Langevin sources in the hydrodynamic equations of a multicomponent plasma

V. V. Belyĭ

Institute of Terrestrial Magnetism, Ionosphere, and Radiowave Propagation, USSR Academy of Sciences

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Dissipation due to interaction of particles of different species adds Langevin sources to the Landau–Lifshitz sources in the hydrodynamic equations. It is shown that the intensities of the sources are determined by the temperatures of the components and by the Spitzer transport coefficients.

I. INTRODUCTION

Hydrodynamic fluctuations are of great interest in the study of scattering and transformation of electromagnetic waves in a conducting medium. Information obtained by the study of scattering can be used to determine properties of the medium such as density, temperature, conductivity, viscosity, and others. The most suitable way of describing fluctuations in both equilibrium and nonequilibrium systems is to introduce Langevin sources into the equations of motion.[1] This method was used, in particular, by Rytov and Levin in the theory of electromagnetic fluctuations.[2] The Langevin approach was first used to describe hydrodynamic fluctuations in a single-component equilibrium gas or liquid by Landau and Lifshitz,[3] who introduced an extraneous stress tensor into the Navier–Stokes equation and an extraneous heat-flux vector into the heat-transport equation. The correlators of the extraneous quantities have been obtained by an entropy method for the thermodynamic equilibrium state, and are determined by appropriate dissipative characteristics (viscosity and heat-conduction coefficients) and by the system temperature.

The hydrodynamic equations of a multicomponent system (e.g., a fully ionized plasma) contain additional dissipation sources, viz., the relaxation of the momentum and energy of one component relative to the other. The thermodynamic equations should therefore contain additional Langevin sources correponding to the new dissipation sources. Another question is that of the dissipative quantities and temperature which determine the correlators of the extraneous stress tensors and of the extraneous heat-flux vector in multicomponent systems with components of unequal temperature.

2. LANGEVIN SOURCE IN THE HYDRODYNAMIC EQUATIONS OF A FULLY IONIZED PLASMA

We begin with a linearized kinetic equation for the fluctuations δf_a of a distribution function with a Langevin source y_a:

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\frac{\partial}{\partial \mathbf{r}}+\mathbf{F}_a\frac{\partial}{\partial \mathbf{p}}-\delta \hat{I}_{a\mathbf{p}}\right)\delta f_a=-\delta\mathbf{F}_a\frac{\partial \bar{f}_a}{\partial \mathbf{p}}+y_a, \tag{1}$$

where

$$\mathbf{F}_a=e_a\mathbf{E}+(e_a/c)\times[\mathbf{vB}],\quad \delta\hat{I}_{a\mathbf{p}}\delta f_a(\mathbf{p})=\sum_c\frac{\partial}{\partial p_i}\int Q_{ij}^{ac}\left(\frac{\partial}{\partial p_j}-\frac{\partial}{\partial p_j'}\right)\times[\bar{f}_a(\mathbf{p})\delta f_c(\mathbf{p}')+\bar{f}_c(\mathbf{p}')\delta f_a(\mathbf{p})]\,d\mathbf{p}' \tag{2}$$

is a linearized operator of collisions with a nucleus, which takes into account the dynamic screening of the potential $\varepsilon(\omega,\mathbf{k})$ (Ref. 4), and

$$Q_{ij}^{ab}=2\int\frac{e_a^2e_b^2k_ik_j}{k^4|\varepsilon(\mathbf{kv},\mathbf{k})|^2}\delta(\mathbf{kv}-\mathbf{kv}')\,d\mathbf{k}.$$

The intensity of the source y_a is given by[5,6]

$$\overline{y_a(\mathbf{x}_1,t_1)y_b(\mathbf{x}_2,t_2)}=\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(t_1-t_2)[-(\delta\hat{I}_{a\mathbf{p}_1}+\delta\hat{I}_{b\mathbf{p}_1})\delta_{ab}\delta(\mathbf{p}_1-\mathbf{p}_2)\bar{f}_a(\mathbf{p}_1)+\delta_{ab}\delta(\mathbf{p}_1-\mathbf{p}_2)I_a+I_{ab}(\mathbf{p}_1\mathbf{p}_2)], \tag{3}$$

where

$$I_{ab}(\mathbf{p}_1\mathbf{p}_2)=\left(\frac{\partial}{\partial p_{1i}}-\frac{\partial}{\partial p_{2i}}\right)Q_{ij}^{ab}\left(\frac{\partial}{\partial p_{1j}}-\frac{\partial}{\partial p_{2j}}\right)\bar{f}_a(\mathbf{p}_1)\bar{f}_b(\mathbf{p}_2),$$

$$I_a=\sum_b\int I_{ab}(\mathbf{p}_1\mathbf{p}_2)\,d\mathbf{p}_2,\qquad \mathbf{x}=(\mathbf{r},\mathbf{p}).$$

