"Naive" Ward identities for the topological gluon current

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The question of which symmetry can be associated with the topological current K_μ of the gluons is discussed. It is shown that in the case of a supersymmetric theory the matrix elements of $\partial_\mu K_\mu$ can be found by varying the effective action with respect to the bare coupling constant. The relation obtained this way is an analog of the "naive" conservation equation for the axial vector current of the fermions: the contribution of the triangle diagram does not satisfy the relation and represents an anomaly. For the case of pure gluodynamics it is shown that one can introduce conserved chiralities, but only at the one-loop level and for special choices of the background field. The vanishing of the corresponding matrix elements of $\partial_\mu K_\mu$ can naturally be considered as "naive" Ward identities for K_μ in this case. It follows from them, in particular, that the contribution of the triangle diagram vanishes.

1. Recently a similarity was discovered between the matrix elements corresponding to triangle diagrams for the chiral current of Dirac and vector particles (Refs. 1–3). In the present paper we attempt to find symmetry arguments which would explain this similarity. However, before everything, one should explain what is meant by a chiral current of vector particles. The chirality of gluons is naturally related to the known topological current K_μ:

$$K_\mu=-g^{-2}\varepsilon_{\mu\nu\alpha\beta}(A_\nu^aD_\alpha A_\beta^a+{}^1/_3f^{abc}A_\nu^aA_\alpha^bA_\beta^c), \qquad (1)$$

where A_μ^a is a nonabelian vector field and g is the coupling constant. Indeed, for a free gluon the value of the corresponding charge W

$$W\equiv\int d^3xK_0(x) \qquad (2)$$

coincides with the helicity of the state. For this and some other reasons which we do not discuss here, it is natural to call W a chirality.

As was already mentioned, the analogy between the properties of the fermionic and gluonic chiralities extends also to the case of triangle diagrams for different forms of the background field. In particular, for the case of a gluon background field itself, the contributions of the appropriate triangle diagrams (see Fig. 1, a, b) differ only by a factor of 2. As a result of this[1)] we obtain

$$(\partial_\mu K_\mu)_\Delta=2(\partial_\mu a_\mu)_\Delta=\frac{T(G)}{8\pi^2}\tilde{G}_{\mu\nu}^aG_{\mu\nu}^a, \qquad (3)$$

where $a_\mu=\lambda^a\sigma_\mu\bar\lambda^a$ is the axial vector current of the two-component gluino field λ^a, $T(G)$ equals half the Dynkin index for the adjoint representation, $\mathrm{Tr}(T^aT^b)=T\delta^{ab}$, and $G_{\mu\nu}^a$ is the field strength of the gluon field, with $\tilde G_{\mu\nu}^a=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}G_{\alpha\beta}^a$.

In the case of a fermionic current a nonvanishing result (3) for the divergence of the matrix element related to a triangle diagram represents the famous chiral anomaly. By analogy it is natural to speak about a triangle anomaly for the current K_μ. The problem, however, is that there are no obvious symmetry considerations which would require a "naive" vanishing of the triangle diagram of Fig. 1, b, so that one could indeed speak of an "anomaly." Thus, apparently we can assert that thre is an "anomaly," but do not know what constitute "normal" Ward identities for the current K_μ.

One can indicate an example of a bosonic theory in which there exists a definite symmetry which predicts the vanishing of the matrix elements of the divergence of an appropriate current K_μ (Refs. 2, 3). We have in mind the theory of an electromagnetic field interacting with a background gravitational field. In this case the chirality of the photons is conserved[4] and as a consequence of this conservation the matrix elements for the transition of $\partial_\mu K_\mu$ into any number of gravitons must vanish (Refs. 2, 3). A nonvanishing result for a triangle diagram signals an anomaly (which is related, as usual, to a violation of chirality conservation in the interactions of the regulator fields).

The case of a gluon field is significantly more complicated. The conservation of chirality is absent already at the classical level.[4] We assert that nevertheless, for supersymmetric theories, the matrix elements of $\partial_\mu K_\mu$ can be determined systematically and that the result of an explicit calculation of the triangle diagram (3) can be compared with the predictions of the general Ward identities for the current K_μ. The main thrust of the derivation of the Ward identities is the observation due to Witten[5] that in supersymmetric theories the operator W [Eq. (2)] generates a shift of the bare coupling constant g_0. Starting from this we show that the matrix elements of $\partial_\mu K_\mu$ can be reduced to a variation of the effective action with respect to g_0.

