Contribution to the theory of weakly turbulent Kolmogorov spectra of a homogeneous magnetized plasma

A. B. Mikhaĭlovskiĭ, E. A. Novakovskaya, V. P. Lakhin,
S. V. Novakovskiĭ, O. G. Onishchenko, and A. I. Smolyakov

Institute of Space Research, USSR Academy of Sciences
(Submitted 14 April 1988)
Zh. Eksp. Teor. Fiz. **95**, 1598–1613 (May 1989)

The possible existence of weakly turbulent Kolmogorov spectra on oscillation modes of a homogeneous magnetized plasma is investigated theoretically. It is shown that these spectra can be related to Alfven, magnetosonic, or magnetized electron Langmuir waves, to whistlers, and to a number of other waves. In addition, the question of weak-turbulence Kolmogorov spectra of ion-sound waves is analyzed in a more general formulation than in preceding studies (the vector nonlinearity is taken into account and the concept of plane turbulence is formulated). The role of backward waves, allowance for which is essential in problems involving turbulence of strongly dispersed waves, is investigated. It is shown that in such problems there can be realized only one Kolmogorov spectrum, rather than the two in the case of weakly dispersed waves.

1. INTRODUCTION

A homogeneous magnetized plasma contains a significant number of oscillation modes, for which the dispersion equations can be written in the form

$$\omega_{\mathbf{k}} \propto k_z^a k_\perp^b, \tag{1.1}$$

where $\omega_{\mathbf{k}}$ is the frequency of the waves in a suitably chosen coordinate frame, k_z and $k_\perp = (k_x^2 + k_y^2)^{1/2}$ are the longitudinal and transverse components of the wave vector $\mathbf{k}$, z is the direction of the equilibrium magnetic field $\mathbf{B}_0$, and a and b are certain constants. According to the general premises advanced in Ref. 1, these modes can be related to weak-turbulence Kolmogorov spectra. The method of finding exact solutions for the kinetic equation of the wave (sometimes called the method of factorizing the collision term in power-law solutions), which is based on the concept of Kolmogorov spectra,[1] has been known for more than two decades (since Zakharov's paper[2]). The existence of numerous modes of type (1.1) is known even longer (see, e.g., the review in Ref. 3). Accordingly, everything necessary to apply this method to problems involving a homogeneous magnetized plasma and to develop a fairly complete theory of weakly turbulent Kolmogorov spectra of such a plasma have long been on hand. No such theory was developed to this day, however. The only significant contribution to the considered problem is a paper by Kuznetsov,[4] dealing only with one variant of waves of type (1.1), ion-sound waves. Our present aim is to assess the possibility of obtaining weakly turbulent Kolmogorov spectra for a larger aggregate of oscillation modes of a homogeneous magnetized plasma. This includes the fundamental modes whose dispersion equations can be written in the form (1.1), and also a class of waves with $k_x \gg k_y$, characterized by a dispersion equation in the form

$$\omega_{\mathbf{k}} \propto k_z^a k_x^{b_1}, \tag{1.2}$$

where b_1 is a certain constant. Waves of type (1.2) can be analyzed by a formalism described in Ref. 5 for Kolmogorov spectra of drift waves (this formalism also stems from Zakharov's method[1]). Development of such a theory is timely, in particular, in view of the need for interpreting the plotted wave-number dependences of the spectral noise-energy density, obtained in present-day computer experiments, as well as in experiments with a real plasma under laboratory and outer-space conditions. Examples of such plots are the spectral characteristics of the turbulence in tokamak experiments,[6] and also computer experiments[7] dealing with drift-wave turbulence.

Section 2 is devoted to an analysis of ion-sound turbulence. This analysis differs from that of Ref. 4 in the following respects. First, we show that in addition to the solutions of the kinetic equation for waves that depend on k_z and $k_\perp$ and correspond to the case of axisymmetric turbulence there are also solutions that depend on k_z, k_x, and k_y and correspond to the case of plane turbulence. The latter solutions can be of greater interest than the axisymmetric ones, for example for a plasma in a magnetic wave with shear. In this case the coordinates x and y are not on a par and, according to Ref. 8, the least sensitive to shear are waves with $k_x \gg k_y$ (x is the direction of the magnetic-field gradient). Second, it was assumed in Ref. 4 that the interaction between the ion-sound waves, which establishes stationary spectra, is due only to the so-called scalar nonlinearity, whereas we, following Ref. 9, take both scalar and vector nonlinearity into account. As shown in Sec. 2, the vector nonlinearity is more substantial than the scalar if $k_z/k_\perp$ is small enough and $k_\perp\rho_0$ is not too small (ρ_0 is the Larmor radius of the ion at the electron temperature), i.e., in the region of greatest interest, when dealing with the buildup of ion-sound waves by drift effects (see, e.g., Refs. 8 and 10).

Dispersion equations of form (1.1) are obtained both for weakly dispersed and strongly dispersed waves. A general prescription for calculating the Kolmogorov-spectra exponents for weakly dispersed waves of type (1.1) was formulated by Zakharov.[1] Its gist is that if the wave frequency is characterized by homogeneity exponents a and b [see (1.1)] and the matrix element of the interaction by homogeneity exponents u and v (relative to the variables k_{zi} and $k_{\perp i}$, where i is the index of the interacting waves), then the "number of photons" N_k in the Kolmogorov spectra

$$N_{\mathbf{k}} \propto |k_z|^\alpha k_\perp^\beta \tag{1.3}$$

is characterized by exponents α and β equal to

 0038-5646/89/050924-09$04.00

$$\alpha^{(1)}=-(1+u),\quad \beta^{(1)}=-(2+v) \tag{1.4}$$

or

$$\alpha^{(2)}=a/2-(^3/_2+u),\quad \beta^{(2)}=b/2-(2+v). \tag{1.5}$$

The use of this prescription is the basis of our following analysis of stationary axisymmetric turbulence of weakly dispersed waves.

A specific feature of the problem of turbulence of strongly dispersed waves of type (1.1) is the need for taking into account the interaction between the forward and backward waves, i.e., waves with different signs of the longitudinal phase velocity $\omega_{\mathbf{k}}/k_z$. In other words, we should consider in this case simultaneously (i.e., within the framework of one and the same problem) two oscillation modes:

$$\omega_{\mathbf{k}}^{(1)}\propto|k_z|^a k_\perp^b \operatorname{sign} k_z, \tag{1.6}$$

$$\omega_{\mathbf{k}}^{(2)}\propto-|k_z|^a k_\perp^b \operatorname{sign} k_z. \tag{1.7}$$

The calculation of the exponents of the Kolmogorov spectra in this situation is considered in the Appendix below. It follows from the Appendix that under conditions (1.6) and (1.7) there can be realized only one Kolmogorov axisymmetric-turbulence spectrum characterized by the exponents (1.4).

According to Ref. 5, two variants of plane turbulence, two-and three-dimensional, can be associated with waves of type (1.2). In the case of weakly dispersive waves, each of these turbulence variants corresponds to two Kolmogorov spectra.

