Note on multi-loop calculations for superstrings in the Neveu-Schwarz-Ramond formalism

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We discuss a relatively simple and attractive prescription for multi-loop calculations. It consists in defining the metric, the Beltrami superdifferentials and summation over the spinor structures with the help of one and the same odd θ-characteristic e_*. At the end of the calculations modular invariance is restored by summing over e_*. It is noted that included in the prescription should be a description of a limiting procedure needed in intermediate stages of the calculations. For a particular choice of this procedure the contribution to the cosmological constant from matter supercurrents is absent, at least for the first five loops ($p \leqslant 5$).

Among the string models the theory of superstrings occupies a special place—it is the simplest string theory which, as is to be expected, is free of divergences in the scattering amplitudes in all orders of perturbation theory. Unfortunately, the proof of finiteness and the calculation itself of multi-loop superstring amplitudes encounters definite difficulties. The approach to the construction of these objects that has been carried the farthest consists of the following (see, e.g., Ref. 1). Use is made of first-quantization formalism for string theory, in which p-loop diagrams are described in terms of Riemann surfaces of genus p. In such a formalism one is really discussing definite correlators in two-dimensional quantum field theory for nontrivial topology of the two-dimensional world surface. Under these conditions one may define theories with local two-dimensional supersymmetry and introduce the concept of super Riemannian surfaces and the space of supermoduli. The simplest model is based on a supergeneralization of the Polyakov action, which is quadratic in the fields in the lightcone gauge (when that gauge can be chosen).

However, the theory of fermionic superstrings obtained in this fashion is not endowed with supersymmetry in the multidimensional spacetime and is not free of divergences. To obtain a theory of superstrings from it the following idea might be used. The super-Riemann surfaces depend on one discrete parameter—the so-called spinor structure or theta characteristic e. In the case of the fermionic string the amplitudes are represented in the form of a sum over e, whose terms are in the form of integrals of the square of the modulus of the corresponding generalized Mumford supermeasure over the supermoduli space. Instead of such a sum of squares of moduli another quantity may be considered. Let us integrate the Mumford supermeasure over the odd moduli. Let us form a linear combination of such integrals with different values of e—this yields a certan measure in the conventional space of moduli. The integral of the square of its modulus over this space also defines a certain amplitude. This Neveu-Schwarz-Ramond (NSR) procedure is well suited to the one-loop case and for an appropriate choice of linear combinations indeed leads to superstring amplitudes satisfying the requirement of unitarity. In this approximation this procedure is easily interpreted as the projection onto the G-even sector of the fermionic string.[2]

Unfortunately in the case of many loops the situation becomes more complicated. The procedure formulated above has two obvious ambiguities: in the choice of the odd moduli to be integrated over in the first stage, and in the choice of the linear combinations in the second stage. The initial expectation that the answer would be independent of the choice of odd moduli has not been confirmed and at this time it is widely believed[3,4] that the NSR formulation at the multi-loop level[1,5–8] must be supplemented by special prescriptions to resolve the ambiguities indicated above. It may be that in the future these prescriptions will be derived from general principles, when the global structure of the space of supermoduli become better studied (see e.g., the papers in Refs. 6 and 9 where definite steps were taken in this direction). However it makes sense to search for the needed prescription in a more heuristic fashion by attempting to develop a procedure which, on the one hand, would be sufficiently simple and allow the performing of the calculations and, on the other hand, would satisfy a number of simple requirements such as modular invariance, factorization, finiteness and the vanishing of certain entities such as, for example, the cosmological constant. In our opinion the propositions made in Refs. 4 and 7 deserve further study as prototypes of this kind of a recipe.

Thus, the main difficulties in the NSR formalism stem from two sources: the arbitrariness in the choice of the Beltrami superdifferentials and the ambiguity in the summation over the spinor structures (θ characteristics). For $p \geqslant 2$ the summation over the spinior structures (the Gliozzi-Scherk-Olive projection) cannot be accomplished in a modular-invariant manner if all the spinor structures are included in the sum with weights equal to ± 1.

This problem might not appear explicitly in the calculation of the vanishing expressions for the 0-, 1-, 2- and 3-point functions. Consequently, for an arbitrary prescription which is not manifestly modular-invariant, the vanishing of the cosmological constant and the correlation functions for 1, 2, and 3 massless particles cannot serve as the decisive criterion for its validity: whether such a prescription is reasonable cannot be decided without evaluating with its help some nonvanishng expressions and verifying its modular invariance.