2. It is convenient to approach the general formulation by considering special examples. In the theory we discuss the initial action has the form

$$S=\int d^4x\left\{\frac{1}{16g_0^2}(G+i\tilde G)^2+\frac{i}{2}\lambda^a(\sigma_\mu D_\mu\bar\lambda)\right\}+\text{h.c.}, \qquad (4)$$

where λ^a is the two-component gluino field, $\sigma_\mu=(1,\sigma)$, and g_0 is the bare coupling constant. We note that the action is written in terms of chiral fields, since the fields $G+i\tilde G$, λ^a, $\widehat{D}\bar\lambda$ have chiralities of the same sign.

Since we make use of a chiral gluon field, a shift of g_0 generates a variation of the classical action (4) proportional to $G\tilde G$. More precisely, if

$$\delta g_0^{-2}=i\eta,$$

where η is a real number, then

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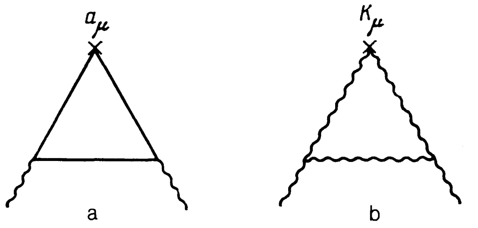

FIG. 1. Triangle diagrams describing the transitions of the chiral currents of a Dirac field a_μ (a) and of gluons K_μ (b) into two gluons. The gluons are denoted by a wavy line and the fermions by straight line. The currents a_μ and K_μ are denoted by $\times$.

$$\delta S=-\frac{\eta}{8}\int d^4x G\tilde{G}=\frac{\eta g_0^2}{4}\int d^4x\partial_\mu K_\mu. \tag{5}$$

The question naturally arises whether one is justified in retaining an integral of an exact divergence in the action. In order to fix such terms we assume, in effect, that g_0 represents a slowly varying function of x.

The relation (5) represents the simplest example of a relation between a variation of the action and a matrix element of $\partial_\mu K_\mu$. Here we are dealing with a classical divergence $\partial_\mu K_\mu$. We stress once again that no matter how trivial the derivation given here may seem, the use of chiral fields is essential already at this stage, since it allows us to relate the coefficients of the terms G^2 and $G\tilde{G}$. These coefficients are determined respectively by the real and imaginary parts of g_0.

The relation between the variations with respect to g_0 and $\partial_\mu K_\mu$ must in fact be general. Indeed, the fact that the operator W generates a shift of g_0^{-2} was first proved by Witten (for supersymmetric theories) based only on the canonical commutation relations (Ref. 5). Therefore the result should not be restricted to any order of perturbation theory. The zeroth order in g_0 considered above allows one to fix the normalization coefficients in this general relation.

Going on to higher orders, we first note that the chiral form of the Lagrangian is conserved even when loop corrections are taken into account. This assertion is particularly obvious if one makes use of superfield notations:

$$S\sim\frac{1}{g_0^2}\int d^4x\, d^2\theta W_\alpha^2+\text{h.c.}, \tag{6}$$

where W_α is a chiral superfield describing the gluino and the field strength of the gluon field. Our preference for the component notation (4) is related to the fact that we make use of different normalizations for the gluon field and the gluino field. Only with the normalization chosen here does the variation of the classical action yield exactly $\partial_\mu K_\mu$. Otherwise we would have obtained a combination of $\partial_\mu K_\mu$ and $\partial_\mu a_\mu$. This difference in normalizations does not interfere, of course, with making use of the general consequences of supersymmetry.

