Determination of ultrarelativistic amplitudes and differential cross sections for $e^{\pm} e^{-} \to e^{\pm} e^{-} \gamma$ processes

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(Submitted 13 December 1988)
Zh. Eksp. Teor. Fiz. **95**, 1921–1928 (June 1989)

The differential cross sections for $e^{\pm} e^{-} \to e^{\pm} e^{-} \gamma$ processes are determined in the ultrarelativistic limit for helicity-polarized initial leptons and photon. It is shown that each cross section can be written as the product of two factors, one of which is universal and is identical with the factor obtained previously in the absence of polarization. The analysis is based on the direct evaluation of the matrix elements in the diagonal spin basis.

There have been a number of investigations of bremsstrahlung processes accompanying the scattering of electrons and positrons by electron.[1–5] Processes involving the emission of hard protons are of interest because they constitute the background for studies of hadronic processes. Moreover, they provide us with a possible test of quantum electrodynamics (QED) in high perturbation–theory orders. In this paper, we report a determination of the differential cross sections for the processes $e^{\pm} e^{-} \to e^{\pm} e^{-} \gamma$ for helicity-polarized initial leptons and proton. The initial and final electrons and positrons are assumed massless. It will be shown below that each cross section for these processes can be written as the product of two factors, one of which is universal and is identical with the factor obtained previously in the absence of polarization.

1. The determination of the cross sections is based on the direct evaluation of QED matrix elements[6–9] in the diagonal spin basis (DSB).[10–12] The essence of the method is as follows (we shall use the notation employed in Ref. 7). In DSB, the spin vectors $s_1 = (\mathbf{s}_1, is_{10})$ and $s_3 = (\mathbf{s}_3, is_{30})$ of a particle participating in the $1 + 2 \to 3 + 4$ reaction with 4-velocity $v_1 = p_1/m$ (prior to interaction) and $v_3 = p_3/m$ (after) interaction are defined by

$$s_1 = \frac{(1+v_1\cdot v_1)v_3}{[(v_1v_3)^2-1]^{1/2}}, \quad s_3 = -\frac{(1+v_3\cdot v_3)v_1}{[(v_1v_3)^2-1]^{1/2}}. \tag{1}$$

The dot between the vectors in (1) and elsewhere denotes a dyad: $(a\cdot b)_{ij} = a_i b_j$ $(i, j = 1, 2, 3, 4)$. Let us introduce the orthonormal basis of vectors n_A (Ref. 11) $n_A n_B = \delta_{AB}$ $(A, B = 1, 2, 3, 4)$:

$$n_1=[n_0\cdot n_3]^{\times}n_2,\quad n_2=[p_1\cdot p_3]^{\times}p_2/\rho, \tag{2}$$

$$n_3=(p_3-p_1)/[(p_3-p_1)^2]^{1/2},\quad n_4=in_0,$$

$$n_0=(p_3+p_1)/[-(p_1+p_3)^2]^{1/2}.$$

where $[a\cdot b] = a\cdot b - b\cdot a = \alpha$, $\tilde{\alpha} = -\alpha$, $\alpha^{\times}_{kl} = i\varepsilon_{klmn}\alpha_{mn}/2$, ε_{klmn} is the Levi-Civita symbol, and ρ is determined from the normalization condition. The following relations[11,12] are valid for the quadruple of vectors n_A in (2):

$$[n_A\cdot n_B]^{\times}n_C = i\varepsilon_{ABCD}n_D. \tag{3}$$

In the above spin basis, the spin component operators σ_1 and σ_3, and also the raising and lowering operators $\sigma_1^{\pm\delta}$ and $\sigma_3^{\pm\delta}$ for the initial and final particles are identical and take the following form for Dirac particles[8,9,12]:

$$\sigma = \sigma_1 = \sigma_3 = \gamma^5\hat{s}_1\hat{v}_1 = \gamma^5\hat{s}_3\hat{v}_3 = \gamma^5\hat{n}_3\hat{n}_0 = i\hat{n}_2\hat{n}_1, \tag{4}$$

$$\sigma^{\pm\delta}=\sigma_1^{\pm\delta}=\sigma_3^{\pm\delta}=i\gamma^5(\hat{n}_1\pm i\delta\hat{n}_2)/2,\quad \delta=\pm1, \tag{5}$$

