Chaos and two-dimensional random walk in periodic and quasiperiodic fields

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(Submitted 19 December 1988)
Zh. Eksp. Teor. Fiz. **95**, 1723–1733 (May 1989)

The chaotic dynamics of a particle in two-dimensional periodic potentials or in potentials with quasicrystalline symmetry is investigated. It is shown that under conditions of deterministic chaos the infinite motion of a particle in a periodic hexagonal potential has the character of "Levy flights," i.e., diffusion motion alternating with periods of almost free motion. A connection is established between the random walk of the particle and diffusion in multifractals. In potentials with higher degree of symmetry (of the quasicrystalline type) the random walk is close to the ordinary diffusion process.

1. INTRODUCTION

The motion of a particle in a one-dimensional potential $V(x)$ is an integrable problem, but starting already with a two-dimensional potential $V(x, y)$ the motion is nonintegrable in general. Regions of stochastic dynamics are present in phase space. Their location and extent are determined by the form of the potential $V(x, y)$ and the values of the parameters in the potential. We write the Hamiltonian for two-dimensional motion in the form

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+V(x, y). \tag{1.1}$$

It may define finite as well as infinite motion. If the stochastic dynamics arises in the region of the finite motion, the chaotic regions are localized in a finite region of the (x, y) space. In the case of infinite dynamics, however, chaos means the appearance of random walk of the particle in space, and this phenomenon has numerous and important applications, which only recently have become an object of study.

One of the clearest models of motion in a periodic potential is provided by the Lorentz gas: a point particle moving among periodically distributed hard discs and reflected from their boundaries upon scattering according to the law of perfectly elastic collision. Such a system is a version of billiards, in which the dynamics of a particle has been shown to be stochastic.[1] The random-walk law of a particle in a periodic triangular Lorentz gas corresponds to ordinary diffusion, provided only that the density of the discs is sufficiently high.[2]

A more complicated potential with threefold symmetry was considered in Ref. 3:

$$V(x, y)=\cos(x+y/3^{1/2})+\cos(x-y/3^{1/2})+\cos(2y/3^{1/2}). \tag{1.2}$$

In this potential there also exists a proper internal stochastic particle motion, not due to the action of random perturbations. It was also shown in Ref. 3 that for energies $E<E_c\approx -0.4$ the random walk is diffusive, while for $E>E_c$ its character is more complicated. However, as will be seen below, the result that a finite diffusion region exists is not confirmed. This is due to the action of chaos mechanisms in such models. Certain other periodic potentials were considered in Refs. 4 and 5.

The appearance of chaos in the dynamics of particles in periodic potentials has numerous applications in various physical problems. Interacting multiplets of waves in a plasma produce very similar conditions for the motion of particles.[6,7] Similar problems occur in the analysis of the motion of passive particles when hydrodynamic or magnetohydrodynamic structures are present.[8–11] To this list one may add the huge range of problems on the motion of adatoms above the surface of solids. A recent paper of the authors[13] indicates one more area in which random walk due to chaos in periodic fields plays an important role: this is Lagrangian turbulence in flows with symmetry or quasisymmetry.

In this way we face in effect a new type of problem with many applications. It consists in the need to investigate the basic properties of dynamic chaos due to motion in a 2D potential. For the potential we will choose one that is endowed with q-symmetry[13,14]:

$$V_q(\mathbf{r})=V_0\sum_{j=1}^{q}\cos(\mathbf{r}\mathbf{e}_j), \tag{1.3}$$

where $\mathbf{r}=(x, y)$, and $\mathbf{e}_j$ is a system of unit vectors forming a regular star. For $q=4$

$$V(\mathbf{r})=2V_0(\cos x+\cos y) \tag{1.4}$$

and we arrive at a potential forming a square grid. For $q=3$ and $q=6$ this potential is equivalent to (1.2). For other values of q ($q\neq 1,2,3,4,6$) the potential (1.3) has quasicrystalline symmetry.[14] It is almost periodic.