The Beltrami superdifferentials are chosen most conveniently in the form of δ-functions:[1,5,7–9]

$$\chi_i(z) = \delta^{(2)}(z-P_i)\, d\bar{z}/(dz)^{1/2}, \quad i=1, \ldots, 2p-2. \tag{1}$$

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The fixing of the $2p-2$ points $\{P_i\}$ on each Riemann surface may also break modular invariance. Moreover, due to the manifest noninvariance of the Riemann identities even a modular-invariant choice of the points P_i could lead to a modular noninvariant answer after the standard noninvariant summation over the spinor structures. Starting from these considerations it was proposed in Ref. 4 that the two indicated sources of "modular anomaly" be compensated. Also, a concrete method for achieving this compensation was proposed.

The Polyakov formulation of string theory includes integration over metrics on the world sheet. In order to show that in the case of the heterotic and supersymmetric strings this reduces to finite-dimensional integration over the moduli space of Riemann surfaces it is necessary to demonstrate the absence of three anomalies: a) the Weyl-Polyakov anomaly; b) the analytic anomaly; c) the modular anomaly.

The first two anomalies are local and have been well-studied; it is known that they are related to each other and are indeed absent in string theory. But in explicit formulas for individual determinants and correlators[10] there appears particulary strikingly the principal value of the singular metrices of the form $g\propto|W(z)|^2\exp[\sigma(z,\bar{z})]$. (Here $W(z)=\widehat{W}(z)dz$ is a holomorphic 1-differential on the Riemann surface. As regards the Weyl factor $\exp[\sigma(z,\bar{z})]$, it disappears from anomaly-free combinations of determinants and correlators.) There exists no particular method for a one-to-one association of $W(z)$ with Riemann surfaces. Let us say that the zeros of $W(z)$, which we agree to denote by $Q_1,...,Q_{2p-2}$, are constrained by the single condition

$$\sum_{i=1}^{2p-2} Q_i=2\Delta,$$

where 2Δ is the so-called canonical divisor by the Jacobian of the Riemann surface. For an arbitrary set of $2p-2$ points $\{Q_i\}$ on the surface, whose Jacobi transformed image satisfies this condition a holomorphic 1-differential can be found with such a set of zeros. There exists a multitude of sets of points $\{Q_i\}$ of this type, and for a unique specification of $W(z)$ one must impose some additional requirements. It is convenient to demand that $W(z)$ have double zeros:

$$Q_1=Q_2=R_1^*,\ldots,\quad Q_{2p-3}=Q_{2p-2}=R_{p-1}^*,$$

i.e., that it be the square of a $\frac{1}{2}$-differential: $W(z)=\nu_*^2(z)$. In this manner one may arrive at an almost unique choice of $W(z)$. However for $p\geqslant 2$ there remains a finite set of possibilities, correlated with odd θ characteristics, which we shall denote by e_* or simply "*".

Thus, this choice of metric breaks modular invariance and only invariance with respect to transformations that do not affect e_* is left. This provides a potential source for modular anomaly. It is known that modular anomaly is cancelled in combinations of determinants, free from the local anomalies a) and b).[11] However the measure on the space of moduli connected with superstrings does not simply reduce to the anomaly-free combination of determinants; it also contains the correlator of $2p-2$ supercurrents located at the points P_i [assuming that the Beltrami superdifferentials are chosen in the form (1)]. In Ref. 7 it was proposed to place the supermoduli at the zeros of the metric: $P_i=Q_i$. If the metric is taken as $g=|\nu_*|^4$ then the location of the points P_i turns out to be specified by the odd θ-characteristics e_* and in this sense modular invariance is explicitly violated. In Ref. 7 it was proposed to also sum over the spinor structures e with weight $\langle e_*,e\rangle$ determined by that same odd θ-characteristics e_*.