General prescriptions for calculating spectral exponents, of both two- and three-dimensional turbulence, were formulated in Ref. 5. In the case of two-dimensional turbulence, when

$$N_{\mathbf{k}}\propto|k_z|^{\alpha}|k_x|^{\beta_1}, \tag{1.8}$$

the spectral exponents are $\alpha^{(1)}$, $\beta_1^{(1)}$ or $\alpha^{(2)}$, $\beta_1^{(2)}$, where $\alpha^{(1)}$, $\alpha^{(2)}$ are given by Eqs. (1.4) and (1.5), while $\beta_1^{(1)}$, $\beta_1^{(2)}$ denote

$$\beta_1^{(1)}=-(1+v_1),\quad \beta_1^{(2)}=b/2-(1+v_1), \tag{1.9}$$

where v_1 is the homogeneity index of the matrix element with respect to k_x. In the case of three-dimensional turbulence, when

$$N_k\propto|k_z|^{\alpha}|k_x|^{\beta_1}|k_y|^{\beta_2}, \tag{1.10}$$

we have two aggregates of spectral exponents, $\alpha^{(1)}$, $\beta_1^{(1)}$, $\beta_2^{(1)}$ or $\alpha^{(2)}$, $\beta_1^{(2)}$, $\beta_2^{(2)}$, where $\alpha^{(1)}$ and $\alpha^{(2)}$ are given by Eqs. (1.4) and (1.5), $\beta_1^{(1)}$, $\beta_1^{(2)}$ by (1.9), and $\beta_2^{(1)}$, $\beta_2^{(2)}$ take the form

$$\beta_2^{(1)}=\beta_2^{(2)}=-(1+v_2), \tag{1.11}$$

where v_2 is the homogeneity index of the matrix element with respect to k_x.

The calculation of the exponents of plane-turbulence Kolmogorov-spectrum exponents of strongly dispersed waves is dealt with in the Appendix. It is shown there that in such waves only one Kolmogorov spectrum can be realized, just as in the axisymmetric turbulence considered above. Two-dimensional turbulence is characterized by spectral exponents $\alpha^{(1)}$, $\beta_1^{(1)}$, and three-dimensional by $\alpha^{(1)}$, $\beta_1^{(1)}$, $\beta_2^{(1)}$ (see the equations above).

Let us dwell also on the case of plane turbulence with $v_2=0$, i.e., when both the oscillation frequency and the matrix element depend only on two wave-number components (on k_z and k_x in the situation considered). Such a turbulence is physically two-dimensional, but it can be treated as three-dimensional by recognizing that the two-dimensional Kolmogorov spectra $W_{kz,kx}$, are physically equivalent to three-dimensional spectra $W_{kz,kx,ky}$ of the form

$$W_{kz,kx,ky}\propto k_y^{-1}W_{kx,kz}. \tag{1.12}$$

It is borne in mind here that

$$W_{kz,kx}\propto\int W_{k_z,k_x,k_y}\,dk_y. \tag{1.13}$$

We apply these ideas concerning weakly dispersed waves, in addition to the above case of ion-sound waves, also to problems involving Alfven and magnetosonic waves. The Alfven turbulence will be investigated in Sec. 3, and the magnetosonic in Sec. 4. Among the strongly dispersed waves considered by us are magnetized electron Langmuir waves (Sec. 5) and whistlers, also called whistling atmospherics (Sec. 6).

It is noted in Sec. 7 that problems involving turbulences of certain other types of waves in a homogeneous magnetized plasma are reducible mathematically to those investigated in the preceding sections. These include electron-sound, short-wave ion-sound, and short-wave Alfven waves. We show by the same token that weak turbulence of waves of this type can also be characterized by Kolmogorov spectra.

2. ION-SOUND TURBULENCE

2.1. Initial equations

By analogy with Ref. 9, we choose as initial for ion-sound waves the continuity and ion longitudinal-motion equations

$$\frac{d_0\tilde{n}}{dt}+n_0\left(\frac{\partial V_z}{\partial z}+\operatorname{div}\mathbf{V}_I\right)+\frac{\partial}{\partial z}(\tilde{n}V_z)=0, \tag{2.1}$$

$$\frac{d_0V_z}{dt}+V_z\frac{\partial V_z}{\partial z}=-\frac{e}{m_i}\frac{\partial\varphi}{\partial z}, \tag{2.2}$$

and choose for the electrons the Boltzmann law written in the approximate form

$$\tilde{n}=\frac{en_0\varphi}{T_e}\left(1+\frac{e\varphi}{2T_e}\right). \tag{2.3}$$

Here n_0 and T_e are the equilibrium plasma density and electron temperature, $\tilde{n}$ and V_z are the perturbations of the plasma density and of the longitudinal ion velocity, φ is the electrostatic potential, and $\mathbf{V}_I$ is the transverse inertial velocity of the ions. The expression for div $\mathbf{V}_I$ is

$$\operatorname{div}\mathbf{V}_I=-\frac{c}{B_0\omega_{Bi}}\frac{d_0}{dt}\Delta_\perp\varphi. \tag{2.4}$$

The operator d_0/dt is defined as

$$\frac{d_0}{dt}=\frac{\partial}{\partial t}+\frac{c}{B_0}[\nabla\varphi,\nabla]_z. \tag{2.5}$$

The remaining notation is: $\omega_{Bi}=eB_0/m_ic$ is the ion cyclotron frequency, e and m_i the ion charge and mass, c the speed of light, and $\Delta_\perp=\partial^2/\partial x^2+\partial^2/\partial y^2$.

We introduce $\zeta = z - c_s t$, i.e., we change over to a reference frame moving along z at the ion-sound velocity $c_s \equiv (T_e/m_i)^{1/2}$. After a number of transformations we arrive at the following equation for φ:

$$2\frac{\partial\varphi}{\partial t}+\rho_0^2\left(c_s\frac{\partial}{\partial\zeta}-\frac{3}{2}\frac{c}{B_0}[\nabla\varphi,\nabla]_z\right)\Delta_\perp\varphi-\frac{e}{m_i c_s}\frac{\partial\varphi^2}{\partial\zeta}=0, \tag{2.6}$$

where $\rho_0 = c_s/\omega_{Bi}$. The terms with $[\nabla\varphi \times \nabla]_z \Delta_\perp \varphi$ and $\partial\varphi^2/\partial\zeta$ correspond to the vector and scalar nonlinearities, respectively.

We change over to a Fourier representation, introducing by the same token the amplitude of the **k**th Fourier harmonic of the potential $\varphi_{\mathbf{k}}(t)$. We obtain from (2.6) the following equation for $\varphi_{\mathbf{k}}$:

$$i\frac{\partial\varphi_{\mathbf{k}}}{\partial t}=-\frac{e}{2m_i c_s}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}\left\{k_z-\frac{3i}{4}\rho_0^3[\mathbf{k}_1\mathbf{k}_2]_z \times(k_{2\perp}^2-k_{1\perp}^2)\right\}\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\exp[-i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}})t], \tag{2.7}$$

where

$$\omega_{\mathbf{k}}=-c_s\rho_0^2 k_z k_\perp^2/2. \tag{2.8}$$

We introduce the normalized potential $C_{\mathbf{k}}$ defined by the relation $|C_{\mathbf{k}}|^2 \propto N_{\mathbf{k}}$. Recognizing that for ion sound we have $W_{\mathbf{k}} \propto |\varphi_{\mathbf{k}}|^2$ ($W_{\mathbf{k}}$ is the spectral density of the wave energy) and for weakly dispersed waves the number of quanta $N_{\mathbf{k}}$ is connected with $W_{\mathbf{k}}$ by the relation

$$N_{\mathbf{k}} \propto W_{\mathbf{k}}/|k_z|, \tag{2.9}$$

we obtain the connection between $C_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$:

$$C_{\mathbf{k}} \propto \varphi_{\mathbf{k}}/|k_z|^{1/2}. \tag{2.10}$$

Writing (2.7) in canonical form[11]

$$i\frac{\partial C_{\mathbf{k}}}{\partial t}=\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)C_{\mathbf{k}_1}C_{\mathbf{k}_2}\exp[-i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}})t], \tag{2.11}$$

we obtain an expression for the interaction matrix element:

$$V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\propto|k_z k_{1z}k_{2z}|^{1/2}\,\mathrm{sign}\,k_z\left\{1-\frac{i}{4}\rho_0^3[\mathbf{k}_1\mathbf{k}_2]_z \times\left(\frac{k_{2\perp}^2-k_{1\perp}^2}{k_z}+\frac{k_\perp^2-k_{1\perp}^2}{k_{2z}}+\frac{k_{2\perp}^2-k_\perp^2}{k_{1z}}\right)\right\}. \tag{2.12}$$

The symmetry properties of the matrix elements are verified by using the obvious equalities $[\mathbf{k}_1 \times \mathbf{k}_2]_z = [\mathbf{k} \times \mathbf{k}_2]_z = [\mathbf{k}_1 \times \mathbf{k}]_z$.