The main results of our paper reduce to the following. For nontrivial q-values ($q\neq 1,2,4$) the motion of a particle in the field $V(\mathbf{r})$, Eq. (1.3), is stochastic and unbounded for energy values in a certain region ΔE. The random walk of the particle takes place inside a certain spatial region forming a regular web in space.

In the article we find the dependence of the random walk velocity on the values of the particle energy for various symmetries of the potential $V(\mathbf{r})$. It turns out that the random walk has a multifractal character, has manifest intermittency properties, and may be formalized with the help of a generalization of Levy's random walk.[15–17]

2. CHAOS IN AN HEXAGONAL POTENTIAL

The dynamics in an hexagonal potential is determined by the Hamiltonian (1.1) with the potential (1.3) for $q=3$:

$$H_3=\frac{1}{2}(p_x^2+p_y^2)+\sum_{j=1}^{3}\cos\left(x\cos\frac{2\pi}{3}j+y\sin\frac{2\pi}{3}j\right). \tag{2.1}$$

The potential V_3 has saddle points for values of the energy $E_s=-1$. For $E<1$ the motion of the particle is confined to

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one of the potential wells. The motion near the bottom of the well is found by expanding V_3 about the minimum where $V_3 = -1.5$. We have

$$V_3 \approx -3/2 + 3(x^2+y^2)/8 + 3^{1/2}x(3y^2-x^2)/16 \qquad (2.2)$$

accurate to terms small to higher order. The potential (2.2) is analogous to the Henon-Heiles model.[18] The stochastic dynamics of the particle is due to the nonlinear coupling of two degrees of freedom in (2.1) and (2.2). Therefore chaos is possible even for $E < E_s$ (Fig. 1). In that case, however, the particle does not execute random walk along the potential grid.

For $E_3 = -1$ special trajectories of the system (separatrices) pass through the saddle points and cover the plane by a triangular net. If $E > E_s$, it becomes possible for the particles to "percolate," with their spread in the (x, y) plane becoming unbounded. Such percolation may be random as well as regular in character. This depends on the particle energy. In Fig. 2 an example if shown of a random walk in the (x, y) plane for $E = -0.95$. In Fig. 2a equipotential lines for $V_3(x, y)$ are shown, i.e., points satisfying the condition $V_3(x, y) = -0.95$. The region bounded by them determines the random-walk zone, which is given in Fig. 2b. Every point on it refers to one and the same trajectory at an instant time when the trajectory crosses the (x, y) plane with $p_x = 0$ and $p_y > 0$ (Poincaré section). A magnified detail of the Poincaré section is shown in Fig. 2c. Analogous results for $E = 0.01$ for equipotential lines for $V_3(x, y)$ and Poincaré sections of the trajectories in the (x, y) plane are shown in Fig. 3a and 3b, respectively.

The energy value $E_c \approx 5.5$ turns out to be critical and for $E > E_c$ chaotic random walk has not been observed. In general the energy region in which stochastic dynamics occurs is determined with the help of the Kolmogorov-Sinai entropy h. Let $d(t)$ be the distance between two trajectories in phase space

$$d(t) = (|x_1-x_2|^2 + |y_1-y_2|^2 + |p_{x_1}-p_{x_2}|^2 + |p_{y_1}-p_{y_2}|^2)^{1/2},$$