In other words it is proposed to specify any odd spinor structure e_* whatsoever and carry out all calculations using the metric $|\nu_*|^4$ with double zeros at the points $R_1^*,...,R_{p-1}^*$, using the Beltrami superdifferentials

$$\chi_\alpha\propto\delta(z-R_\alpha^*),\qquad \tilde{\chi}_\alpha\propto\delta'(z-R_\alpha^*),\qquad \alpha=1,\ldots,p-1,$$

and in the sum over the spinor structures e use as weights $\langle e_*,e\rangle$. Generally speaking such a procedure will produce an answer ϕ_*, dependent on e_*. For this reason the premise of Ref. 4 is to sum these answers over the odd θ-characteristics e_* and consider as the final answer $\phi\equiv\Sigma_*\phi_*$. Obviously ϕ is modular-invariant but, clearly, it is not known *a priori* whether this procedure agrees with the factorization requirement and whether it results in vanishing answers for the 0-, 1-, 2- and 3-point functions. It could happen that this is not so and one would have to admit that this prescription is erroneous. Its advantage, however, is in being constructive, which allows performing calculations and verifying the adequacy of this way of action. It should also be noted that, due to the additional summation over e_* which restores modular invariance, the procedure under discussion does not fall into the class considered in Ref. 9, and for this reason we refrain from commenting on those papers. We also do not repeat the arguments that were advanced in Ref. 4 in favor of the proposed prescription.

Instead we briefly discuss the formal apparatus needed in this approach. The point is that the limit $P_2\to P_1\to R_1^*$, $P_4\to P_3\to R_2^*,...P_{2p-2}\to P_{2p-3}\to R_{p-1}^*$ is strongly singular and one must resolve a number of ambiguities of the type 0/0. Below we confine ourselves to the case when the odd spinor structures do not contribute to the Gliozzi-Scherk-Olive sum.

To discuss the singular limit prior to summing over the spinor structures is most difficult. But to perform the sum over e in the general case is not easy either since the denominator contains the superghost determinant $\det_{(ij)}\zeta_i^e(P_j)\det_e\bar{\partial}_{3/2}$ ($\zeta_1^e,...,\zeta_{2p-2}^e$ are holomorphic 3/2-differentials), which is proportional to $\theta_e\ (\Sigma_{i=1}^{2p-2}\mathbf{P}_i-2\Delta)$. The sum over even characteristics e looks as follows:

$$\sum_e\langle e_*,e\rangle\frac{\theta_e(\mathbf{a}_1)\ldots\theta_e(\mathbf{a}_4)\theta_e(0)}{\theta_e(\Sigma\mathbf{P}_i-2\Delta)}\tag{2}$$

($\mathbf{a}_1...\mathbf{a}_4$ are combinations of the same vectors $\mathbf{P}_i$ that occur in the Jacobian, resulting from the correlators of the $\frac{1}{2}$-differentials ψ. The combination $\mathbf{a}_5$ vanishes since otherwise the odd spinor structures would not disapear from the sum).

The problem has to do with the fact that the Riemann identities cannot be used to evaluate (2) as long as θ_e remains in the denominator. In order to evaluate (2) in the general case we need some as yet unknown identities of the type discovered in Ref. 12 for the case of $p=1$. If, however, we set

$$P_2=P_1=R_1^*,\ldots,\quad P_{2p-2}=P_{2p-3}=R_{p-1}^*,$$

then

$$\sum_{i=1}^{2p-2} \mathbf{P}_i = 2\sum_{\alpha=1}^{p-1} \mathbf{R}_\alpha^* = 2\Delta$$

and the argument of the θ-function in the denominator of (2) vanishes. This, in truth, is insufficient since (2) is multiplied by an e-independent factor which is singular in the limit $P_2 \to P_1, \ldots P_{2p-2} \to P_{2p-3}$. Consequently a special limiting procedure is needed.[7,13] The idea is to introduce "regularization" with a parameter ε, to set $P_i = Q_i$ for the metric

$$g = |W|^2 = |\nu_*^2 + \varepsilon\omega|^2 \tag{3}$$

with pairwise different zeros $Q_1, \ldots, Q_{2p-2}$. In that case the above-mentioned factor is finite, we still have $\Sigma \mathbf{P}_i = \Sigma \mathbf{Q}_i = 2\Delta$ and the sum (2) is easily calculated. The answer is

$$\theta_*\left(\frac{\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4}{2}\right)\theta_*\left(\frac{\mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_3-\mathbf{a}_4}{2}\right)$$
$$\times\theta_*\left(\frac{\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3-\mathbf{a}_4}{2}\right)\theta_*\left(\frac{\mathbf{a}_1-\mathbf{a}_2-\mathbf{a}_3+\mathbf{a}_4}{2}\right). \tag{4}$$

