An electrovacuum solution of the general-relativity equations that has the Schwarzschild limit

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An exact, asymptotically plane solution of the electrostatic equations of the general theory of relativity is obtained that goes over to the Schwarzschild solution in the case of zero electric field.

Among the exact solutions of the Einstein-Maxwell equations those of greatest interest are the solutions which go over to the Schwarzschild solution in the absence of an electric field. From among the few known metrics of this type that describe the external gravitational fields of charged sources, one should note the Reissner-Nordström metric[1,2] and the metrics found in the papers of Herlt[3] and Hoenselaers[4] and also in Ref. 5.

In the present paper, using the method developed in Refs. 5 and 6 for solving the static Einstein-Maxwell equations, we find one more electrostatic solution, which contains two arbitrary parameters and has as its vacuum limit the Schwarzschild solution.

As is well known (see, e.g., Ref. 5), the metric interval describing a static, axially symmetric gravitational field can be chosen, without loss of generality, in the form

$$ds^2=k^2f^{-1}\left[e^{2\gamma}(x^2-y^2)\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)\,d\varphi^2\right]-f\,dt^2, \tag{1}$$

where the unknown metric functions f and γ depend only on the two prolate ellipsoidal coordinates (x, y) (these coordinates are connected with the curvature coordinates r and ϑ by the relations $x=(r-M)/k$ and $y=\cos\vartheta$, where k and M are constants). The electrostatic equations can then be written as follows[5]:

$$(\varepsilon_1+\varepsilon_2)\Delta\varepsilon_1=2(\nabla\varepsilon_1)^2,\quad (\varepsilon_1+\varepsilon_2)\Delta\varepsilon_2=2(\nabla\varepsilon_2)^2. \tag{2}$$

Here the functions ε_1 and ε_2 are related to the metric coefficient f and the electric component A_4 of the four-potential of the electromagnetic field by the formulas

$$\varepsilon_1=f^{1/2}+A_4,\quad \varepsilon_2=f^{1/2}-A_4. \tag{3}$$

The operators Δ and ∇ are given by the expressions

$$\Delta\equiv k^{-2}(x^2-y^2)^{-1}\left\{\frac{\partial}{\partial x}\left[(x^2-1)\frac{\partial}{\partial x}\right]+\frac{\partial}{\partial y}\left[(1-y^2)\frac{\partial}{\partial y}\right]\right\},$$
$$\nabla\equiv k^{-1}(x^2-y^2)^{-1/2}\left(\mathbf{x}_0(x^2-1)^{1/2}\frac{\partial}{\partial x}+\mathbf{y}_0(1-y^2)^{1/2}\frac{\partial}{\partial y}\right), \tag{4}$$

where $\mathbf{x}_0$ and $\mathbf{y}_0$ are unit vectors.

The metric coefficient γ is found from the known ε_1 and ε_2 from the following system of first-order differential equations:

$$\gamma_{,x}=\frac{4(1-y^2)}{(x^2-y^2)(\varepsilon_1+\varepsilon_2)^2}[x(x^2-1)\varepsilon_{1,x}\varepsilon_{2,x}-x(1-y^2)\varepsilon_{1,y}\varepsilon_{2,y}-y(x^2-1)(\varepsilon_{1,x}\varepsilon_{2,y}+\varepsilon_{1,y}\varepsilon_{2,x})];$$
$$\gamma_{,y}=\frac{4(x^2-1)}{(x^2-y^2)(\varepsilon_1+\varepsilon_2)^2}[y(x^2-1)\varepsilon_{1,x}\varepsilon_{2,x}-y(1-y^2)\varepsilon_{1,y}\varepsilon_{2,y}+x(1-y^2)(\varepsilon_{1,x}\varepsilon_{2,y}+\varepsilon_{1,y}\varepsilon_{2,x})] \tag{5}$$

(the comma denotes the partial derivative with respect to the coordinate x or y).

In Ref. 5 it was shown that Eqs. (2) are satisfied by potentials ε_1 and ε_2 of the form

$$\varepsilon_1=e^{\psi}\left[1-\frac{2(1-A)(1-B)}{x(1-AB)+y(B-A)+(1-A)(1-B)}\right],$$
$$\varepsilon_2=e^{\psi}\left[1-\frac{2(1+A)(1+B)}{x(1-AB)+y(A-B)+(1+A)(1+B)}\right], \tag{6}$$

where ψ is an arbitrary solution of the equation

$$\Delta\psi=0, \tag{7}$$

and A and B obey the following system of differential equations:

$$A_{,x}=A(x-y)^{-1}[(xy-1)\psi_{,x}+(1-y^2)\psi_{,y}],$$
$$A_{,y}=A(x-y)^{-1}[-(x^2-1)\psi_{,x}+(xy-1)\psi_{,y}],$$

$$B_{,x}=-B(x+y)^{-1}[(xy+1)\psi_{,x}+(1-y^2)\psi_{,y}],$$
$$B_{,y}=-B(x+y)^{-1}[-(x^2-1)\psi_{,x}+(xy+1)\psi_{,y}].$$

We choose the solution of Eq. (7) in the form

$$\psi=\frac{1}{2}\ln\frac{x+1}{x-1}. \tag{9}$$

In this case, integrating (8) we find

$$A=\frac{\alpha(x^2-1)^{1/2}}{x-y},\quad B=\frac{\beta(x^2-1)^{1/2}}{x+y}, \tag{10}$$

where α and β are two real arbitrary constants.