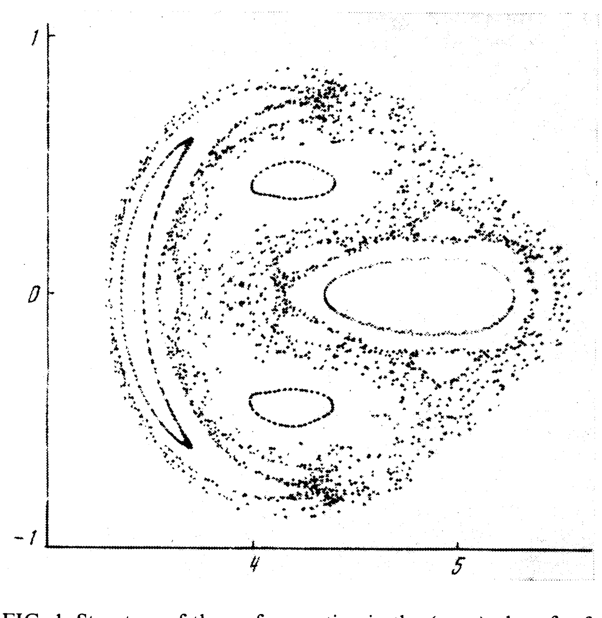

FIG. 1. Structure of the surface section in the (x,p_x) plane for finite motion in the hexagonal potential, $E = -1.1 < E_s$.

where the coordinates x_i, y_i and momenta p_{x_i}, p_{y_i} are taken at the time instant t. Then the quantity h is defined by the expression

$$h = \lim_{t\to\infty} \lim_{d(0)\to 0} \frac{1}{t} \left\langle \ln \frac{d(t)}{d(0)} \right\rangle, \qquad (2.3)$$

where the averaging is over different trajectories. In the calculations we have taken $d(0) = 10^{-6}$ and the time t not very large. The results of the calculations are shown in Fig. 4 (solid curve). The region of existence of chaos is fairly broad and exceeds significantly the height of the potential barrier $V_3 = 3$.

Let us stop to consider the particle's random walk in the plane in the case when its energy is in the region in which chaotic motion and percolation are allowed. Let $r = (x^2 + y^2)^{1/2}$. Numerical calculations show the random-walk law to be

$$\langle r^2 \rangle = Dt^{1+\beta}, \qquad (2.4)$$

where D and β are some constants depending on the value of the energy E. For $\beta = 0$ we have ordinary random walk with Gaussian distribution up to the point r. However, as can be seen from the values of D and β below, ordinary diffusion is absent:

$$E=-0.95,\ \beta=0.33,\ D=1.9\cdot 10^{-3},$$
$$E=0.01,\ \beta=0.42,\ D=3.5\cdot 10^{-2},$$
$$E=2.0,\ \beta=0.46,\ D=2.2\cdot 10^{-1},$$
$$E=4.0,\ \beta=0.26,\ D=8.6. \qquad (2.5)$$

In the case of free motion the parameter β in Eq. (2.4) equals 1. Therefore the results (2.5) indicate that the random walk in the potential $V_3(x, y)$ is intermediate between free motion and diffusion. The best description of this motion is provided by Fig. 5. It shows points (particle positions) at instants of time corresponding to the Poincaré section (i.e., positions at times when $p_x = 0$, $p_y > 0$). Long flights of the particle can be seen, whose length reaches $\sim 300\pi$. The existence of such flights allows the classification of the process as Levy random walk ("Levy flights"). The reason for the appearance of long flights is the presence in the plane of channels along which the potential $V_3(x, y)$ has very little influence on the dynamics of the particle. For example, for the Lorentz gas the existence of such channels is obvious. In turn such channels appear as a result of the symmetry of the problem. We shall discuss Levy flights in Sec. 4.

3. CHAOS IN A POTENTIAL WITH QUASISYMMETRY

The last remark of the preceding section indicates the important role played by symmetry in the character of stochastic dynamics. As a next step we consider the almost-periodic potential V_q defined by Eq. (1.3). For $q = 5,7,8,...$ it defines a field with quasicrystalline symmetry.[13,14,19] We give below numerical results for $q = 5$. The dashed curve in Fig. 4 shows the dependence of the entropy h on the energy E. Chaos exists in a very broad interval from $E = -1$ to $E \sim 27$, although max $V_5(x, y) = 5$. For $E = 1$ we have a