In contrast to single-component gasdynamics, in a multicomponent plasma the contribution of a Langevin source to the momentum balance and kinetic-energy balance equations is not zero. In fact, in multicomponent systems the conservation laws for both the average quantities and the fluctuations are satisfied only for the system as a whole, and not for the individual components. A convolution of expression (3) with the corresponding polynomials suffices to calculate the intensity of the Langevin sources.

Using for an arbitrary function the identities

$$\left(\frac{\partial}{\partial \mathbf{p}_1}+\frac{\partial}{\partial \mathbf{p}_2}\right)\delta(\mathbf{p}_1-\mathbf{p}_2)\Phi(\mathbf{p}_1-\mathbf{p}_2)=\delta(\mathbf{p}_1-\mathbf{p}_2)\frac{\partial}{\partial \mathbf{p}_1}\Phi(\mathbf{p}_1\mathbf{p}_2),$$

$$\left(\frac{\partial^2}{\partial \mathbf{p}_1^2}+\frac{\partial^2}{\partial \mathbf{p}_2^2}\right)\delta(\mathbf{p}_1-\mathbf{p}_2)\Phi(\mathbf{p}_1\mathbf{p}_2)=-2\frac{\partial}{\partial \mathbf{p}_1}\frac{\partial}{\partial \mathbf{p}_2}\Phi(\mathbf{p}_1\mathbf{p}_2)\delta(\mathbf{p}_1-\mathbf{p}_2)+\delta(\mathbf{p}_1-\mathbf{p}_2)\frac{\partial^2}{\partial \mathbf{p}_1^2}\Phi(\mathbf{p}_1\mathbf{p}_2), \tag{4}$$

we can represent the expression for the hydrodynamic intensity moments of the Langevin sources (y_ay_b) in the form

$$\int\varphi_a(\mathbf{p}_1)\psi_b(\mathbf{p}_2)(y_ay_b)_{\omega\mathbf{k}\mathbf{p}_1\mathbf{p}_2}d\mathbf{p}_1\,d\mathbf{p}_2=2\sum_c\int Q_{ij}^{ab}(\mathbf{v}_1\mathbf{v}_2)\bar{f}_a(\mathbf{p}_1)\bar{f}_b(\mathbf{p}_2)\left[\delta_{ab}\frac{\partial}{\partial p_{1i}}\varphi_a(\mathbf{p}_1)\times\frac{\partial}{\partial p_{1j}}\psi_b(\mathbf{p}_1)-\delta_{bc}\frac{\partial}{\partial p_{1i}}\varphi_a(p_1)\frac{\partial}{\partial p_{2j}}\psi_b(p_2)\right]d\mathbf{p}_1\,d\mathbf{p}_2 \tag{5}$$

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for arbitrary $\varphi_a(p)$ and $\psi_b(p)$. This expression, summed over all the particle species, can be rewritten in the form

$$\sum_{ab}\int \varphi_a(\mathbf{p}_1)\psi_b(\mathbf{p}_2)(y_a y_b)_{\omega\mathbf{k}\mathbf{p}_1\mathbf{p}_2} d\mathbf{p}_1\, d\mathbf{p}_2$$
$$=\sum_{ab}\int Q_{ij}^{ab}(\mathbf{v}_1\mathbf{v}_2)\bar{f}_a(\mathbf{p}_1)\bar{f}_b(\mathbf{p}_2)$$
$$\times\left[\frac{\partial}{\partial p_{1i}}\varphi_a(\mathbf{p}_1)-\frac{\partial}{\partial p_{2i}}\varphi_b(\mathbf{p}_2)\right]$$
$$\times\left[\frac{\partial}{\partial p_{1j}}\psi_a(\mathbf{p}_1)-\frac{\partial}{\partial p_{2j}}\psi_b(\mathbf{p}_2)\right]d\mathbf{p}_1\, d\mathbf{p}_2, \quad (6)$$

which satisfies the conservation laws for the number of particles, the momentum, and the energy of the entire system as a whole. Expression (6) vanishes if φ or ψ are equal to 1, $\mathbf{p}$, and $p^2/2m$.