2.2. Spectra due to vector nonlinearity

If

$$k_z/k_\perp<\rho_0^3 k_\perp^3 \tag{2.13}$$

the scalar nonlinearity is insignificant, i.e., the first term in the curly brackets of (2.12) can be neglected. The matrix element becomes then scale-invariant with exponents

$$u={}^1/_2,\quad v=4, \tag{2.14}$$

$$v_1=3,\quad v_2=1. \tag{2.15}$$

We recognize also that according to (2.8) we have for the ion sound [cf. Eqs. (1.1) and (1.2)]

$$a=1,\quad b=b_1=2. \tag{2.16}$$

Using (2.16), (1.4), and (1.5) we obtain the axisymmetric-turbulence spectral exponents:

$$\alpha^{(1)}=\alpha^{(2)}=-{}^3/_2,\quad \beta^{(1)}=-6,\quad \beta^{(2)}=-5. \tag{2.17}$$

According to (2.9), these exponents correspond to the energy spectra

$$W_{\mathbf{k}} \propto (k_z^{-1/2}k_\perp^{-6},\ k_z^{-1/2}k_\perp^{-5}). \tag{2.18}$$

In the case of plane turbulence we obtain similarly

$$\beta_1^{(1)}=-4,\quad \beta_1^{(2)}=-3,\quad \beta_2^{(1)}=\beta_2^{(2)}=-2. \tag{2.19}$$

As a result we arrive at the following equations for the energy spectra of a plane ion-sound turbulence

$$W_{\mathbf{k}} \propto (k_z^{-1/2}k_x^{-4}k_y^{-2},\ k_z^{-1/2}k_x^{-3}k_y^{-2}). \tag{2.20}$$

At the validity limits of the approximation of small k_y/k_x, i.e., at $k_x \approx k_y \approx k_\perp$ Eqs. (2.20) change into (2.18).

2.3. Spectra due to scalar nonlinearity

We assume now

$$k_z/k_\perp>\rho_0^3 k_\perp^3. \tag{2.21}$$

In addition,

$$u={}^3/_2,\quad v=v_1=v_2=0. \tag{2.22}$$

With the aid of (2.22), (2.16), (1.4), and (1.5) we find that in the case of axisymmetric turbulence

$$\alpha^{(1)}=\alpha^{(2)}=-{}^5/_2,\quad \beta^{(1)}=-2,\quad \beta^{(2)}=-1. \tag{2.23}$$

These exponents correspond to energy spectra

$$W_{\mathbf{k}} \propto (k_z^{-3/2}k_\perp^{-2},\ k_z^{-3/2}k_\perp^{-1}). \tag{2.24}$$

The results (2.23) and (2.24) are due to Kuznetsov.[4]

Since $v_2 = 0$, we can speak, in accord with the statements in Sec. 1, of two variants of plane turbulence, three-dimensional and two-dimensional. The expressions for the three-dimensional planar-turbulence spectra take the form

$$W_{\mathbf{k}} \propto (k_z^{-3/2}k_x^{-1}k_y^{-1},\ k_z^{-3/2}k_y^{-1}). \tag{2.25}$$

For $k_x \approx k_y \approx k_\perp$ these spectra go over into (2.24). For two-dimensional turbulence, on the other hand, we have in place of (2.25)

$$W_{\mathbf{k}} \propto (k_z^{-3/2}k_x^{-1},\ k_z^{-3/2}). \tag{2.26}$$

Relations (2.25) and (2.26) illustrate the statements made in Sec. 1 with respect to Eq. (1.12).

Note also that at the limits of validity of the inequalities (2.13) and (2.21), i.e., at $k_z \propto k_\perp^4$, Eqs. (2.24) have qualitatively the same meaning as (2.18). A similar remark holds for (2.20) and (2.25). In other words, the spectra due to the vector and scalar nonlinearities change smoothly into each other in the region where these linearities are of equal importance.

3. ALFVEN TURBULENCE

We take the sought equations for nonlinear Alfven waves from Ref. 12. These are the vorticity equation

$$\frac{d_0}{dt}\left(\Delta_\perp\varphi+\frac{3}{4}\rho_i^2\Delta_\perp^2\varphi\right)+\frac{c_A^2}{c}\nabla_\parallel\Delta_\perp A=0, \tag{3.1}$$

the electron longitudinal-motion equation

$$\frac{\partial A}{\partial t}+c\nabla_\parallel\left(\varphi-\frac{T_e}{en_0}\tilde{n}\right)=0, \tag{3.2}$$

and the continuity equation for the electrons

$$\frac{d_0\tilde{n}}{dt}+\frac{c}{4\pi e}\nabla_\parallel\nabla_\perp A=0. \tag{3.3}$$

Here A is the z component of the vector potential

$$\nabla_\parallel=\frac{\partial}{\partial z}-\frac{1}{B_0}[\nabla A,\nabla]_z, \tag{3.4}$$

$c_A=B_0/(4\pi n_0 m_i)^{1/2}$ is the Alfven velocity, and $\rho_i=(T_e/T_i)^{1/2}\rho_0$ is the ion Larmor radius.

We introduce $\zeta=z-c_A t$, meaning a change to a coordinate frame moving along z with velocity c_A (cf. Sec. 2). We introduce in place of A the function $\tilde{A}$ defined by the relation

$$\tilde{A}=A-\frac{c}{c_A}(\varphi-\rho_0^2\Delta_\perp\varphi). \tag{3.5}$$

Assuming small dispersion and weak nonlinearity of the waves, we transform the system (3.1)–(3.3) into

$$\frac{\partial}{\partial t}\Delta_\perp\varphi-\frac{\lambda^2}{2}\left(c_A\frac{\partial}{\partial\zeta}-\frac{c}{B_0}[\nabla\varphi,\nabla]_z\right)\Delta_\perp^2\varphi$$

$$-\frac{c_A}{B_0}\operatorname{div}([\nabla\tilde{A},\nabla]_z\nabla\varphi)=0, \tag{3.6}$$

$$\frac{\partial\tilde{A}}{\partial\zeta}=\frac{c}{c_A^2}\frac{\partial\varphi}{\partial t}, \tag{3.7}$$

where $\lambda^2=\rho_0^2+\frac{3}{4}\rho_i^2$. Changing in (3.6) and (3.7) to a Fourier representation, we arrive at the following equation for $\varphi_{\mathbf{k}}(t)$:

$$\frac{\partial\varphi_{\mathbf{k}}}{\partial t}\propto\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}[\mathbf{k}_1\mathbf{k}_2]_z(k_\perp^2+k_{1\perp}^2+k_{2\perp}^2)$$

$$\times\frac{k_{1\perp}^2-k_{2\perp}^2}{k_\perp^2}\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\exp[-i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}})t], \tag{3.8}$$

where

$$\omega_{\mathbf{k}}=\lambda^2 c_A k_z k_\perp^2/2. \tag{3.9}$$

We have left out of the right-hand side of (3.8) an inessential factor of order λ^2.