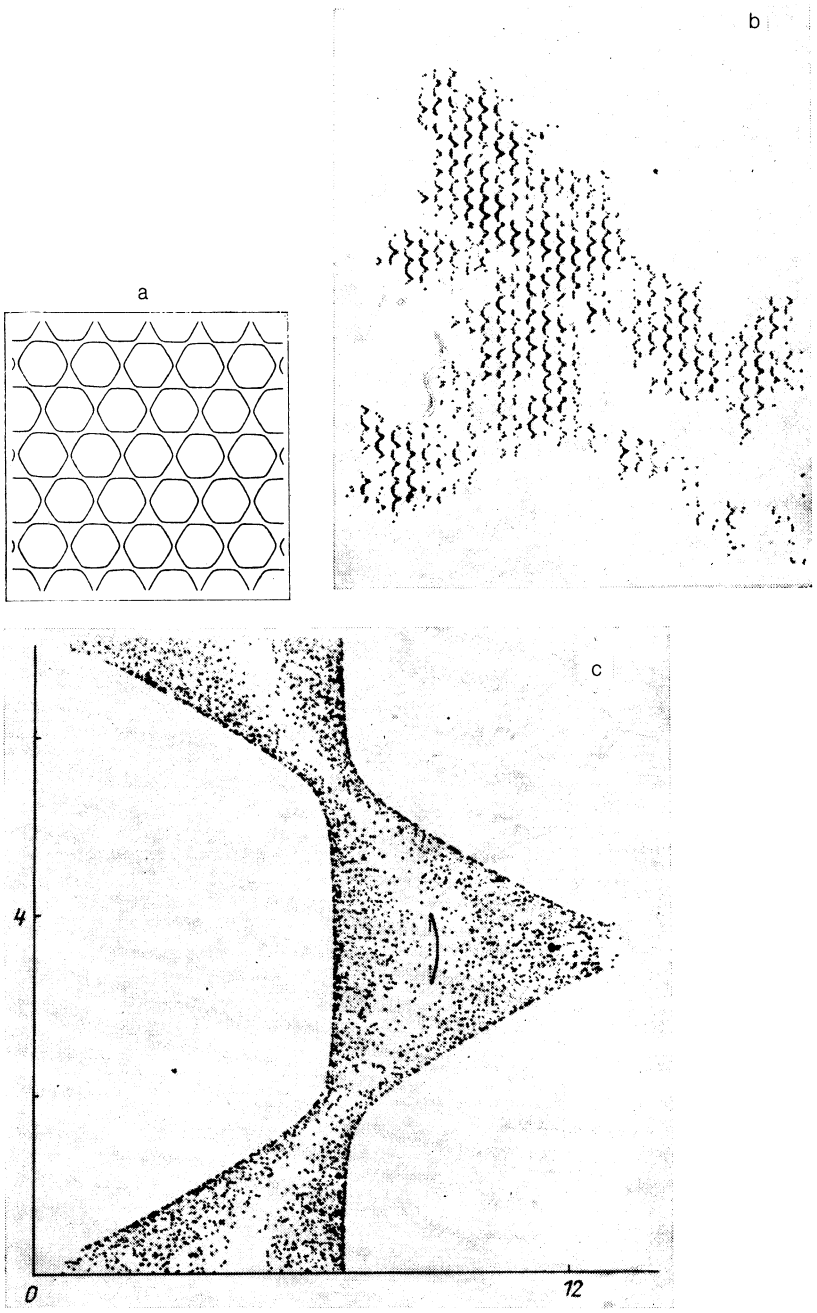

FIG. 2. Spatial diffusion in the hexagonal potential for $E = -0.95 > E_s$: a) level lines for the potential (1.2). Square size $20\pi \times 20\pi$; b) structure of the surface section in the (x, y) plane. Square size $70\pi \times 70\pi$; c) structure of the surface section of one cell in the (x, y) plane.