In the one-loop approximation we obtain, in particular

$$S_1=\int d^4x\left[\frac{1}{16g_0^2}(G_{\mu\nu}^a+i\tilde{G}_{\mu\nu}^a)^2+\frac{i}{2}\lambda^a\sigma_\mu D_\mu\bar{\lambda}^a\right]\left(1-\frac{bg_0^2}{8\pi^2}\ln M_0\right)+\text{h.c.}, \tag{7}$$

where b is the one-loop coefficient in the β-function, $b=3T(G)$, M_0 is the ultraviolet cutoff. Shifting g_0^{-2} we obtain a relation for $\partial_\mu K_\mu$ which must be valid in the one-loop approximation

$$(\partial_\mu K_\mu)_1=-(g_0^2 b/4\pi^2)\partial_\mu a_\mu \ln M_0. \tag{8}$$

The relation (8) should, obviously, correspond to an explicit calculation of the diagram of Fig. 2, which describes mixing of the operators K_μ and a_μ. The change from the current matrix element to its divergence is unimportant in this case, since the matrix element of the current is described by a unique form factor. The relation (8) indeed reproduces the result of a direct calculation.[6]

The one-loop relation (8) means that the diagram b in Fig. 1 with external gluon lines must vanish. This assertion agrees with the absence of the factor $\ln M_0$ in the explicit calculation (3) (see below for a further discussion of this point).

The nontrivial character of the numerical coincidence for the matrix element describing the transition of the current K_μ into a_μ is a strong argument in favor of the validity of the whole procedure. We stress once again that this derivation used in effect the assumption that g_0 depends on x, but in a weak manner, so that in the momentum representation one may retain only the first term in the expansion with respect to the momentum q_μ of the current. Indeed, one usually assumes that g_0 in Eq. (4) is real. As long as g_0 is constant this assumption is inessential, since by going over to complex g_0 we would obtain the additional so-called θ-term, proportional to $\theta G\tilde{G}$. As long as g_0 is a constant this additional term can be neglected, since the contribution of a total derivative to perturbation theory vanishes. In the derivation of the relations for $\partial_\mu K_\mu$ it is as if we consider the contribution of this θ-term to first order in θ which depends weakly on the coordinates. The contribution of the θ-term can be neglected, as before, in the derivation of the expression for the effective action. Moreover, outside perturbation theory the proposed recipe for the calculation of $\partial_\mu K_\mu$ does not have to be correct, since outside the framework of perturbation theory there can arise a real dependence of the effective action on θ on account of the contribution of the topologically nontrivial classical solutions of the Yang-Mills equations.

3. The procedure described above obviously allows one to obtain further predictions for the matrix elements of $\partial_\mu K_\mu$. First, considering multiloop generalizations of the relation (7) one can find a closed expression for the transition matrix elements of the current K_μ into two gluons or two fermions in all orders of perturbation theory. Indeed, the variation of the multiloop effective action with respect to g_0^{-2} reduces to the following expression:

$$\partial_\mu K_\mu=-\frac{1}{2}\frac{\alpha_0^2}{\alpha^2}\frac{\beta(\alpha)}{\beta(\alpha_0)}G\tilde{G}+\frac{2\alpha_0^2}{\alpha^2}\left[1-\frac{\beta(\alpha)}{\beta(\alpha_0)}\right]\partial_\mu a_\mu, \tag{9}$$

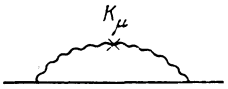

FIG. 2. A triangle diagram describing the mixing of the currents K_μ and a_μ at the one-loop level.

where α_0 is the bare coupling constant, $\alpha_0 \equiv g_0^2/4\pi$, α is the "running" coupling constant. Moreover, the β function itself is known in perturbation theory to all orders[7]:

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi}\left[1-\alpha\frac{T(G)}{2\pi}\right]^{-1}, \tag{10}$$

so that the relation (9) has indeed a numerical character. In particular, in the two-loop approximation we obtain the following prediction:

$$(\partial_\mu K_\mu)_2 = -\frac{3T(G)\alpha_0^2 \ln M_0}{4\pi^2} G\tilde{G}$$
$$-\frac{3T(G)\alpha_0^2}{4\pi^2}\left[2-\frac{T(G)}{2\pi}\right]\partial_\mu a_\mu \ln M_0. \tag{11}$$

The first term describes the renormalization of the θ-term in supersymmetric theories, i.e., a mixing of the operators K_μ and a_μ at the two-loop level. The relation (11) demonstrates most intuitively that the "naive" vanishing of the contribution of triangle diagrams with two external lines at the one-loop level is an exception rather than the rule within the framework of the theory under consideration.