$$\sigma u^{\delta}(p)=\delta u^{\delta}(p),\quad \sigma^{\pm\delta}u^{\mp\delta}(p)=u^{\pm\delta}, \tag{6}$$

where $\hat{n} = n_k\gamma^k$, γ^k, γ^5 are the Dirac matrices and $u^{\delta}(p)$ are bispinors satisfying the equation

$$(i\hat{p}+m)u^{\delta}(p)=0. \tag{7}$$

We note that the coincidence of the spin operators for the initial and final particles is a consequence of the realization in DSB of the small Lorentz group that is common to particles 1 and 3 (Refs. 7 and 10). The bispinors of the initial and final states $u^{\delta}(p_1)$ and $u^{\delta}(p_3)$ can be related to one another with the aid of the transition operators T_{31} and $T_{13} = T_{31}^{-1}$ (Refs. 6–8):

$$u^{\delta}(p_3)=T_{31}u^{\delta}(p_1),\quad \bar{u}^{\delta}(p_3)=\bar{u}^{\delta}(p_1)T_{13}, \tag{8}$$

where $\bar{u}^{\delta}(p) = (u^{\delta}(p))^*\gamma^4$, $\bar{u}^{\delta}(p)u^{\delta}(p) = m$. The explicit form of the operators in DSB was obtained in Ref. 12. The Dirac equation can be used to reduce them to the same form:

$$T_{31}=T_{13}=-i\hat{n}_0. \tag{9}$$

The projection operator for the particle state is

$$\tau^{\delta}=u^{\delta}(p)\cdot\bar{u}^{\delta}(p)=(m-i\hat{p})(1+i\delta\gamma^5\hat{s})/4. \tag{10}$$

In DSB, the operator τ_1^{δ} and τ_3^{δ} have the form

$$\tau_1^{\delta}=[m-i(\xi_+\hat{n}_0-\xi_-\hat{n}_3)+i\delta\gamma^5(-\xi_-\hat{n}_0+\xi_+\hat{n}_3-im\hat{n}_3\hat{n}_0)]/4, \tag{11}$$

$$\tau_3^{\delta}=[m-i(\xi_+\hat{n}_0+\xi_-\hat{n}_3)+i\delta\gamma^5(\xi_-\hat{n}_0+\xi_+\hat{n}_3-im\hat{n}_3\hat{n}_0)]/4, \tag{12}$$

where $\xi_{\pm} = [(-p_1p_3 + m^2)/2]^{1/2}$. We must now construct the operators[7–9]

$$P_{31}^{\delta,\delta}=u^{\delta}(p_1)\cdot\bar{u}^{\delta}(p_3)=u^{\delta}(p_1)\cdot\bar{u}^{\delta}(p_1)T_{13}=\tau_1^{\delta}T_{13}=T_{13}\tau_3^{\delta},$$

$$P_{31}^{-\delta,\delta}=u^{\delta}(p_1)\cdot\bar{u}^{-\delta}(p_3)$$

$$=\sigma^{+\delta}u^{-\delta}(p_1)\cdot\bar{u}^{-\delta}(p_3)=\sigma^{\delta}P_{31}^{-\delta,-\delta}=\sigma^{\delta}\tau_1^{-\delta}T_{13}.$$

The explicit form of the operators $P_{31}^{\pm\delta,\delta}$ in DSB was obtained in Ref. 12:

$$P_{31}^{\delta,\delta}=[\xi_+ - im\hat{n}_0 + \xi_-\hat{n}_3\hat{n}_0 + \delta\gamma^5(\xi_- + im\hat{n}_3 + \xi_+\hat{n}_3\hat{n}_0)]/4, \tag{13}$$

$$P_{31}^{-\delta,\delta}=i\delta(\xi_- - imn_3+\xi_+\delta\gamma^5)\hat{n}_{\delta}/4,\quad n_{\delta}=n_1+i\delta n_2. \tag{14}$$

These expressions for $P_{31}^{\pm\delta,\delta}$ can be used to reduce the evaluation of the matrix elements $M^{\pm\delta,\delta} = \bar{u}^{\pm\delta}(p_3)Qu^{\delta}(p_1)$ to