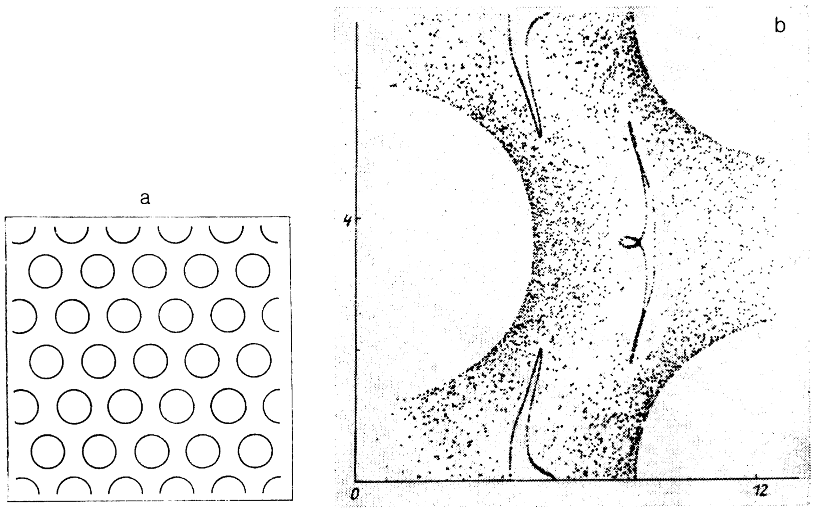

FIG. 3. Infinite motion in the hexagonal potential for $E = 0.01 > E_s$: a) potential lines; b) structure of the surface section of one cell in the (x, y) plane.

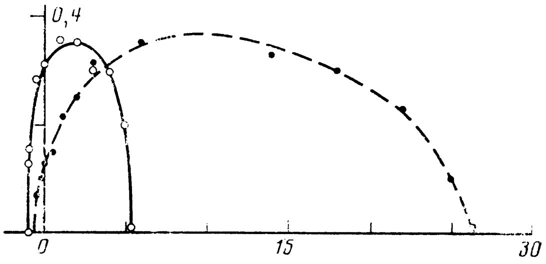

FIG. 4. Kolmogorov-Sinai entropy as a function of the particle energy in the potential (1.3); $q = 3$—solid line, $q = 5$—dashed line.

potential barrier encompassing the center (Fig. 6). Therefore the random walk is bounded by the region of this barrier. For $E > 1$, however, percolation occurs, resulting in unbounded random walk in the plane.

For the random walk law (2.4) we obtain $\beta = 0.02$ for $E = 1.13$. This shows that it is very close to ordinary Brownian motion. The quasicrystalline symmetry is substantially higher than crystalline,[14,19] and this is right away reflected in the disappearance of long flights of the particle.

4. LEVY FLIGHTS ON MULTIFRACTALS

A peculiarity of the Levy process consists of the significantly faster excursion of the particle ("flight") than in the case of Brownian motion. Therefore the quantity $\langle r^2 \rangle / t$ is proportional to t^α ($\alpha > 0$), and not to a constant ($d = 0$). This means that the quadratic variation of the coordinate in one step of the random walk tends to infinity with increasing t, i.e., as the particle moves farther and farther away. The formal expression of this fact arises as follows.[20,21]