Diagrams with additional external gluon or fermion lines no longer depend on ln M_0. The matrix elements of $\partial_\mu K_\mu$ can be determined if one has previously calculated the effective action. It is, of course, necessary that the momentum q_μ of the current should be much smaller than the other characteristic momenta.

4. We stress once again the fact that the variation of the one-loop effective action does not contain the result (3) for the divergence of the triangle diagram with two external gluon lines. In this sense one is entitled to say that Eq. (3) expresses the anomaly in agreement with the fact, already mentioned above, that the contributions of the triangle diagrams in Fig. 1, a, b are algebraically indistinguishable.[1]

Moreover, since the operations of calculating $\partial_\mu K_\mu$ and the variation with respect of g_0^{-2} are so immediately related to each other, it is apparent that one can talk about the anomaly in the commutation relations for composite operators which were used by Witten[5] in his derivation of the assertion that the operator W shifts the coupling constant.

In our opinion it is interesting that it is possible to consider the quantity $\partial_\mu K_\mu$ as a result of a variation with respect to g_0^{-2} including the effect of the anomaly. In order to realize this possibility it is necessary to assume that in the expression (7) ln M_0 has to be replaced by:

$$\ln M_0 \to \ln(M_0 g_0^{-2/3}). \tag{12}$$

In this case the result (3) for the triangle anomaly is reproduced as before by a shift (5) of the coupling constant. Although we have no rigorous justification for the substitution (12) within the framework of the concrete regularization procedure, we remark on the encouraging fact that this substitution allows one to go from the one-loop β-function to the exact one to all loops.[7,1] Thus, the anomaly can be reproduced by adding to the effective action a term which is singular in g_0. Taking this term into account, the operator W generates a shift in g_0^{-2}. Thus, the operator W is defined at the quantum level taking the regulators into acount.

5. It seems somewhat puzzling that the proof of the "naive" vanishing of the contribution of the diagram b in Fig. 1 succeeded only in supersymmetric theories, in spite of the fact that this diagram does not contain any fermions. We therefore turn directly to the case of gluodynamics and try to determine the chiral symmetry which would require the vanishing of the contribution of the triangle diagram. We show that for a definite choice of the background field such a symmetry does indeed exist. We note that our discussion here is close to the treatment of photons in an external gravitational field in Ref. 3.

For calculations by means of the background field method, the vector potential A_μ^a is replaced by the sum of a background field and a quantized field: $A_\mu^a + a_\mu^a$. The part of the action which is quadratic in a_μ^a has the form

$$S^{(2)} = (1/2g_0^2)\left[a_\mu^c D^2 a_\mu^c + (D_\mu a_\mu^a)^2 + 2f^{abc}G_{\mu\nu}^a a_\mu^b a_\nu^c\right]. \tag{13}$$

In the sequel we make use of the light-cone gauge, since it is the most convenient for the introduction of the chirality of the gluon field. In spinor notations, $a_{\alpha\dot\alpha} \equiv (\sigma_\mu)_{\alpha\dot\alpha} a_\mu$, the appropriate gauge condition has the following form

$$a_{2\dot2}^a = 0. \tag{14}$$

Further, the equation of motion for the component $a_{1\dot1}^a$ does not contain "time" derivatives $D_{1\dot1}$ so that $a_{1\dot1}^a$ is not a dynamical variable and can be expressed in terms of the complex field a, where

$$a \equiv 2^{-1/2} a_{2\dot1}, \tag{15}$$

and its complex conjugate field $\bar a$. Explicitly

$$a_{1\dot1}^a = \frac{1}{D_{2\dot2}}(\bar D a^a + D\bar a^a)^2 + \frac{2^{1/2}}{(D_{2\dot2})^2} f^{abc}(G_{22}^b \bar a^c + \bar G_{\dot2\dot2}^b a^c), \tag{16}$$

where D and $G_{\alpha\beta}$ are defined as follows:

$$D \equiv 2^{-1/2} D_{1\dot2}, \quad \bar D = D^*, \quad G_{\alpha\dot\alpha,\beta\dot\beta} \equiv \varepsilon_{\alpha\beta}\bar G_{\dot\alpha\dot\beta} + \varepsilon_{\dot\alpha\dot\beta}G_{\alpha\beta}. \tag{17}$$