Putting in (5) $\varphi=\psi=\mathbf{p}$ and integrating over the momenta, we obtain an expression for the intensity ξ^a of a random source in the momentum-density balance equation for the component a:

$$\frac{\partial}{\partial t}m_a n_a u_{ai}+\frac{\partial}{\partial r_j}m_a n_a u_i^a u_j^a=-\frac{\partial P^a}{\partial r_i}-\frac{\partial \pi_{ij}^a}{\partial r_j}+n_a F_{ai}$$
$$-\sum_b \nu_{ab} m_a n_a (u_i^a-u_i^b)$$
$$+\frac{3}{5}\sum_b \nu_{ab}\frac{\mu_{ab}}{\Theta_{ab}}\left(q_i^a-\frac{m_a n_a}{m_b n_b}q_i^b\right)+\xi_i^a, \quad (7)$$

where

$$\mu_{ab}=\frac{m_a m_b}{m_a+m_b},\quad \Theta_{ab}=\frac{m_a\Theta_b+m_b\Theta_a}{m_a+m_b},$$
$$\nu_{ab}=\frac{2^{5/2}}{3}\pi^{1/2}e_a^2 e_b^2 n_b\frac{\mu_{ab}^{1/2}}{\Theta_{ab}^{3/2}}\frac{L}{m_a},$$

L is the Coulomb logarithm, Θ_a and Θ_b are the temperatures of the respective components in energy units, π_{ij} is the extraneous stress tensor, and

$$(\xi_k^a\xi_l^b)_{\omega\mathbf{k}}=2\sum_c \nu_{ac} m_a \Theta_{ac} n_a \delta_{kl}(\delta_{ab}-\delta_{bc}). \quad (8)$$

It is easy to verify that

$$\sum_{ab}(\xi_k^a\xi_l^b)_{\omega k}=0.$$

In a two-component plasma the expression for the source, for example in the equation of motion for the electrons, takes the form

$$(\xi_k^e\xi_l^e)_{\omega\mathbf{k}}=2m_e n_e\Theta_e\delta_{kl}\nu_{ei}. \quad (9)$$

Putting in (5) $\varphi=\psi=p^2/2m$ we obtain an expression for the source intensity ζ^a in the balance equation for the kinetic-energy density of the component:

$$\frac{\partial}{\partial t}\left(\frac{n_a u_a^2}{2}+\frac{3}{2}\frac{n_a\Theta_a}{m_a}\right)+\frac{\partial}{\partial r_i}\left[u_i^a\left(\frac{n_a\mathbf{u}_a^2}{2}+\frac{3}{2}\frac{n_a}{m_a}\Theta_a+\frac{P_a}{m_a}\right)\right.$$
$$\left.+\pi_{ij}^a u_j^a+q_i^a\right]=\frac{n_a}{m_a}(\mathbf{F}_a\mathbf{u}_a)-\sum_b 2\nu_{ab}\frac{\mu_{ab}}{m_a+m_b}\frac{3}{2}n_a(\Theta_a-\Theta_b)$$
$$+\sum_b \nu_{ab} n_a\frac{\mu_{ab} m_b\Theta_a}{m_a\Theta_b+m_b\Theta_a}(\mathbf{u}_a-\mathbf{u}_b)^2+\zeta^a, \quad (10)$$

$$(\zeta^a\zeta^b)_{\omega\mathbf{k}}=6\sum_c \nu_{ac}\frac{n_a\mu_{ac}}{m_c}\Theta_a\Theta_c(\delta_{ab}-\delta_{bc}), \quad (11)$$

from which it follows that

$$(\zeta^e\zeta^e)_{\omega\mathbf{k}}=(\zeta^i\zeta^i)_{\omega\mathbf{k}}=-(\zeta^e\zeta^i)_{\omega\mathbf{k}}=3\nu_{ei}^T n_e\Theta_e\Theta_i,$$
$$\nu_{ei}^T=2\nu_{ei}\frac{m_e}{m_e+m_i},\qquad \sum_{ab}(\zeta^a\zeta^b)_{\omega\mathbf{k}}=0. \quad (12)$$