Let ξ be a vector determining the state of the particle in

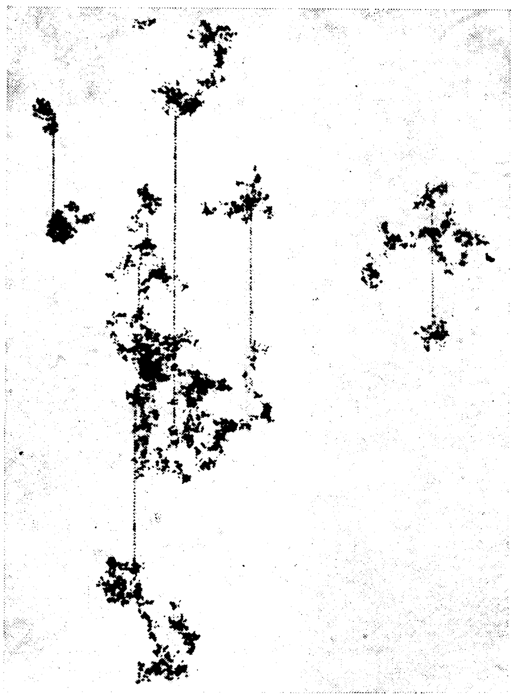

FIG. 5. Chaotic random walk with intermittency in a hexagonal potential. Structure of the surface section in the (x, y) plane for $E = 2.0$. Square size $1400\pi \times 1400\pi$.

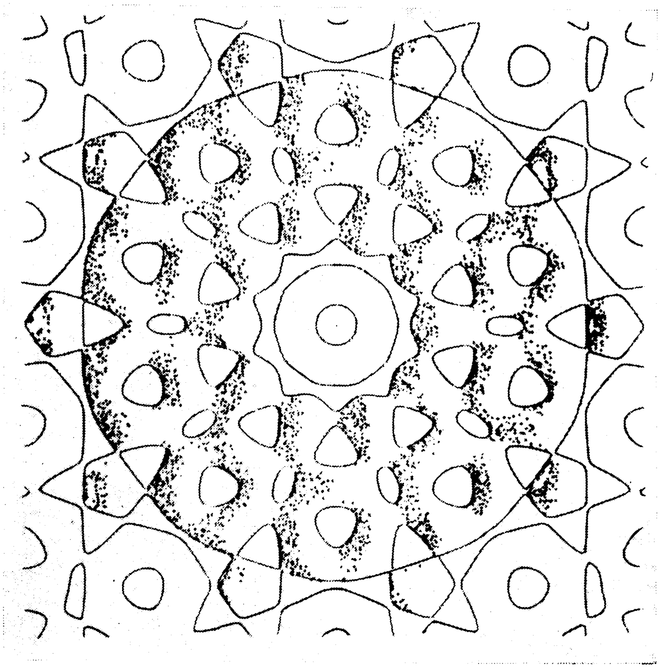

FIG. 6. Diffusion localization by the ring-like energy barrier of the potential V_5. Particle energy $E = 0.99$. Square size $20\pi \times 20\pi$.

the random-walk space, and $p_t(\xi)$ the probability that the value ξ is realized at the time instant t. The random-walk equation is of the form

$$p_{t+1}(\xi) = \sum_{\xi'} W(\xi - \xi') p_t(\xi'), \tag{4.1}$$

where $W(\xi - \xi')$ is the transition probability, depending only on the difference of the states. We make use of the Fourier expansions

$$\begin{aligned} p_t(\xi) &= \frac{1}{2\pi} \int_0^{2\pi} dk\, e^{-ik\xi} p_t(k); \\ W(\xi) &= \frac{1}{2\pi} \int_0^{2\pi} dk\, e^{-ik\xi} W(k), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} p_t(k) &= \sum_{\xi} e^{ik\xi} p_t(\xi), \\ W(k) &= \sum_{\xi} e^{ik\xi} W(\xi). \end{aligned} \tag{4.3}$$

With the help of the relations (4.2) and (4.3) the random-walk equation (4.1) takes the form

$$p_{t+1}(k) = W(k) p_t(k). \tag{4.4}$$

It is readily solved and gives with the initial condition

$$p_0(\xi) = \delta(\xi)$$

the result

$$p_t(k) = [W(k)]^t = \exp\{t \ln[1 + kW_k'(0) + k^2 W_{kk}''(0)/2 + \ldots]\}. \tag{4.5}$$

It is not hard to see from (4.3) that

$$W_k'(0)=i\sum_{\xi}\xi W(\xi)=i\langle\xi\rangle,$$
$$W_{kk}''(0)=-\langle\xi^2\rangle. \tag{4.6}$$