Now one can pass to the limit $\varepsilon \to 0$ without any difficulties, relying on the Riemann theorem on zeros (for details see Refs. 7 and 13). We note that in this procedure all differences $\xi_\alpha \equiv P_{2\alpha} - P_{2\alpha-1}$ are of the same order of smallness ε. [Furthermore, the procedure proposed here has no need for assumptions of the type $\theta_*(\xi_1+\xi_2)/\theta_*(\xi_1) \sim 1$ for $\xi_1 \sim \xi_2$, which have to be invoked in Ref. 13.]

The passage to the limit defined by the regularization (3) turns out to be much more informative than would appear at first glance. We note that whenever $P_i = Q_i$ and $\Sigma_{i=1}^{2p-2}\mathbf{P}_i = \Sigma_{i=1}^{2p-2}\mathbf{Q}_i = 2\Delta$, the vector $\frac{1}{2}\Sigma_{i=1}^{2p-2}\mathbf{P}_i$ coincides with some half-period in the Jacobian. Considerations of continuity in the parameter ε show that this half-period equals $\Delta_* = \Sigma_{\alpha=1}^{p-1}\mathbf{R}\alpha^*$. Therefore

$$\frac{1}{2}\sum_{i=1}^{2p-2}\mathbf{P}_i = \Delta_*. \tag{5}$$

This is a very useful relation. The point is that each of the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ in expression (2) for the contribution of matter supercurrents to the cosmological constant is a difference of the form $\mathbf{a} = \mathbf{P}_1 - \mathbf{P}_2$ or $\mathbf{a} = \mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_3 + \mathbf{P}_4 - \mathbf{P}_5 - \mathbf{P}_6$ etc., with each $\mathbf{P}_i$ entering one and only one of the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$. This means that the arguments of the θ_*-functions in (4) are sums of $p-1$ vectors $\{\mathbf{P}_\alpha\}$ minus $p-1$ vectors $\{\tilde{\mathbf{P}}_\alpha\}$, divided by two [see (5)]:

$$\frac{1}{2}\left(\sum_{\alpha=1}^{p-1}\mathbf{P}_\alpha - \sum_{\alpha=1}^{p-1}\tilde{\mathbf{P}}_\alpha\right) = \sum_{\alpha=1}^{p-1}\mathbf{P}_\alpha - \Delta_*.$$

But this is precisely the argument for which θ_* vanishes identically according to the Riemann theorem on zeros:

$$\theta_*\left(\sum_{\alpha=1}^{p-1}\mathbf{P}_\alpha - \Delta_*\right) \equiv 0.$$

Thus using the regularization (3) allows us to apply this theorem much more effectively than in Refs. 7 and 13.

For example, for genus $p = 2$

$$\mathbf{a}_1 = \mathbf{P}_1 - \mathbf{P}_2, \quad \mathbf{a}_2 = \mathbf{a}_3 = \mathbf{a}_4 = \mathbf{a}_5 = 0$$

and the expression (4) has the form

$$\frac{1}{\xi^4}\theta_*\left(\frac{\mathbf{P}_1 - \mathbf{P}_2}{2}\right).$$

We have explicitly shown the independent singular factor ξ^{-4}, $\xi = \mathbf{P}_1 - \mathbf{P}_2$, which was omitted in (2) and (4). The Riemann theorem on zeros means that for any $P_1 \to P_2$ this product behaves like $\xi^{-4}\xi^{12} = \xi^8$ and vanishes in the limit $\xi = 0$. However, this discussion is not valid for $p > 3$, since the singular factor is constructed as $\xi^{-4(p-1)}$, while the product of θ_*-functions continues to go like ξ^{12} in the limit $P_1 \to P_2, \ldots, P_{2p-3} \to P_{2p-2}$. It turns out that for the regularization (3) and under the condition $P_i = Q_i$ the θ_*-functions vanish identically even for $\xi \neq 0$, since for $\frac{1}{2}(\mathbf{P}_1 + \mathbf{P}_2) = \Delta_*$ we have

$$\theta_*^4\left(\frac{\mathbf{P}_1 - \mathbf{P}_2}{2}\right) = \theta_*^4(\mathbf{P}_1 - \Delta_*) \equiv 0,$$

and this discussion applies to all genera p.