In the case of Alfven waves we have

$$W_{\mathbf{k}}\propto k_\perp^2|\varphi_{\mathbf{k}}|^2. \tag{3.10}$$

The connection between $W_{\mathbf{k}}$ and $N_{\mathbf{k}}$ is given by (2.9). Therefore [cf. (2.10)]

$$C_{\mathbf{k}}\propto k_\perp\varphi_{\mathbf{k}}/|k_z|^{1/2}. \tag{3.11}$$

Taking (3.11) into account, we reduce (3.8) to the form (2.11), and arrive as a result at the following expression for the Alfven-wave interaction matrix element [cf. (2.12)]:

$$V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\propto i|k_z k_{1z} k_{2z}|^{1/2}[\mathbf{k}_1\mathbf{k}_2]_z\times\frac{k_\perp^2+k_{1\perp}^2+k_{2\perp}^2}{k_\perp k_{1\perp} k_{2\perp}}\left(\frac{k_{1\perp}^2-k_{2\perp}^2}{k_z}+\frac{k_\perp^2-k_{2\perp}^2}{k_{1z}}+\frac{k_{1\perp}^2-k_\perp^2}{k_{2z}}\right)\operatorname{sign}k_z. \tag{3.12}$$

We see that this matrix element is scale invariant with respect to k_{zi} and $k_{\perp i}$, with exponents

$$u=1/2,\quad v=3. \tag{3.13}$$

The wave frequency (3.9) is scale invariant with exponents (2.16). From this follows, in accordance with (1.3) and (1.4), the feasibility of Kolmogorov spectra of an Alfven axisymmetric turbulence, with exponents

$$\alpha^{(1)}=-3/2,\quad \beta^{(1)}=-5, \tag{3.14}$$

$$\alpha^{(2)}=-3/2,\quad \beta^{(2)}=-4. \tag{3.15}$$

Taking (2.9) into account, we find that these exponents correspond to energy spectra in the form

$$W_{\mathbf{k}}\propto(k_z^{-1/2}k_\perp^{-5},\ k_z^{-1/2}k_\perp^{-4}). \tag{3.16}$$

We consider now a plane Alfven turbulence. From (3.12) we have

$$v_1=2,\quad v_2=1. \tag{3.17}$$

Taking into account (3.17) and the statements in Sec. 1, we get

$$\beta_1^{(1)}=-3,\quad \beta_2^{(1)}=\beta_1^{(2)}=\beta_2^{(2)}=-2. \tag{3.18}$$

We arrive as a result at the following expressions for the energy spectra of a plane Alfven turbulence:

$$W_{\mathbf{k}}\propto(k_z^{-1/2}k_x^{-3}k_y^{-2},\ k_z^{-1/2}k_x^{-2}k_y^{-2}). \tag{3.19}$$

For $k_x\approx k_y\approx k_\perp$ Eqs. (3.19) go over into (3.16).

4. TURBULENCE OF HIGH-FREQUENCY MAGNETOSONIC WAVES

Consider high-frequency ($\omega\gg\omega_{Bi}$) magnetosonic waves propagating across an equilibrium magnetic field in the x-axis direction, and assume that the field of these waves has also a weak dependence on z and y [$(k_z, k_y)\ll k_x$]. We use a coordinate frame moving with Alfven velocity along the x axis. We introduce correspondingly the coordinate $\xi=x-c_A t$. We put $k_z/k_x\gg(m_e/m_i)^{1/2}$, where m_e is the electron mass. Using the results of Ref. 13, we find that under these assumptions and in the weak-nonlinearity approximation the magnetosonic waves are described by the equations

$$2\frac{\partial\varphi}{\partial t}+\frac{3}{2}\frac{e}{m_i c_A}\frac{\partial\varphi^2}{\partial\xi}-\frac{cc_A^2}{\omega_{pi}^2}\nabla_\parallel\frac{\partial^2 A}{\partial\xi^2}=0, \tag{4.1}$$

$$\frac{\partial A}{\partial\xi}-\frac{c}{c_A}\nabla_\parallel\varphi=0, \tag{4.2}$$

where $\omega_{pi}^2=4\pi e^2 n_0/m_i$ is the squared ion Langmuir frequency.

Changing to a Fourier representation, we obtain with the aid of (4.1) and (4.2)

$$i\frac{\partial\varphi_{\mathbf{k}}}{\partial t}=\frac{ek_x}{2m_ic_A}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}\left[\frac{3}{2}-i\frac{c^3}{\omega_{pi}^3}(k_yk_z^2-k_{1y}k_{1z}^2-k_{2y}k_{2z}^2)\right]\varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\times\exp[-i(\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}})t], \quad (4.3)$$

where

$$\omega_{\mathbf{k}}=c_Ac^2k_xk_z^2/2\omega_{pi}^2. \quad (4.4)$$

Just as for ion-sound waves, we have in this case $W_{\mathbf{k}}\propto|\varphi_{\mathbf{k}}|^2$. The connection between $W_{\mathbf{k}}$ and $N_{\mathbf{k}}$ is

$$N_{\mathbf{k}}\propto W_{\mathbf{k}}/|k_x|, \quad (4.5)$$

therefore

$$C_{\mathbf{k}}\propto\varphi_{\mathbf{k}}/|k_x|^{1/2}. \quad (4.6)$$

Using (4.6), we reduce (4.3) to the form (2.11). We obtain then

$$V(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\propto|k_xk_{1x}k_{2x}|^{1/2}\left[\frac{3}{2}-i\frac{c^3}{\omega_{pi}^3}\times(k_yk_z^2-k_{1y}k_{1z}^2-k_{2y}k_{2z}^2)\right]\operatorname{sign}k_x. \quad (4.7)$$

The first and second terms in the square brackets of the right-hand side of (4.7) are due respectively to scalar and vector nonlinearity. Evidently the scalar nonlinearity exceeds the vector nonlinearity if

$$k_yk_z^2<(\omega_{pi}/c)^3. \quad (4.8)$$

In this case the matrix element is scale-invariant with respect to k_{xi} with exponent 3/2. Recognizing also the scale invariance of the oscillation frequency [see (4.4)], we obtain by the above method the following two Kolmogorov-type energy spectra [cf. (2.26)]:

$$W_{\mathbf{k}}\propto(k_x^{-3/2}k_z^{-1},k_x^{-5/2}). \quad (4.9)$$

Equation (4.9) is written in two-dimensional space, i.e., it is indicative of a two-dimensional spectrum. It is clear from the statements in Sec. 1 that these two-dimensional spectra correspond to three-dimensional spectra of the form [cf. (1.12), (2.25)]

$$W_{\mathbf{k}}\propto(k_x^{-3/2}k_z^{-1}k_y^{-1},k_x^{-5/2}k_y^{-1}). \quad (4.10)$$

If

$$k_yk_z^2>(\omega_{pi}/c)^3 \quad (4.11)$$

we can neglect the contribution of the scalar linearity to the right-hand side of (4.7). The matrix element is also found to be scale invariant, so that under the condition (4.11) there can also be realized a pair of Kolmogorov spectra. In this case

$$W_{\mathbf{k}}\propto(k_x^{-3/2}k_z^{-3}k_y^{-2},k_x^{-5/2}k_z^{-2}k_y^{-2}). \quad (4.12)$$

The spectra (4.12) have the same meaning as in (4.9) at the limits of validity of the inequality (4.11).