Without loss of generality we may set $W_k'(0)=0$ and then (4.5) becomes, with (4.6) taken into account,

$$p_t(k)=\exp\{-tk^2\langle\xi^2\rangle/2+\ldots\}. \tag{4.7}$$

Introduction of (4.7) into (4.2) gives the answer. For $t\to\infty$ the contribution to the integral (4.2) for $p_t(\xi)$ is determined by the limit $k\to 0$ so that the expression (4.7) is sufficient to obtain the well-known answer

$$p_t(\xi)=p_0\exp(-\xi^2/2t\langle\xi^2\rangle). \tag{4.8}$$

However, for processes when "flights" are possible the expansion for $W(k)$ has an entirely different form:

$$W(k)=\exp(-Ck^\alpha)\quad(k>0), \tag{4.9}$$

where α and C are some constants and the remaining terms in the expansion are omitted to simplify the notation.

If the exponent α is contained between the limits $1<\alpha<2$ then formula (4.9) determines a random walk in the velocity interval between free motion ($\alpha=1$) and diffuse motion ($\alpha=2$). Now (4.5) takes the following form

$$p_t(k)\sim\exp(-tCk^\alpha) \tag{4.10}$$

where (4.9) and the first of Eqs. (4.6) were taken into account, and α corresponds to the dimension of the fractal consisting of the points of the random walk.[21]

Using the first formula (4.2) and (4.10) we obtain

$$p_t(\xi)=\int_0^{2\pi}dk\exp(-ik\xi-tCk^\alpha). \tag{4.11}$$

From here follows the similarity law

$$\langle\xi^\alpha\rangle\propto t. \tag{4.12}$$

For $\alpha=2$ formula (4.12) defines diffusion, for $\alpha=1$ it defines free motion, and for $1<\alpha<2$ it defines Levy random walk in a fractal.

In reality the situation with respect to random walks in a two-dimensional potential $V(x,y)$ is more complicated. Real systems experiencing chaotic dynamics are not fractal but multifractal (see Refs. 22–24). This means that in formula (4.9) for the transition probability not one but at least several values for the exponent α appear. Let us carry out the corresponding generalization.

Let the random walk of the particle proceed in some multifractal manifold S. In rough terms this means the following. The full trajectory of the system may be broken up into certain segments, each of which belongs to a submanifold (fractal) S_α. The manifold S consists of all the fractals S_α. We shall suppose that these segments of the trajectory are sufficiently long and will ignore transitions from one fractal S_α to another $S_{\alpha'}$. Then the random walk equation (4.1) may be replaced by the following equation:

$$p_{t+1}(\xi)=\sum_{\xi'}W(\xi-\xi',\alpha_t)p_t(\xi'), \tag{4.13}$$

where α_t is the characteristic of the fractal exponent in the transition probability $W(k,\alpha_t)$ at the instant time t:

$$W(k,\alpha_t)=\exp[-C(\alpha_t)k^{\alpha_t}]\quad(k>0). \tag{4.14}$$

The expression (4.14) is the generalization of formula (4.9).

Performing in (4.13), as before, a Fourier transform in ξ we obtain analogously to (4.5)

$$p_t(k)=\prod_{j=1}^{t}W(k,\alpha_j)=\exp\Big[\sum_{j=1}^{t}\ln W(k,\alpha_j)\Big]$$
$$=\exp\Big[-\sum_{j=1}^{t}C(\alpha_j)k^{\alpha_j}\Big]\quad(k>0). \tag{4.15}$$

The summation in the argument of the exponential in (4.15) can be carried out in the spirit of the general theory of averaging on multifractals.[23–25] To this end the terms in the sum should be regrouped so as to unite all those that have the same value of α_j, i.e., that belong to the same fractal.