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the evaluation of the trace[6–9,12]

$$M^{\pm\delta,\delta}=\bar{u}^{\pm\delta}(p_3)Qu^{\delta}(p_1)=(u^{\delta}(p_1)\cdot\bar{u}^{\pm\delta}(p_3)Q)_t=(P_{31}^{\pm\delta,\delta}Q)_t,$$

where the subscript t represents the trace and Q is the interaction operator. The operators $P_{31}^{\pm\delta,\delta}$ determine the structure of the spin dependence of the matrix elements for transitions with $M^{-\delta,\delta}$ and without $M^{\delta,\delta}$ spin flip.

We now reproduce the following relations that will be useful in the evaluation of the matrix elements in DSB:

$$\begin{aligned}&\hat{a}u^{\delta}(p_1)=-i[(a_0+a_3\delta\gamma^5)u^{\delta}(p_3)+an_\delta\gamma^5u^{-\delta}(p_1)],\\&\bar{u}^{\delta}(p_3)\hat{a}=-i[\bar{u}^{\delta}(p_1)(a_0-a_3\delta\gamma^5)-an_\delta^*\bar{u}^{-\delta}(p_3)\gamma^5],\end{aligned}\qquad(15)$$

where a is an arbitrary 4-vector, $a_0=an_0$, $a_3=an_3$, $n_\delta^*=n_1-i\delta n_2$.

The circular polarization vector e_λ of a photon of momentum k, emitted by the particle in the $p_1\to p_3$ transition, is naturally expressed in terms of the 4-vectors p_1, p_3, $k(e_\lambda=e_{\lambda 13})$:

$$e_{\lambda 13}=\frac{[n_0\cdot n_3]k+i\lambda[n_0\cdot n_3]^\times k}{2^{1/2}\rho},\qquad [n_0\cdot n_3]=\frac{[p_1\cdot p_3]}{2\xi_+\xi_-},\qquad(16)$$

$$\rho=-\{([p_1\cdot p_3]k)^2\}^{1/2}/2\xi_+\xi_-.\qquad(17)$$

The operators $\hat{e}_{\lambda 13}$ and $\hat{e}^*_{\lambda 13}$ can be written in the form

$$\begin{aligned}&\hat{e}_{\lambda 13}=N_{13}(\hat{p}_3\hat{p}_1\hat{k}(1-\lambda\gamma^5)-\hat{k}\hat{p}_3\hat{p}_1(1+\lambda\gamma^5)-2p_1p_3\lambda\gamma^5\hat{k}),\\&\hat{e}^*_{\lambda 13}=N_{13}(\hat{p}_3\hat{p}_1\hat{k}(1+\lambda\gamma^5)-\hat{k}\hat{p}_3\hat{p}_1(1-\lambda\gamma^5)+2p_1p_3\lambda\gamma^5\hat{k}),\qquad(18)\\&N_{13}^{-1}=2^{1/2}\{-8p_1p_3\cdot p_1k\cdot p_3k-m^2[(2p_1k)^2+(2p_3k)^2]\}^{1/2}.\end{aligned}$$

2. In the massless case, the operators (11)–(14) assume the form

$$\tau_1^{\delta}=-i\hat{p}_1(1-\delta\gamma^5)/4,\quad \tau_3^{\delta}=-i\hat{p}_3(1+\delta\gamma^5)/4,\qquad(19)$$

$$P_{31}^{\delta\delta}=\xi(1+\delta\gamma^5)(1+\hat{n}_3\hat{n}_0)/4,\quad P_{31}^{-\delta,\delta}=i\delta\xi(1+\delta\gamma^5)\hat{n}_\delta/4,\qquad(20)$$

where $\xi=(-p_1p_3/2)^{1/2}$. It is readily verified that the operators τ_1^δ and τ_3^δ in (19) satisfy the relations

$$\begin{aligned}&\gamma^5\tau_1^{\delta}=\delta\tau_1^{\delta},\quad \gamma^5\tau_3^{\delta}=-\delta\tau_3^{\delta},\\&\tau_1^{\delta}\gamma^5=-\delta\tau_1^{\delta},\quad \tau_3^{\delta}\gamma^5=\delta\tau_3^{\delta},\end{aligned}\qquad(21)$$