5. TURBULENCE OF MAGNETIZED ELECTRON LANGMUIR WAVES

Consider purely electronic electrostatic waves in a cold magnetized plasma. We start from the standard hydrodynamic continuity and electron-motion equations

$$\partial n/\partial t+\operatorname{div}(n\mathbf{V})=0, \quad (5.1)$$

$$m_e\frac{d\mathbf{V}}{dt}=-e_e\nabla\varphi+\frac{e_e}{c}[\mathbf{V}\mathbf{B}_0] \quad (5.2)$$

and the Poisson equation

$$\Delta\varphi=-4\pi e_e\tilde{n}. \quad (5.3)$$

Here $n=n_0+\tilde{n}$, $\tilde{n}$ and $\mathbf{V}$ are the density perturbations and the electron velocities, $d/dt=\partial/\partial t+\mathbf{V}\cdot\nabla$, and e_e is the electron charge. After a number of simplifications (cf. Sec. 2) we reduce the system (5.1)–(5.3) to the form

$$\frac{d_0}{dt}\Delta_\perp\varphi-\frac{\omega_{pe}^2}{1+\omega_{pe}^2/\omega_{Be}^2}\frac{m_e}{e_e}\frac{\partial V_z}{\partial z}=0, \quad (5.4)$$

$$\frac{d_0V_z}{dt}=-\frac{e_e}{m_e}\frac{\partial\varphi}{\partial z}, \quad (5.5)$$

where $\omega_{pe}^2=4\pi e_e^2n_0/m_e$ is the squared electron Langmuir frequency, and $\omega_{Be}=e_eB_0/m_ec$ is the electron cyclotron frequency. It is assumed that the characteristic frequency of the waves is low compared with the electron cyclotron frequency, $\partial/\partial t\ll\omega_{Be}$, and the longitudinal wave numbers are considerably smaller than the transverse ones, $\partial^2/\partial z^2\ll\Delta_\perp$. Note also that we have neglected the scalar nonlinearity in (5.4) and (5.5), i.e., account is taken of only the vector nonlinearity [which is connected with the operator d_0/dt, see (2.5)].

The change to the Fourier representation in (5.4), (5.5) is with the aid of the equation

$$\varphi=\sum_{i=1,2}\sum_{\mathbf{k}}\varphi_{\mathbf{k}}^{(i)}\exp(i\mathbf{k}\mathbf{r}-i\omega_{\mathbf{k}}^{(i)}t)+\text{c.c.}, \quad (5.6)$$

where the superscript i labels oscillation modes with different signs of the frequency [cf. (1.6), (1.7)], and the remaining notation is obvious. We express V_z in similar form. Neglecting the nonlinear terms and the time dependence of $\varphi_{\mathbf{k}}^{(i)}(t)$ we obtain from (5.4) and (5.5) a dispersion equation that determines the frequencies $\omega_{\mathbf{k}}^{(i)}$:

$$\omega_{\mathbf{k}}^{(1,2)}=\pm\frac{|\omega_{pe}|}{(1+\omega_{pe}^2/\omega_{Be}^2)^{1/2}}\frac{k_z}{k_\perp}. \quad (5.7)$$

It follows from a comparison of (5.7) with (1.6) and (1.7) that in our case

$$a=1,\quad b=-1. \quad (5.8)$$

Allowance for the nonlinear terms and for the terms with $\partial\varphi_{\mathbf{k}}^{(i)}/\partial t$ in (5.4) and (5.5) leads to the following equation for $\varphi_{\mathbf{k}}^{(i)}$;

$$\frac{\partial\varphi_{\mathbf{k}}^{(i)}}{\partial t}=\frac{c}{4B_0k^2}\sum_{j,l}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}[\mathbf{k}_1\mathbf{k}_2]_z\left[k_2^2-k_1^2+\frac{k^2\omega_{\mathbf{k}}^{(i)}}{k_z}\left(\frac{k_{2z}}{\omega_{\mathbf{k}_2}^{(l)}}-\frac{k_{1z}}{\omega_{\mathbf{k}_1}^{(j)}}\right)\right]\times\varphi_{\mathbf{k}_1}^{(j)}\varphi_{\mathbf{k}_2}^{(l)}\exp[-i(\omega_{\mathbf{k}_1}^{(j)}+\omega_{\mathbf{k}_2}^{(l)}-\omega_{\mathbf{k}}^{(i)})t]. \quad (5.9)$$

To simplify the notation we omit here and in several equations below the subscript "⊥" of $k_\perp$, $k_{1\perp}$, and $k_{2\perp}$.

We recognize that the spectral energy density W_k of the considered waves is connected with φ_k by relation (3.10). According to (5.7), the connection between N_k and W_k is

$$N_k \propto kW_k/|k_z|, \qquad (5.10)$$

therefore

$$C_k \propto k^{1/2}\varphi_k/|k_z|^{1/2}. \qquad (5.11)$$

Changing in (5.9) to the variables C_k and expressing the result in a form similar to (2.11), we obtain an expression for the interaction matrix elements

$$V^{ijl}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) \propto i(-1)^i[\mathbf{k}_1\mathbf{k}_2]_z \frac{|k_z k_{1z} k_{2z}|^{1/2}}{(kk_1k_2)^{1/2}}$$

$$\times\left\{\frac{(-1)^i}{k_z}\left[\frac{(-1)^l}{k_1}-\frac{(-1)^j}{k_2}\right]+\frac{(-1)^j}{k_{1z}}\left[\frac{(-1)^l}{k}-\frac{(-1)^i}{k_2}\right]+\frac{(-1)^l}{k_{2z}}\left[\frac{(-1)^i}{k_1}-\frac{(-1)^j}{k}\right]\right\}$$

$$\times[(-1)^i k+(-1)^j k_1+(-1)^l k_2]\,\mathrm{sign}\,k_z. \qquad (5.12)$$

The meaning of the superscripts i, j, and l of the matrix elements is clear from the foregoing and from the Appendix.

The matrix elements (5.12) are scale-invariant in the longitudinal and transverse wave numbers with exponents

$$u={}^1/_2, \quad v={}^1/_2. \qquad (5.13)$$

We conclude therefore, with allowance for the statements made in Sec. 1, that an axisymmetric Kolmogorov turbulence can be realized in this case, with exponents

$$\alpha^{(1)}=-{}^3/_2, \quad \beta^{(1)}=-{}^5/_2. \qquad (5.14)$$

According to (5.10), the spectral energy density in such a turbulence is of the form

$$W_k \propto k_z^{-1/2} k_\perp^{-7/2}. \qquad (5.15)$$

For $k_x \gg k_y$ the oscillation frequency (5.7) and the matrix elements (5.12) are scale invariant in k_z, k_x, and k_y. In this case

$$v_1=-{}^1/_2, \quad v_2=1, \qquad (5.16)$$

so that, in accord with (1.9) and (1.11), we are dealing now with spectral exponents

$$\alpha^{(1)}={}^3/_2, \quad \beta_1{}^{(1)}=-{}^1/_2, \quad \beta_2{}^{(1)}=-2. \qquad (5.17)$$

Taking (5.10) into account, we find that these spectral exponents correspond to a plane turbulence with an energy spectrum

$$W_k \propto k_z^{-1/2} k_x^{-3/2} k_y^{-2}. \qquad (5.18)$$

We note also that at the limits of its applicability, i.e., for $k_x \approx k_y \approx k_\perp$, Eq. (5.18) is in qualitative agreement with (5.15).