Let us denote by n_α the total number of steps in the random walk which proceed in the fractal S_α in the sum in (4.15). We may then introduce the number $\rho(\alpha)$ of fractals whose exponent lies in the interval $(\alpha,\alpha+d\alpha)$. We have, by definition of n_α:

$$t=\sum_{n_\alpha}n_\alpha=t\sum_{\alpha}n_\alpha/t=t\int_{\alpha_1}^{\alpha_2}d\alpha\,\rho(\alpha). \tag{4.16}$$

Formula (4.16) shows how to introduce integration over the normalized density $\rho(\alpha)$, and (α_1,α_2) is the range of variation of the exponent α. In analogy with (4.16) we obtain from (4.15)

$$p_t(k)=\exp\Big[-\sum_{\alpha}n_\alpha C(\alpha)k^\alpha\Big]$$
$$=\exp\Big[-t\int_{\alpha_1}^{\alpha_2}d\alpha\,\rho(\alpha)C(\alpha)k^\alpha\Big]\quad(k>0). \tag{4.17}$$

If the function $C(\alpha)$ does not change sign in the region (α_1,α_2) then (4.17) can be represented in the form

$$p_t(k)=\exp(-\mathrm{const}\,tk^{\bar\alpha}), \tag{4.18}$$

where $\bar\alpha$ is some average value of α.

Expression (4.18) determines the asymptotic random walk law

$$\langle|\xi|\rangle\sim t^{1/\bar\alpha}. \tag{4.19}$$

If, for example, there are altogether several different fractals S_α forming the full manifold S, then

$$\rho(\alpha)=\sum_s\rho_s\delta(\alpha-\alpha_s).$$

Therefore (4.17) takes on the form

$$p_t(k)=\exp\Big(-t\sum_{s=1}^{N}C_sk^{\alpha_s}\Big), \tag{4.20}$$

where C_s are some new constants. Expression (4.20) determines several different intermediate asymptotics

$$\langle|\xi\rangle|\propto t^{1/\alpha_s}. \tag{4.21}$$

To smaller values of α_s correspond asymptotics which occur at longer times. Therefore for $t\to\infty$ the asymptotic that survives is

$$\langle |\xi| \rangle \propto t^{1/\min \alpha_s}. \quad (4.22)$$

Formulas (4.21) and (4.22) show why there exists a great variety of random walk exponents, depending on the nature of the problem. In particular, the observed strong dependence on the particle energy and the symmetry of the potential is naturally explained by differences in the chaotic dynamics. The determination of the relation between the exponents α and the distributions $\rho(\alpha)$ with the concrete properties of the dynamic system presents a much more complex task.

CONCLUSION

In conclusion we note what are, in our opinion, the principal results of general character of this work. The numerical analysis of the particle dynamics in a periodic hexagonal potential has shown the existence of the anomalous nature of the random-walk process, accompanied by the intermittency property. The process under consideration is significantly different in this way from the ordinary model of Brownian motion. Investigation of such random walks is needed for an understanding of the properties of dynamic chaos and turbulence in liquids and plasma.

The intermittency phenomenon in nonintegrable systems is related to the multifractal nature of the dynamics of such systems. The phase space of the Hamiltonian system is constructed in an extraordinarily complicated manner. It contains an infinite hierarchy of different regions of regular and chaotic motion and cantori. Cantori are fractal objects, which appear in place of distorted invariant tori. They are localized in the chaotic region, mainly near the boundary with a region of regular motion, and play the role of a kind of trap or barrier that is hard to traverse by a particle. Upon penetration of such a barrier a particle may find itself for the duration of a rather long time in a regime of almost free motion. Averaged over a long interval of time the motion becomes as if intermediate between diffusion and free. It is important to emphasize that anomalous processes of this kind, connected with the multifractal nature of the dynamics of nonintegrable systems, are to a larger or lesser extent common to a majority of dynamical systems with chaos. Therefore the investigation of the properties of the random walk permits a deeper understanding of certain fine points of deterministic chaos.

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Translated by Adam M. Bincer