Putting in (5)

$$\varphi_a=m_a(\delta v_{1i}^a\delta v_{1j}^a-{}^1/_3\delta_{ij}\delta v_1^{a2}),\quad \psi_b=m_b(\delta v_{2k}^b\delta v_{2l}^b-{}^1/_3\delta_{kl}\delta v_2^{b2}),$$

we obtain an expression for the extraneous stress-tensor intensity in the form

$$(\delta\pi_{ij}^a\delta\pi_{kl}^b)_{\omega\mathbf{k}}=\frac{\eta_a\eta_b}{P_aP_b}\int\varphi_a(\mathbf{p}_1)\psi_b(\mathbf{p}_2)(y_a y_b)_{\omega\mathbf{k}\mathbf{p}_1\mathbf{p}_2}d\mathbf{p}_1\, d\mathbf{p}_2$$
$$=\Delta_{ij}^{kl}\frac{\eta_a\eta_b}{P_aP_b}\sum_c \nu_{ac}\frac{8}{5}\frac{\Theta_a\Theta_c}{m_c}\mu_{ac}n_a\left(\delta_{ab}-\delta_{bc}+\frac{3}{2}\frac{\Theta_{ac}m_c}{\mu_{ac}\Theta_c}\right),$$

$$\Delta_{ij}^{kl}=(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-{}^2/_3\delta_{ij}\delta_{kl}). \quad (13)$$

Substituting the values of the viscosity coefficients of a two-component plasma

$$\eta_e={}^5/_6P_e/(\nu_{ee}+\nu_{ei}),\quad \eta_i={}^5/_6P_i/\nu_{ii},$$

we get

$$(\delta\pi_{ij}^e\delta\pi_{kl}^e)_{\omega\mathbf{k}}=2\Delta_{ij}^{kl}\eta_e\Theta_e, \quad (14)$$
$$(\delta\pi_{ij}^i\delta\pi_{kl}^i)_{\omega\mathbf{k}}=2\Delta_{ij}^{kl}\eta_i\Theta_i, \quad (15)$$
$$(\delta\pi_{ij}^e\delta\pi_{kl}^i)_{\omega\mathbf{k}}=2\Delta_{ij}^{kl}\eta_i\Theta_e P_e/P_i, \quad (16)$$
$$(\delta\pi_{ij}^i\delta\pi_{kl}^e)_{\omega\mathbf{k}}=2\Delta_{ij}^{kl}\eta_e\Theta_i P_i/P_e. \quad (17)$$

We obtain similarly an expression for the intensity of the extraneous heat-flux vector in a multicomponent plasma:

$$(\delta q_k^a\delta q_l^b)_{\omega\mathbf{k}}$$
$$=\frac{\lambda_a\lambda_b m_a m_b}{25\varkappa_B^2 P_aP_b}\int\delta v_{1k}^a\delta v_{2l}^b\left(\delta v_1^{a2}-\frac{5\Theta_a}{m_a}\right)\left(\delta v_2^{b2}-\frac{5\Theta_b}{m_b}\right)$$
$$\times(y_a y_b)_{\omega\mathbf{k}\mathbf{p}_1\mathbf{p}_2}d\mathbf{p}_1\, d\mathbf{p}_2=2\frac{\lambda_a\lambda_b m_a m_b}{25\varkappa_B^2 P_aP_b}\delta_{kl}\sum_c \nu_{ac}\frac{n_a m_a}{(m_a+m_b)^2}$$
$$\times\frac{\Theta_a^2\Theta_c^2}{\Theta_{ac}}\left[25(\delta_{ab}-\delta_{bc})+\frac{(m_a\Theta_c+m_c\Theta_a)(3m_a\Theta_c+13m_c\Theta_a)}{m_a^2\Theta_c^2}\right] \quad (18)$$

($\varkappa_B$ is the Boltzmann constant). In an electron–ion plasma

$$\lambda_e=\frac{P_e 25\varkappa_B}{m_e(13\nu_{ei}+8\nu_{ee})},\quad \lambda_i=\frac{25P_i\varkappa_B}{8\nu_{ii}m_i}.$$

Thus,

$$(\delta q_k^e\delta q_l^e)_{\omega\mathbf{k}}=2\delta_{kl}\lambda_l\Theta_e^2/\varkappa_B, \quad (19)$$
$$(\delta q_k^i\delta q_l^i)_{\omega\mathbf{k}}=2\delta_{kl}\lambda_i\Theta_i^2/\varkappa_B, \quad (20)$$

$$(\delta q_k^e \delta q_l^i)_{\omega\mathbf{k}} = 2\lambda_i \delta_{kl} \Theta_e^2 P_e m_e / \varkappa_B P_i m_i, \tag{21}$$

$$(\delta q_k^i \delta q_l^e)_{\omega\mathbf{k}} = 2\delta_{kl} \lambda_e \Theta_i^2 P_i m_i / \varkappa_B P_e m_e. \tag{22}$$

It follows from (14), (15), (19), and (20) that the correlators of the extraneous quantities within any one component are described by Landau–Lifshitz equations that differ only in the fact that the dissipative parameter, be it λ or η, is determined both by collisions within the component and by collisions of different particle species. So definite a result could not possibly be obtained without taking into account the nonequilibrium terms I_{ab} and $\delta_{ab}\delta(p_1 - p_2) I_a$ in the expression for the source intensity in the kinetic equation. Without these terms, the expressions for the source intensities ξ^a and ζ^a in the nonequilibrium state would likewise not be as compact and lucid.