6. TURBULENCE OF WHISTLERS

The electrostatic approximation assumed in Sec. 5 is valid only if $k_\perp \gg c/\omega_{pe}$. We assume now that $k_\perp \ll c/\omega_{pe}$. We regard the perturbations as quasineutral. In the case of purely electronic waves, i.e., neglecting the ion motion, this means $\tilde{n}=0$. Equation (5.1) is then simply the condition that the electron component of the plasma be incompressible,

$$\mathrm{div}\,\mathbf{V}=0. \qquad (6.1)$$

We modify the electron equation of motion (5.2) as follows. On the one hand, we neglect in it the inertia, and on the other we take into account the unperturbed magnetic field $\tilde{\mathbf{B}}$ and the fact that the electric field $\mathbf{E}$ is not potential. In other words, we replace (5.2) by

$$\mathbf{E}+\frac{1}{c}[\mathbf{VB}]=0, \qquad (6.2)$$

where $\mathbf{B}=\mathbf{B}_0+\tilde{\mathbf{B}}$ is the total magnetic field. We use also the equation (cf. Ref. 14)

$$\Delta_\perp \tilde{B}_z=-4\pi e_e n_0\,\mathrm{rot}_z\,\mathbf{V}/c, \qquad (6.3)$$

which follows from the Maxwell equations and from the expression $\mathbf{j}=e_e n_0 \mathbf{V}$ for the electric current. We obtain from (6.1)–(6.3) the system of nonlinear equations

$$\frac{\partial\varphi}{\partial t}-\frac{c^3\omega_{Be}{}^2}{\omega_{pe}{}^4}\nabla_\parallel\Delta_\perp A=0, \qquad (6.4)$$

$$\partial A/\partial t+c\nabla_\parallel\varphi=0. \qquad (6.5)$$

Obviously, (6.5) is a particular case of (3.2) as $T_e\to 0$ [cf. also Eq. (4.2)]. At the same time, (6.4) is the analog of (4.1).

In analogy with (5.7), it follows from (6.4) and (6.5) that

$$\omega_k^{(1,2)}=\pm|\omega_{Be}|c^2k_zk_\perp/\omega_{pe}{}^2, \qquad (6.6)$$

so that now [cf. (5.8)]

$$a=1, \quad b=1. \qquad (6.7)$$

At the same time we have in the present case in place of (5.9)

$$\frac{\partial\varphi_k^{(i)}}{\partial t}=-\frac{1}{4}\frac{c^5\omega_{Be}^2}{\omega_{pe}^4 B_0}\times\sum_{j,l}\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}}[\mathbf{k}_1\mathbf{k}_2]_z\left[(k_2{}^2-k_1{}^2)\frac{k_{1z}k_{2z}}{\omega_{k_1}^{(j)}\omega_{k_2}^{(l)}}+\frac{k^2k_z}{\omega_k^{(i)}}\left(\frac{k_{1z}}{\omega_{k_1}^{(j)}}-\frac{k_{2z}}{\omega_{k_2}^{(l)}}\right)\right]\varphi_{k_1}^{(j)}\varphi_{k_2}^{(l)}$$

$$\times\exp[-i(\omega_{k_1}^{(j)}+\omega_{k_2}^{(l)}-\omega_k^{(i)})t]. \qquad (6.8)$$

Just as in the cases of ion-sound and magnetosonic waves (see Secs. 2 and 4), the connection between W_k and φ_k is $W_k \propto |\varphi_k|^2$. Taking (6.6) into account, we obtain

$$N_k \propto W_k/|k_z|k_\perp. \qquad (6.9)$$

Consequently

$$C_k \propto \varphi_k/(|k_z|k_\perp)^{1/2}. \qquad (6.10)$$

With the aid of (6.8) and (6.10) we get, by analogy with (5.12),

$$V^{ijl}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\propto i(-1)^i \operatorname{sign} k_z \frac{|k_z k_{1z} k_{2z}|^{1/2}}{(kk_1k_2)^{1/2}}[\mathbf{k}_1\mathbf{k}_2]_z$$

$$\times\left\{\frac{(-1)^i}{k_z}[(-1)^j k_2-(-1)^l k_1]\right.$$

$$\left.-\frac{(-1)^j}{k_{1z}}[(-1)^l k-(-1)^i k_2]-\frac{(-1)^l}{k_{2z}}[(-1)^i k_1-(-1)^j k]\right\}$$

$$\times[(-1)^i k+(-1)^j k_1+(-1)^l k_2]. \qquad (6.11)$$

Evidently, the matrix elements are scale invariant in k_z and $k_\perp$, with exponents

$$u={}^1/_2, \quad v={}^5/_2, \quad v_1={}^3/_2, \quad v_2=1. \qquad (6.12)$$

With allowance for (6.12) and (1.4) we find that the spectral exponents of the Kolmogorov turbulence of whistlers are

$$\alpha^{(1)}=-{}^3/_2, \quad \beta^{(1)}=-{}^9/_2, \quad \beta_1^{(1)}=-{}^5/_2, \quad \beta_2^{(1)}=-2. \qquad (6.13)$$

Using (6.13) and (6.9) we conclude that the considered turbulence is characterized by a spectral energy density of exactly the same form as in the case of magnetized electron Langmuir waves, i.e., it is given by (5.15). As for plane turbulence of whistlers, the spectral energy of such a turbulence is determined by Eq. (5.18).

7. KOLMOGOROV SPECTRA OF CERTAIN OTHER TYPES OF WAVES

7.1. Electron-sound waves

Electron-sound waves[5] occur in the case of a plasma with hot ions, $T_i \gg T_e$. They are characterized by a dispersion equation

$$\omega^2=k_z^2c_e^2(1-\rho_{0e}^2k_\perp^2), \qquad (7.1)$$

where $c_e^2 = T_i/m_e$ is the squared electron-sound velocity, and ρ_{0e} is the electron Larmor radius calculated from the ion temperature. It is assumed that the terms with $k_\perp^2$ in (7.1) are small dispersion corrections. In a coordinate frame moving along the magnetic field with velocity c_e, the dispersion equation (7.1) is replaced by

$$\omega_k=-c_e\rho_{0e}^2k_zk_\perp^2/2. \qquad (7.2)$$

Apart from the interchange of the subscripts labeling the species of the charges, Eq. (7.2) is the same as the dispersion increment (2.8) to the ion-sound frequency. The nonlinear equations for the electron-sound waves are obtained from (2.1)–(2.4) likewise by interchanging the subscripts. The results of the analysis of Sec. 2 are consequently valid also for the problem of electron-sound waves. It is clear from the foregoing that the electron-sound waves can be the cause of weak turbulence with Kolmogorov spectra in the form (2.18), (2.20), and (2.24)–(2.26).

7.2. Short-wave ion-sound waves

In Sec. 2 we have considered the turbulence of waves described by the dispersion equation (2.8). This equation is the long-wave limit ($k_\perp\rho_0 \ll 1$) of the general dispersion equation of ion-sound waves[15]:

$$\omega^2=k_z^2c_s^2/(1+k_\perp^2\rho_0^2). \qquad (7.3)$$

In the short-wave limit $k_\perp\rho_0 \gg 1$ we get from (7.3) in place of (2.8)

$$\omega_{\mathbf{k}}^{(1,2)}=\pm\omega_{Bi}k_z/k_\perp. \qquad (7.4)$$

A comparison of (7.4) with (5.7) reveals the formal analogy of the dispersion properties of short-wave ion-sound waves and magnetized electron Langmuir waves. The initial nonlinear equations of the short-wave ion-sound waves are similar in form to (5.4) and (5.5). Therefore, taking the analysis of Sec. 5 into account, we conclude that waves in question can be related to weak turbulence with Kolmogorov spectra of the form (5.15) or (5.18).