In this manner we have demonstrated that the contribution of matter supercurrents to the cosmological constant vanishes for $p \leqslant 5$. [Beginning with $p = 6$ odd spinor structures start to contribute and they require a separate analysis. For odd structures the superghost determinant in the denominator of (2) vanishes for $P_i = Q_i$. Therefore even for $p = 5$ one must first consider the case $P_i \neq Q_i$ (with the odd spinor structures not contributing because $\mathbf{a}_5 = 0$), and then go to the "limit" $P_i = Q_i$, in which the contribution from the odd structures continues to be absent by continuity, and only following this type of argumentation can the sum (2) be analyzed by the method indicated above.] For those who believe in the ansatz of Ref. 14 for the superstring measure, based on ignoring the contribution from ghost supercurrents, the present result is sufficient to conclude that for $p \leqslant 5$ the measure corresponding to the cosmological constant vanishes pointwise on the space of supermoduli. However, the evaluation of the contribution of the ghost supercurrents, which we continue to consider relevant, is more complicated than the evaluation of (2) (in this connection see Appendix).

The fact that we have not completely exhausted the freedom in the choice of the limiting procedure is also worth mentioning. The choice of the holomorphic 1-differential ω in (3) has not been restricted in our discussions in any way (except for the one requirement that the points $Q_1, \ldots, Q_{2p-2}$ be pairwise different). It might turn out to be useful to choose ω in some special manner. This possibility should be kept in mind.

In this manner we have discussed the necessity and utility of the concentration of the limiting procedure to refine the definition of the prescription for multiloop superstring calculations proposed in Refs. 4 and 7. We see no obstacles within the framework of this procedure to carrying out all the necessary calculations and determining all necessary modifications. In our opinion considerations along these lines deserve further study.

We are grateful to O. Lechtenfeld for sending us his paper,[13] which served as stimulus for the publication of these remarks.

APPENDIX

We describe here technical details for dealing with correlators of ghost supercurrents. In the general case, what arises in the multiloop calculations for superstrings is not $[\det\zeta_i^e(P_j)\det_e\bar{\partial}_{3/3}]^{-1}$ but the more complicated correlator

$$G_e(P_0,\ \{P_i\}\,|\,\{z_i\})\equiv\langle\xi(P_0)\ldots\xi(P_{2p-2})\gamma(z_1)\ldots\gamma(z_{2p-2})\rangle_e. \tag{A1}$$

In what follows we make use of the description of this correlator given in Ref. 8, and of the notation from that paper:

$$G_e(P_0,\ \{P_i\}\,|\,\{z_i\})=\prod_{i=1}^{2p-2}\langle\zeta(P_0)\ldots\zeta(P_{2p-2})\hat{\gamma}(z_i)\rangle_e \Big/\prod_{\mu=0}^{2p-2}\langle\zeta(P_1)\ldots\zeta(\check{P}_\mu)\ldots\zeta(P_{2p-2})\rangle_e \tag{A2}$$

(it should be noted that in the correlator in the denominator the field $\zeta(P_\mu)$ is omitted. The subscripts run over the values $\mu=0,\ldots,2p-2$; $i=1,\ldots,2p-2$). The additional point P_0 appears in these formulas, but in fact G_e is a sum of terms, each of which is independent of any of the points P_μ:

$$G_e(P_0,\{P_i\}\,|\,\{z_i\})=\sum_{\mu=0}^{2p-2}\frac{h_e(P_0\ldots\check{P}_\mu\ldots P_{2p-2}|z_i)}{\langle\zeta(P_0)\ldots\zeta(\check{P}_\mu)\ldots\zeta(P_{2p-2})\rangle_e}=\frac{h_e(\{P_i\}\,|\,\{z_i\})}{\langle\zeta(P_1)\ldots\zeta(P_{2p-2})\rangle_e}+\ldots, \tag{A3}$$

and only one term, independent of P_0, contributes to the formulas for superstrings. However, the representation (A3) also has disadvantages as compared to (A2): $h_e(\{P_i\}|\{z_i\})$ turns out to be a rather complicated expression (see below). Let us introduce the notation