Note that in multicomponent systems there exists also a cross correlation of the sources in the balance equations for the velocity and for the extraneous heat-flux tensor

$$\begin{aligned}(\xi_k^a \delta q_l^b)_{\omega\mathbf{k}} &= \frac{\lambda_b}{5\varkappa_B P_b} \int m_a \delta v_{1k}{}^a m_b \delta v_{2l}{}^b \left(\delta v_2^{b2} - \frac{5\Theta_b}{m_b} \right) \\ &\times (y_a y_b)_{\omega\mathbf{k}\mathbf{p}_1\mathbf{p}_2} d\mathbf{p}_1 \mathbf{p}_2 \\ &= \frac{\lambda_b}{25\varkappa_B P_e} \sum_c \nu_{ac} \frac{6n_a}{m_a + m_c} m_c \Theta_a^2 \delta_{kl} (\delta_{bc} - \delta_{ab}).\end{aligned} \tag{23}$$

In a two-component plasma $(a,b = e,i)$

$$(\xi_k^e \delta q_l^e)_{\omega\mathbf{k}} = -6\delta_{kl} \nu_{ei} P_e \Theta_e / m_e (13\nu_{ei} + 8\nu_{ee}), \tag{24}$$

$$(\xi_k^i \delta q_l^i)_{\omega\mathbf{k}} = -30\delta_{kl} P_i \Theta_i m_e \nu_{ie} / 8 m_i m_i \nu_{ii}, \tag{25}$$

$$(\xi_k^e \delta q_l^i)_{\omega\mathbf{k}} = 30\delta_{kl} \nu_{ei} P_e \Theta_e / 8\nu_{ii} m_i, \tag{26}$$

$$(\xi_k^i \delta q_l^e)_{\omega\mathbf{k}} = 30\delta_{kl} \Theta_i P_i \nu_{ie} / m_i (8\nu_{ce} + 13\nu_{ei}) \tag{27}$$

The reason for this fact is that Eq. (7) for the momentum density contains both the relaxation of the average velocity and the relaxation of the heat-flux vector. A similar dissipation is present also in the equation for the heat-flux vector. The contribution of the heat flux $\mathbf{q}_k^a$ to the equation of motion means allowance for the second Chapman–Enskog approximation. As shown first by Braginskiĭ,[7] allowance for the higher Chapman–Enskog approximations leads to corrections to the kinetic coefficients. Exact values of the coefficients were obtained by Spitzer and Härm[8] by numerically solving an integral equation. Thus, for example, for a fully ionized plasma with singly charged ions, the Spitzer results lead to a numerical factor 0.51 in the expression for the friction coefficient. In the second Chapman–Enskog approximation this factor is 0.52 (Ref. 7). The linkage of the sources in the hydrodynamic equations leads, by analogy with the Spitzer renormalization of the kinetic coefficients, to a renormalization of the intensity of the Langevin sources. Combining the source ξ^a with $\delta\mathbf{q}^a$ in Eq. (7):

$$\tilde{\xi}_k^a = \xi_k^a + \frac{3}{5} \sum_c \frac{\mu_{ac}}{\Theta_{ac}} \nu_{ac} \left(\delta q_k^a - \frac{m_a}{m_c} \frac{n_a}{n_c} \delta q_k^c \right), \tag{28}$$

we have for a two-component plasma $(a,b,c = e,i)$, for example,

$$(\tilde{\xi}_k^e \tilde{\xi}_l^e)_{\omega\mathbf{k}} = 2\delta_{kl} m_e n_e \Theta_e \nu_{ei} \cdot 0.52,$$

$$(\xi_k^e \delta q_l^e)_{\omega\mathbf{k}} = 0. \tag{29}$$

The intensity of a Langevin source in the equation of motion is thus determined by a dissipative parameter with a Spitzer correction coefficient. Similar correction factors will appear also in expressions (14), (15), (19), and (20) when account is taken of higher moments.

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