7.3. Short-wave Alfven waves

Short-wave Alfven (SA) waves are continuations of modes of long-wave Alfven waves (see Sec. 3) into the region $k_\perp\rho_i > 1$. They are characterized by a dispersion equation

$$\omega_{\mathbf{k}}^{(1,2)}=\pm c_A\rho_0(1+T_i/T_e)^{1/2}k_zk_\perp. \qquad (7.5)$$

This equation is similar in structure to the whistler equation (6.6). In the SA wave problem the electron continuity equation differs from (6.1) by the presence of a contribution of the perturbed density $\tilde{n}$:

$$d_0\tilde{n}/dt+n_0 \operatorname{div} \mathbf{V}=0. \qquad (7.6)$$

We obtain the perturbed plasma density $\tilde{n}$ with the aid of the Boltzmann law for ions

$$\tilde{n}=-en_0\varphi/T_i. \qquad (7.7)$$

In contrast to (6.2), we take into account in the equation of motion of the electrons a term with a gradient pressure, and neglect the temperature perturbations, i.e., we use the equation

$$\frac{T_e\nabla\tilde{n}}{en_0}+\mathbf{E}+\frac{1}{c}[\mathbf{VB}]=0. \qquad (7.8)$$

We neglect the perturbation of the longitudinal magnetic field $\tilde{B}_z$; this is justified if $\beta_p \ll 1$, where β_p is the ratio of the plasma pressure to the magnetic-field pressure. As a result we arrive at the system of equations (cf. Ref. 17)

$$\frac{\partial\varphi}{\partial t}-\frac{cT_i}{4\pi e^2n_0}\nabla_\parallel\Delta_\perp A=0, \qquad (7.9)$$

$$\frac{\partial A}{\partial t}+c\left(1+\frac{T_e}{T_i}\right)\nabla_\parallel\varphi=0. \qquad (7.10)$$

Apart from the constants, this system is equivalent to (6.4) and (6.5). It is clear therefore that the considered problem of the turbulence of SA waves reduces mathematically to the problem of whistler turbulence. We conclude then without further analysis that SA waves can give rise to a weak Kolmogorov turbulence with spectrum exponents of form (6.13) and with spectral density distributions of form (5.15) and (5.18).

8. CONCLUSION

The foregoing analysis attests to the presence in a homogeneous magnetized plasma of a rather great variety of oscillation modes on which weakly turbulent Kolmogorov spectra can be realized. We have confined ourselves to finding the stationary turbulence spectra of the corresponding modes. It is also of interest to study the dynamic properties of these spectra (in particular, analysis of the spectral fluxes and of the problem of locality of the turbulence); this can be the subject of future research.

APPENDIX

KOLMOGOROV STATIONARY SOLUTIONS WITH ALLOWANCE FOR THE EFFECT OF COUNTERPROPAGATING WAVES

Consider the interaction between waves of type $i = 1, 2$ [see (1.6), (1.7)] having a wave vector $\mathbf{k}$ and waves of type $(j, l) = 1, 2$ having vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. Such an interaction is described by the kinetic equations for the waves (cf. Ref. 11):

$$\frac{\partial N_\mathbf{k}}{\partial t}=\sum_{j,l}\int d\mathbf{k}_1\, d\mathbf{k}_2 U^{ijl}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) \times [N_{\mathbf{k}_1}N_{\mathbf{k}_2}-N_\mathbf{k}N_{\mathbf{k}_1}\operatorname{sign}(\omega_\mathbf{k}^{(i)}\omega_{\mathbf{k}_2}^{(l)}) -N_\mathbf{k}N_{\mathbf{k}_2}\operatorname{sign}(\omega_\mathbf{k}^{(i)}\omega_{\mathbf{k}_1}^{(j)})]\delta(\omega_\mathbf{k}^{(i)}-\omega_{\mathbf{k}_1}^{(j)}-\omega_{\mathbf{k}_2}^{(l)}) \times\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,d\mathbf{k}_1\,d\mathbf{k}_2. \quad \text{(A1)}$$

Here

$$U^{ijl}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)=|V^{ijl}(\mathbf{k}_1,\mathbf{k}_1,\mathbf{k}_2)|^2, \quad \text{(A2)}$$

and $V^{ijl}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)$ is the matrix element of the interaction. It is assumed that the forward and backward waves have equal numbers of quanta.

Changing over in the right-hand side of (A1) to positive frequencies and integrating over positive k_{1z} and k_{2z}, we get

$$\frac{\partial N_\mathbf{k}}{\partial t}=\int dk_{1z}\,dk_{2z}\int d\mathbf{k}_{1\perp}\,d\mathbf{k}_{2\perp}\delta(\mathbf{k}_\perp-\mathbf{k}_{1\perp}-\mathbf{k}_{2\perp}) \times\{Z_1(N_{\mathbf{k}_1}N_{\mathbf{k}_2}-N_\mathbf{k}N_{\mathbf{k}_1}-N_\mathbf{k}N_{\mathbf{k}_2}) \times\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2})+Z_2(N_{\mathbf{k}_1}N_{\mathbf{k}_2}+N_\mathbf{k}N_{\mathbf{k}_1}-N_\mathbf{k}N_{\mathbf{k}_2})\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}) +Z_3(N_{\mathbf{k}_1}N_{\mathbf{k}_2}-N_\mathbf{k}N_{\mathbf{k}_1}-N_\mathbf{k}N_{\mathbf{k}_2})\delta(\omega_\mathbf{k}+\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2})\}. \quad \text{(A3)}$$

Here

$$Z_1=U^{111}(k_z,\mathbf{k}_\perp;\,k_{1z},\mathbf{k}_{1\perp};\,k_{2z},\mathbf{k}_{2\perp})\delta(k_z-k_{1z}-k_{2z})+U^{121}(k_z,\mathbf{k}_\perp;\,-k_{1z},\mathbf{k}_{1\perp};\,k_{2z},\mathbf{k}_{2\perp}) \times\delta(k_z+k_{1z}-k_{2z})+U^{112}(k_z,\mathbf{k}_\perp;\,k_{1z},\mathbf{k}_{1\perp};\,-k_{2z},\mathbf{k}_{2\perp})\delta(k_z-k_{1z}+k_{2z}), \quad \text{(A4a)}$$

$$Z_2=U^{112}(k_z,\mathbf{k}_\perp;\,k_{1z},\mathbf{k}_{1\perp};\,k_{2z},\mathbf{k}_{2\perp}) \times\delta(k_z-k_{1z}-k_{2z})+U^{122}(k_z,\mathbf{k}_\perp;\,-k_{1z},\mathbf{k}_{1\perp};\,k_{2z},\mathbf{k}_{2\perp})\delta(k_z+k_{1z}-k_{2z})+U^{111}(k_z,\mathbf{k}_\perp;\,k_{1z},\mathbf{k}_{1\perp};\,-k_{2z},\mathbf{k}_{2\perp})\delta(k_z-k_{1z}+k_{2z}), \quad \text{(A4b)}$$

$$Z_3=U^{121}(k_z,\mathbf{k}_\perp;\,k_{1z},\mathbf{k}_{1\perp};\,k_{2z},\mathbf{k}_{2\perp})\delta(k_z-k_{1z}-k_{2z})+U^{111}(k_z,\mathbf{k}_\perp;\,-k_{1z},\mathbf{k}_{1\perp};\,k_{2z},\mathbf{k}_{2\perp}) \times\delta(k_z+k_{1z}-k_{2z})+U^{122}(k_z,\mathbf{k}_\perp;\,k_{1z},\mathbf{k}_{1\perp};\,-k_{2z},\mathbf{k}_{2\perp})\delta(k_z-k_{1z}+k_{2z}). \quad \text{(A4c)}$$

The use of Zakharov's factorization method[1] permits transformation of the right-hand side of (A3), which contains three different frequency δ functions into a form containing only one δ function, say $\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}1}-\omega_{\mathbf{k}2})$. In this case (A3) takes the form

$$\frac{\partial N_\mathbf{k}}{\partial t}=\int_0^\infty dk_{1z}\,dk_{2z}\int d\mathbf{k}_{1\perp}\,d\mathbf{k}_{2\perp}\delta(\mathbf{k}_\perp-\mathbf{k}_{1\perp}-\mathbf{k}_{2\perp}) \times Z_1K(N_{\mathbf{k}_1}N_{\mathbf{k}_2}-N_\mathbf{k}N_{\mathbf{k}_1}-N_\mathbf{k}N_{\mathbf{k}_2})\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}). \quad \text{(A5)}$$

