rangle=0$ (y_1 is a random quantity), we have, correspondingly, the following expression for the total self-energy, averaged over y_1:

$$\langle T_s\rangle=\rho_0\gamma^2\frac{\pi a_0^2}{2}\Big(1-\frac{\langle y_1^2\rangle}{2}\Big)\int_0^b dr\, r\delta_0^2(r), \quad (A1)$$

where

$$a_0 2\pi\int_0^b dr\, r\delta_0(r)=1.$$

In (A5) we have $\langle y_1^2\rangle\leqslant2$, and at $\langle y_1^2\rangle=2$ we have $\langle T_s\rangle=0$.

3. In the three-dimensional case, the Helmholtz equation for the vortex field $\omega(\mathbf{x},t)$ is given for the case of an ideal, incompressible fluid by[1]

$$\frac{\partial\omega_i}{\partial t}+u_l\frac{\partial\omega_i}{\partial x_l}=\omega_l\frac{\partial u_i}{\partial x_l},\qquad \boldsymbol{\omega}=\text{rot}\,\mathbf{u}. \tag{A6}$$

Substituting the expression for $\boldsymbol{\omega}$ from (8) into (A6), and multiplying both sides of (A6) by an arbitrary smooth function $\varphi(\mathbf{x})$, we can carry out the integration over the entire space, including the region in which point vortex dipoles (8) are concentrated. From (A6) we find ($\dot{\boldsymbol{\gamma}}^m \equiv d\boldsymbol{\gamma}^m/dt$, $\dot{\mathbf{x}}^m \equiv d\mathbf{x}^m/dt$)

$$\int d^3x\delta(\mathbf{x}-\mathbf{x}^m)\left[-\frac{d\varphi}{\partial x_p}\varepsilon_{ipl}\dot{\gamma}_l^m-\varepsilon_{ijl}\gamma_l^m\dot{x}_p^m\frac{\partial^2\varphi}{\partial x_j\partial x_p}+\varepsilon_{ijl}\gamma_l^m\left(\frac{\partial u_p}{\partial x_j}\frac{\partial\varphi}{\partial x_p}+u_p\frac{\partial^2\varphi}{\partial x_p\partial x_j}\right)+\varepsilon_{lpb}\gamma_b^m\frac{\partial u_i}{\partial x_l}\frac{\partial\varphi}{\partial x_p}\right]=0, \tag{A7}$$

where $m=1,2,...N$. From (A7) we find

$$\dot{x}_p^m=u_p(\mathbf{x}^m), \tag{A8}$$

$$\varepsilon_{ipl}\dot{\gamma}_l^m=\varepsilon_{ijl}\gamma_l^m\frac{\partial u_p}{\partial x_j}+\varepsilon_{lpb}\gamma_b^m\frac{\partial u_i}{\partial x_l}. \tag{A9}$$

Multiplying (A9) by ε_{ipd}, we find

$$\dot{\gamma}_d^m=-\gamma_l^m\frac{\partial u_l(\mathbf{x}^m)}{\partial x_d^m}, \tag{A10}$$

since we have $\varepsilon_{ipd}\varepsilon_{ijl}=\delta_{pj}\delta_{dl}-\delta_{pl}\delta_{dj}$, etc., and div $\mathbf{u}=0$. The singular self-induced velocity also enters u in (A8) and (A10) [i.e., $\mathbf{u}$ is of the form in (4), if we again replace $\tilde{\delta}$ by a δ-function in (4)]. This velocity can be eliminated by means of, for example, the regularization of $\mathbf{u}$ which is used in the theory of generalized functions for the functional $\partial\Phi_3/\partial x_i$ (Ref. 6). From (A8) and (A10) we find system (9), (9′).

We also note that system (A8), (A10) [and, correspondingly, (9), (9′)] was derived in Ref. 15 for the particular case of a weight function by carrying out a vector multiplication of (A6) [for ω from (8)] by $\mathbf{x}$, followed by integration over the entire volume. Furthermore, the procedure used above is analogous to the derivation of a weak solution for two-dimensional point vortex dipoles which has been carried out by V. M. Gryanik (private communication) for the two-dimensional Helmholtz equation [in which case (A6) does not have the term on the right side].

[1)]In the two-dimensional case, a pair of point vortices which have opposite signs and which are separated by an arbitrarily small distance in the plane (for example) correspond to a point vortex dipole. In the three-dimensional case, a vortex dipole may be thought of as an infinitesimal vortex ring or a spherical Hill vortex.

[2)]I wish to thank P. G. Saffman for furnishing a refinement of the conclusions of Ref. 3 concerning the dynamic system (9), (9′).

[3)]In contrast with average balance equation (16), for example, the Helmholtz equations (A6) may give rise, in particular, to delocalization at $t>0$ of an initial structure of the type in (5), (7) [i.e., $b=b(\mathbf{x},t)$ at $t>0$, although $b=$ const at $t=0$]. On the other hand, to analyze the evolution of the moments $J_p=\int d^3x\tilde{\omega}^{2p}$ ($p=2,3,...$) we should also introduce the corresponding average scales of the smearing, $\tilde{b}_p$ (which regularizes J_p) which are different for the moments of different index p.

[4)]If the parameter y is imaginary and random [$y=iy_1$, $\langle y_1\rangle=0$, and $\langle y_1^2\rangle=7$ (i.e., $\langle T_s\rangle=0$), we have $A_0=0.7$; Ref. 16]. This result holds only if $\alpha\gg1$, in which case all the terms in (19) change sign. In this case we have $I>0$ only for $-0.5<\alpha<-0.17$ (with $\alpha=-0.2$ $A_0\approx27$), but for $\alpha\leqslant1/2$ the integral in (19) which describes the effect of the viscosity diverges.

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