Consequently, for ε_1 and ε_2 we obtain the expressions

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$$\varepsilon_1=\frac{[x^2-y^2-\alpha\beta(x+1)^2](x-1)^{1/2}+[\alpha(x+y)(1-y)+\beta(x-y)(1+y)](x+1)^{1/2}}{[x^2-y^2-\alpha\beta(x-1)^2](x+1)^{1/2}-[\alpha(x+y)(1+y)+\beta(x-y)(1-y)](x-1)^{1/2}},$$
$$\varepsilon_2=\frac{[x^2-y^2-\alpha\beta(x+1)^2](x-1)^{1/2}-[\alpha(x+y)(1-y)+\beta(x-y)(1+y)](x+1)^{1/2}}{[x^2-y^2-\alpha\beta(x-1)^2](x+1)^{1/2}+[\alpha(x+y)(1+y)+\beta(x-y)(1-y)](x-1)^{1/2}} \tag{11}$$

by means of which, using the formulas (3), we find f and A_4:

$$f=(x^2-1)a^2b^{-2},\quad A_4=2cb^{-1}, \tag{12}$$

where

$$\begin{aligned} a&\equiv[x^2-y^2-\alpha\beta(x^2-1)]^2+(1-y^2)[\alpha(x+y)\\ &\quad\times\beta(x-y)]^2,\\ b&\equiv(x+1)[x^2-y^2-\alpha\beta(x-1)^2]^2-(x-1)[\alpha(x+y)(1+y)\\ &\quad+\beta(x-y)(1-y)]^2,\\ c&\equiv(\alpha+\beta)(x^2-y^2)^2-\alpha\beta(x^2-1)[\alpha(x+y)^2+\beta(x-y)^2]. \end{aligned} \tag{13}$$

Taking (11) into account and integrating (5), we also find the metric coefficient γ:

$$e^{2\gamma}=\frac{x^2-1}{x^2-y^2}\frac{a^4}{(1-\alpha\beta)^8(x^2-y^2)^8}. \tag{14}$$

Thus, the formulas (12)–(14) completely determine the metric (1).

It follows from (12) and (13) that the total mass M and total charge Q of the source have the form

$$M=\frac{k(1-3\alpha\beta)}{1-\alpha\beta},\quad Q=\frac{2k(\alpha+\beta)}{1-\alpha\beta}. \tag{15}$$

It is easy to see that for $\alpha=\beta=0$ the solution obtained goes over into the Schwarzschild solution

$$f=\frac{x-1}{x+1},\quad e^{2\gamma}=\frac{x^2-1}{x^2-y^2}, \tag{16}$$

or, in the curvature coordinates,

$$f=1-\frac{2M}{r},\quad e^{2\gamma}=\frac{r^2-2Mr}{r^2-2Mr+M^2\sin^2\vartheta}$$

(here, we have taken into account that $k=M$ in the case of a static vacuum gravitational field).

Setting $\alpha=0$ in (12)–(14) we arrive at the solution found in Ref. 5:

$$\begin{aligned} f&=\frac{(x^2-1)[(x+y)^2+\beta^2(1-y^2)]^2}{[(x+1)(x+y)^2-\beta^2(x-1)(1-y)^2]^2},\\ A_4&=\frac{2\beta(x+y)^2}{(x+1)(x+y)^2-\beta^2(x-1)(1-y)^2},\\ e^{2\gamma}&=\frac{x^2-1}{x^2-y^2}\left[1+\frac{\beta^2(1-y^2)}{(x+y)^2}\right]^4. \end{aligned} \tag{17}$$

Finally, the case when the constants α and β are connected by the relation $\alpha=-\beta$ is of special interest. In this case, as can be seen from (15), the total charge Q vanishes and (12)–(14) go over into the following formulas:

$$\begin{aligned} f&=\frac{x-1}{x+1}\left\{\frac{[x^2-y^2+\alpha^2(x^2-1)]^2+4\alpha^2x^2(1-y^2)}{[x^2-y^2+\alpha^2(x-1)^2]^2-4\alpha^2y^2(x^2-1)}\right\}^2,\\ A_4&=\frac{8\alpha^3xy(x-1)}{[x^2-y^2+\alpha^2(x-1)^2]^2-4\alpha^2y^2(x^2-1)},\\ e^{2\gamma}&=\frac{x^2-1}{x^2-y^2}\frac{\{[x^2-y^2+\alpha^2(x^2-1)]^2+4\alpha^2x^2(1-y^2)\}^4}{(1+\alpha^2)^8(x^2-y^2)^8}. \end{aligned} \tag{18}$$

The expressions (18) describe the external gravitational field of a massive electric dipole with dipole moment p equal to

$$p=8k^2\alpha^3/(1+\alpha^2)^2. \tag{19}$$

On the other hand, in the case when $\alpha\neq-\beta$ the metric coefficients f and γ calculated from the formulas (12)–(14) determine the metric of the axially symmetric gravitational field of a star endowed, in accordance with (15), with charge Q.

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