In the case of axisymmetric turbulence, the function K is given by[1]

$$K=1-\left(\frac{k_{1z}}{k_z}\right)^\xi\left(\frac{k_{1\perp}}{k_\perp}\right)^\eta-\left(\frac{k_{2z}}{k_z}\right)^\xi\left(\frac{k_{2\perp}}{k_\perp}\right)^\eta, \quad \text{(A6)}$$

where

$$\xi=a-2(1+u+\alpha),\quad \eta=b-2(2+v+\beta). \quad \text{(A7)}$$

For plane turbulence with $k_x \gg k_y$ we have in place of (A6) (Ref. 5)

$$K=1-\left(\frac{k_{1z}}{k_z}\right)^\xi\left(\frac{k_{1x}}{k_x}\right)^\eta\left(\frac{k_{1y}}{k_y}\right)^\zeta-\left(\frac{k_{2z}}{k_z}\right)^\xi\left(\frac{k_{2x}}{k_x}\right)^\eta\left(\frac{k_{2y}}{k_y}\right)^\zeta, \quad \text{(A8)}$$

where ξ, as before, is determined by the first equation of (A7), while η and ζ stand for

$$\eta=b_1-2(1+v_1+\beta_1),\quad \zeta=-2(1+v_2+\beta_2). \quad \text{(A9)}$$

Let $\alpha=\alpha^{(2)}$ and $\beta=\beta^{(2)}$, where $\alpha^{(2)}$ and $\beta^{(2)}$ are defined by (1.5). The function K defined by (A6) then takes the form

$$K=K^{(2)}=\frac{1}{k_z}(k_z-k_{1z}-k_{2z}). \quad \text{(A10)}$$

Neglecting the backward waves, i.e., taking into account in Z_1 only the term with U^{111}, we have

$$Z_1\propto\delta(k_z-k_{1z}-k_{2z}). \quad \text{(A11)}$$

Consequently

$$Z_1K^{(2)}=0, \quad \text{(A12)}$$

i.e., in the absence of backward waves the integrands in (A5) vanish at the indicated values of α and β. This corresponds to stationary spectra with $\alpha=\alpha^{(2)}$ and $\beta=\beta^{(2)}$. If, however, backward-wave effects are taken into account, i.e., it is assumed that $(U^{121}, U^{112})\neq 0$, we have according to (A4a) and (A10)

$$Z_1K^{(2)}\neq 0. \quad \text{(A13)}$$

In this case no stationary solutions with $\alpha=\alpha^{(2)}$ and $\beta=\beta^{(2)}$ are realized.

Equation (A6) can be written also in the form

$$K=\frac{1}{\omega_\mathbf{k}}\left[\omega_\mathbf{k}-\omega_{\mathbf{k}_1}\left(\frac{k_{1z}}{k_z}\right)^{-2(\alpha-\alpha^{(1)})}\left(\frac{k_{1\perp}}{k_\perp}\right)^{-2(\beta-\beta^{(1)})}-\omega_{\mathbf{k}_2}\left(\frac{k_{2z}}{k_z}\right)^{-2(\alpha-\alpha^{(1)})}\left(\frac{k_{2\perp}}{k_\perp}\right)^{-2(\beta-\beta^{(1)})}\right], \quad \text{(A14)}$$

where $\alpha^{(1)}$ and $\beta^{(1)}$ are given by Eqs. (1.4). For $\alpha=\alpha^{(1)}$ and $\beta=\beta^{(1)}$ it follows hence that

$$K=K^{(1)}=\frac{1}{\omega_\mathbf{k}}(\omega_\mathbf{k}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}). \quad \text{(A15)}$$

Since $\delta(\omega_\mathbf{k}-\omega_{\mathbf{k}1}-\omega_{\mathbf{k}2})$, in contrast to $\delta(k_z-k_{1z}-k_{2z})$, is a common factor of the integrand in (A5), the case $K = K^{(1)}$ corresponds to stationary solutions both in the absence and in the presence of backward waves. In other words, solutions with $\alpha = \alpha^{(1)}$ and $\beta = \beta^{(1)}$ are not sensitive to effects of the backward waves.

Returning to Eq. (A8), it can be verified that a similar picture obtains also in the case of plane turbulence.

[1]V. E. Zakharov, in: *Basic Plasma Physics*, A. A. Galeev and R. Sudan, eds., North Holland, 1985, Vol. 2, p. 3.

[2]V. E. Zakharov, Zh. Eksp. Teor. Fiz. **51**, 688 (1966) [Sov. Phys. JETP **24**, 455 (1967)].

[3]V. D. Shafranov, in: *Reviews of Plasma Physics*, M. A. Leontovich, ed., Vol. 3, Consultants Bureau, 1967.

[4]E. A. Kuznetsov, Zh. Eksp. Teor. Fiz. **62**, 584 (1972) [Sov. Phys. JETP **35**, 310 (1972)].

[5]A. B. Mikhaĭlovskiĭ, S. V. Novakovskiĭ, V. P. Lakhin, *et al.*, *ibid.* **94**, No. 1, 159 (1988) [67, 1386 (1988)].

[6]D. L. Brower, W. A. Pebles, and N. C. Luhman, Nucl. Fusion **27**, 2055 (1987).

[7]A. Hasegawa, C. G. Maclennan, and Y. Kodama, Phys. Fluids **22**, 2122 (1979).

[8]A. B. Mikhaĭlovskiĭ, *Theory of Plasma Instabilities*, Plenum, 1974, Vol. 2.

[9]A. B. Mikhaĭlovskiĭ, V. P. Lakhin, O. G. Onishchenko, and A. I. Smolyakov, Zh. Eksp. Teor. Fiz. **88**, 798 (1985) [Sov. Phys. JETP **61**, 469 (1985)].

[10]B. B. Kadomtsev, *ibid.*, **43**, 1688 (1962) [**16**, 1191 (1963)].

[11]A. A. Galeev and R. Z. Sagdeev, in: *Reveiws of Plasma Physics*, M. A. Leontovich, ed., Vol. 7, Consultants Bureau, 1979.

[12]V. P. Lakhin, A. B. Mikhaĭlovskiĭ, and A. I. Smolyakov, Zh. Eksp. Teor. Fiz. **92**, 1601 (1987) [Sov. Phys. JETP **65**, 898 (1987)].

[13]A. B. Mikhaĭlovskiĭ, V. P. Lakhin, and L. A. Mikhaĭlovskaya, *ibid.* **87**, 1221 (1984) [**60**, 698 (1984)].

[14]A. B. Mikhaĭlovskiĭ, V. P. Lakhin, L. A. Mikhaĭlovskaya, Zh. Eksp. Teor. Fiz. **89**, 482 (1985) [Sov. Phys. JETP **62**, 273 (1985)].

[15]A. B. Mikhailovskii, op. cit.[8] Vol. 1, Chap. 7.

[16]A. B. Mikhaĭlovskiĭ and L. I. Rudakov, Zh. Eksp. Teor. Fiz. **44**, 912 (1963) [Sov. Phys. JETP **17**, 621 (1963)].

[17]V. P. Lakhin, A. B. Mikhaĭlovskiĭ, and S. V. Novakovskiĭ, Fiz. Plazmy **12**, 566 (1986) [Sov. J. Plasma Phys. **12**, 326 (1986)].

Translated by J. G. Adashko