$$\langle\zeta(P_0)\ldots\zeta(P_{2p-2})\hat{\gamma}(z)\rangle_e \equiv\Gamma_e^{(3/2)}(P_0,\{P_i\}\,|\,z)\propto\theta_e\Big(\sum_{i=2}^{2p-2}\mathbf{P}_i+\mathbf{P}_0-\mathbf{z}-2\mathbf{\Delta}\Big),$$

$$\langle\zeta(P_1)\ldots\zeta(P_{2p-2})\rangle_e\equiv\Gamma_e^{(3/2)}(\{P_i\})\propto\theta_e\Big(\sum_{i=1}^{2p-2}\mathbf{P}_i-2\mathbf{\Delta}\Big). \tag{A4}$$

The correlators Γ_e depend on the metric on the surface.[10] For $g=|W|^2$ with zeros Q_i we have $\Sigma_{i=1}^{2p-2}\mathbf{Q}_i=2\mathbf{\Delta}$, and, from the point of view of dependence on the spinor structure e, the correlator $\Gamma_e^{(3/2)}(\{P_\mu\}|z)$ is equivalent to the analogous correlator of $\frac{1}{2}$-differentials:

$$\Gamma_e^{(1/2)}(\{P_\mu\}\,|\,z,\ \{Q_i\}) \equiv\langle\psi(P_0)\ldots\psi(P_{2p-2})\bar{\psi}(z)\bar{\psi}(Q_1)\ldots\bar{\psi}(Q_{2p-2})\rangle_e \propto\theta_e\Big(\sum_{i=1}^{2p-2}\mathbf{P}_i+\mathbf{P}_0-\mathbf{z}-2\mathbf{\Delta}\Big).$$

One may therefore use $\Gamma_e^{(1/2)}$ to establish the dependence of h_e on e in (A3). For $\Gamma_e^{(1/2)}$ Wick's theorem is valid[15]:

$$\frac{\Gamma_e^{(1/2)}(\{P_\mu\}\,|\,z,\{Q_i\})}{\theta_e(0)}=\det\Big[\frac{\Gamma_e^{(1/2)}(P_\mu|z)}{\theta_e(0)},\frac{\Gamma_e^{(1/2)}(P_\mu|Q_i)}{\theta_e(0)}\Big],$$

$$\frac{\Gamma_e^{(1/2)}(\{P_i\}\,|\,\{Q_i\})}{\theta_e(0)}=\det{}_{(ij)}\Big[\frac{\Gamma_e^{(1/2)}(P_i|Q_j)}{\theta_e(0)}\Big], \tag{A5}$$

with the two-point correlator equal to

$$\frac{\Gamma_e^{(1/2)}(a|b)}{\theta_e(0)}\equiv\langle\psi(a)\bar{\psi}(b)\rangle_e=\frac{\theta_e(\mathbf{a}-\mathbf{b})}{\theta_e(0)E(a,b)}. \tag{A6}$$

The limit $g=|\nu_*|^4$ is singular from the point of view of the relations (A5), and passage to this limit requires the use of the regularization (3) (see examples in Ref. 8; we also note that the singularities of $\Gamma_e^{(3/2)}$ in this limit differ from those of $\Gamma_e^{(1/2)}$; only the dependence on e is the same).

In order to evaluate the sum over the spinor structures for the correlators of supercurrents in most general form we need to find

$$\sum_e\langle e_*,e\rangle\theta_e(\mathbf{a}_1)\ldots\theta_e(\mathbf{a}_4)\theta_e(0)\,h_e(\{P_i\}\,|\,\{z_i\})\Big/\theta_e\Big(\sum_{i=1}^{2p-2}\mathbf{P}_i-2\mathbf{\Delta}\Big), \tag{A7}$$

but for matter supercurrents the integration over z_i is so designed that $h_e(\{P_i\}|\{z_i\})$ can be effectively replaced by unity and a sum of the form (2) results. We now present two simple examples, that illustrate how one deals with the sum (A7) in our regularization procedure.