The action $S^{(2)}$ becomes, after the substitution (16),

$$S^{(2)} = \frac{2}{g_0^2}\int d^4x \left\{\bar a D_\mu^2 a + \frac{1}{D_{2\dot2}} f^{abc}(G_{22}^b \bar a^c + \bar G_{\dot2\dot2}^b a^c)^2 \right.$$
$$\left. + \left[f^{abc}G_{12}^a \bar a^b a^c + f^{abc}G_{22}^a a_{1\dot1}^b \bar a^c + \text{h.c.}\right]\right\}. \tag{18}$$

The first term is obviously invariant with respect to chiral rotations

$$a \to a e^{i\alpha}, \quad \bar a \to \bar a e^{-i\alpha}. \tag{19}$$

This invariance can be maintained also in the subsequent terms if one imposes the following condition on the background field:

$$G_{22} = 0, \quad \bar G_{\dot2\dot2} = 0. \tag{20}$$

Thus one can achieve chirality conservation in the quadratic action (which is adequate for the computation of all one-loop-diagrams). Moreover, it is easy to verify that the quantum part of the current $\partial_\mu K_\mu$ has nonvanishing chirality. Consequently, one can assert that all matrix elements of $\partial_\mu K_\mu$ must vanish in the approximation under consideration:

$$(\partial_\mu K_\mu)_1 = 0 \quad (G_{22} = \bar{G}_{\dot{2}\dot{2}} = 0). \tag{21}$$

It is easy to understand that the Ward identities (21) require, in particular, the vanishing of the contribution of the diagram of Fig. 1, b. Indeed, this diagram is described by a single structure which is proportional to $G\widetilde{G}$. On the other hand $G\widetilde{G} \propto G^2_{\alpha\beta} - G^2_{\dot{\alpha}\dot{\beta}}$ and the requirement (20) by no means leads to a kinematic annihilation of $G\widetilde{G}$. Therefore the relation (21) signifies that the corresponding form-factor must vanish. It is exactly this consequence of the Ward identities (21) that is violated by the triangle anomaly (3).

As far as more complicated diagrams, with a larger number of gluon lines, are concerned, it is possible that there are structures which vanish for a special choice of the background field (20) but do not vanish in the general case. Then the Ward identities (21) do not require the vanishing of the appropriate matrix elements. Moreover, we have succeeded in introducing the notion of conserved chirality only at the one-loop level, which agrees qualitatively with the assertion made above that the triangle diagrams are also different from zero in higher order approximations [cf. Eq. (11)].

6. We see, thus, that there is no universal answer to the question whether there is a hidden symmetry behind the "naive" vanishing of the contribution of triangle diagrams for the topological current K_μ. The answer varies from theory to theory. The simplest case is an abelian field in an external gravitational field.[2,3] The vanishing of all quantum corrections to $\partial_\mu K_\mu$ is related to the conservation[4] of the chirality of the photons. For a nonabelian field the concept of conserved chirality can only be introduced to a limited degree, in the one-loop approximation and for a definite choice of the background field. Nevertheless, this chiral invariance suffices for the derivation of definite relations in the general case too. In particular, we predict a "naive" vanishing of the contribution of the triangle diagram. Therefore the result (3) signifies that, in particular, this chiral symmetry is inevitably violated in the interaction of the regulators.

If one adds fermions in such a way that the theory becomes supersymmetric, the predictive power of the theory increases. However, it turns out that the matrix elements of $\partial_\mu K_\mu$ are fixed not by some symmetry considerations, but by the circumstance that the operator $W = \int d^3x K_0$ generates shifts of the coupling constant g_0^{-2}. The triangle anomaly implies an additional dependence on g_0 of the effective action, singular with respect to g_0. It is remarkable that supersymmetry allows one to fix in the action the integrals over the divergence $\partial_\mu K_\mu$. In other words, there exists some kind of an analog of analyticity, which allows one, if one knows a function (the effective action) for real values of g_0, to predict its value for a small shift into the complex plane.

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[1)] See Sec. 4 of Ref. 1 based on results obtained jointly with V. A.Novikov.

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