The simplest nontrivial expression, similar in structure to (A2) (literally it corresponds to genus $p=3/2$), has the form

$$\frac{\Gamma_e^{(1/2)}(P_0,P_1|z,Q_1)}{\Gamma_e^{(1/2)}(P_0|Q_1)\Gamma_e^{(1/2)}(P_1,Q_1)}=-\frac{\Gamma_e^{(1/2)}(P_1|z)}{\Gamma_e^{(1/2)}(P_1|Q_1)}+\frac{\Gamma_e^{(1/2)}(P_0|z)}{\Gamma_e^{(1/2)}(P_0|Q_1)}. \tag{A8}$$

The role of the term of interest from (A3) is played by the first term on the right-hand side of (A8). It is seen that in this case h_e does not contain a θ-function in the denominator and the sum (A7) is easily evaluated.

A more complicated and realistic example corresponds to $p=2$:

$$\frac{\Gamma_e^{(1/2)}(P_0,P_1,P_2|z_1,Q_1,Q_2)\Gamma_e^{(1/2)}(P_0,P_1,P_2|z_2,Q_1,Q_2)}{\Gamma_e^{(1/2)}(P_0,P_1|Q_1,Q_2)\Gamma_e^{(1/2)}(P_0,P_2|Q_1,Q_2)\Gamma_e^{(1/2)}(P_1,P_2|Q_1,Q_2)}$$
$$=\frac{1}{\Gamma_e^{(1/2)}(P_1,P_2|Q_1,Q_2)}\Big\{-\Gamma_e^{(1/2)}(P_1|z_1)\Gamma_e^{(1/2)}(P_2|z_2)$$
$$-\Gamma_e^{(1/2)}(P_2|z_1)\Gamma_e^{(1/2)}(P_1|z_2)$$
$$+\Gamma_e^{(1/2)}(P_1|z_1)\Gamma_e^{(1/2)}(P_1|z_2)\frac{\Gamma_e^{(1/2)}(P_2|Q_2)}{\Gamma_e^{(1/2)}(P_1|Q_2)}$$
$$+\Gamma_e^{(1/2)}(P_2|z_1)\Gamma_e^{(1/2)}(P_2|z_2)\frac{\Gamma_e^{(1/2)}(P_1|Q_1)}{\Gamma_e^{(1/2)}(P_2|Q_1)}\Big\}+\ldots. \tag{A9}$$

In its e-dependence the combination in braces coincides with $h_e(P_1,P_2|z_1,z_2)$ in (A3) (for a metric $g=|W|^2$), but this time it contains Γ_e in the denominator. It is important that (A9) can be rewritten in such a way that the last two terms in h_e become

$$\Gamma_e^{(1/2)}(P_1|z_1)\Gamma_e^{(1/2)}(P_1|z_2)\frac{\Gamma_e^{(1/2)}(P_2|Q_1)}{\Gamma_e^{(1/2)}(P_1|Q_1)}$$

$$+\Gamma_e^{(1/2)}(P_2|z_1)\Gamma_e^{(1/2)}(P_2|z_2)\frac{\Gamma_e^{(1/2)}(P_1|Q_2)}{\Gamma_e^{(1/2)}(P_2|Q_2)} \qquad \text{(A10)}$$

(it is understood that the omitted terms, indicated in (A9) by dots, are also modified).

In the process of evaluating the correlator of ghost supercurrents we must set $z_2=z_1=z$ and integrate over z along a contour encircling P_2. After that we must differentiate with respect to P_1. Then we must add the analogous expression with P_1 and P_2 exchanged. After these operations are performed the terms omitted in (A3) and (A9) make no contribution. At the end of the calculations following our prescription one should let $P_i\to Q_i$, followed by $g=|W|^2\to g=|\nu_*|^4$.

The first of the "dangerous" terms in (A10) has no singularities for $z\to P_2$ and vanishes upon integration over z along a contour encircling P_2. The second term in (A10) gives a nonzero contribution, but in this case the $\Gamma_e^{(1/2)}(P_2|Q_2)$ that appears in the denominator need not be differentiated with respect to P_1. The limit $P_2\to Q_2$ is nonsingular as long as $g\neq|\nu_*|^4$ and all the points Q_i are different. Therefore the Γ_e in the denominator may be considered equal to $\theta_e(\mathbf{0})$ and it cancels against the $\theta_e(\mathbf{0})$ in the numerator in (A7). (When the integral over z is taken around P_1, while the differentiation is with respect to P_2, the nonzero contribution is connected with the first term in (A10) and the same argument applies.) Apparently the same analysis is possible for higher genera as well, assuming reasonable restriction of the arbitrariness in the regularization (3).

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