which signify that, in the massless case, the initial and final states have positive and negative helicities, respectively.[13] We can now use (21) to express (15) in the form

$$\begin{aligned}&\hat{a}u^{\delta}(p_1)=-i[(a_0-a_3)u^{\delta}(p_3)-\delta an_\delta u^{-\delta}(p_1)],\\&\bar{u}^{\delta}(p_3)\hat{a}=-i[(a_0+a_3)\bar{u}^{\delta}(p_1)+\delta an_\delta^*\bar{u}^{-\delta}(p_3)].\end{aligned}\qquad(22)$$

Terms containing $\gamma^5\hat{k}$ in (18) can be discarded because of gauge invariance. The operators $\hat{e}_{\lambda 13}$ and $\hat{e}^*_{\lambda 13}$ are therefore given by the following expressions used by the CALKUL group[14]:

$$\begin{aligned}&\hat{e}_{\lambda 13}=N_{13}[\hat{p}_3\hat{p}_1\hat{k}(1-\lambda\gamma^5)-\hat{k}\hat{p}_3\hat{p}_1(1+\lambda\gamma^5)],\\&\hat{e}^*_{\lambda 13}=N_{13}[\hat{p}_3\hat{p}_1\hat{k}(1+\lambda\gamma^5)-\hat{k}\hat{p}_3\hat{p}_1(1-\lambda\gamma^5)],\qquad(23)\\&N_{13}^{-1}=4(-p_1p_3\cdot p_1k\cdot p_3k)^{1/2}.\end{aligned}$$

It is readily verified that

$$\begin{aligned}&\hat{e}^*_{\lambda 13}u^{\delta}(p_1)=(1+\delta\lambda)2p_1kN_{13}\hat{p}_3u^{\delta}(p_1),\\&\bar{u}^{\delta}(p_3)\hat{e}^*_{\lambda 13}=-(1+\delta\lambda)2p_3kN_{13}\bar{u}^{\delta}(p_3)\hat{p}_1.\end{aligned}\qquad(24)$$

If photon emission occurs during the $p_A\to p_B$, transitions, the replacements $(p_1,p_3)\to(p_A,p_B)$ in (23) result in operators $\hat{e}_{\lambda AB}$ whose effect on the bispinors differs from the effect of $\hat{e}_{\lambda 13}$ only by a phase factor:

$$\hat{e}_{\lambda 13}=\hat{e}_{\lambda AB}e^{i\varphi_{AB}},\qquad(25)$$

$$e^{i\varphi_{AB}}=-i\lambda 2^{1/2}e_{\lambda 13}n_{2(AB)},\qquad(26)$$

where $n_{2(AB)}$ are the unit vectors

$$n_{2(AB)}=[p_A\cdot p_B]^\times k/\rho_{(AB)},\quad \rho_{(AB)}=-(4p_Ak\cdot p_Bk)^{1/2}/(-2p_Ap_B)^{1/2}.\qquad(27)$$

In the calculations reproduced below, we shall frequently encounter expressions for $\bar{P}_{31}^{\pm\delta,\delta}=\gamma_\mu P_{31}^{\pm\delta,\delta}\gamma_\mu$, of the form

$$\bar{P}_{31}^{\delta,\delta}=\xi(1-\delta\gamma^5),\quad \bar{P}_{31}^{-\delta,\delta}=-i\delta\xi(1-\delta\gamma^5)\hat{n}_\delta/2.\qquad(28)$$

3. We now turn to the evaluation of the matrix elements for the process

$$e^-(p_1)+e^-(p_2)\to e^-(p_3)+e^-(p_4)+\gamma(k),\qquad(29)$$

assuming that the initial and final electrons are massless ($p_A^2=0$, $A=1,2,3,4$). Next, we use the 4-momenta of the electrons and the photon to construct two orthogonal bases of vectors n_A and n'_A (2)

$$\begin{aligned}&n_1=[n_0\cdot n_3]^\times n_2,\ n_1'=[n_0'\cdot n_3']^\times n_2'.\\&n_2=[n_0\cdot n_3]^\times k/\rho_{(13)},\ n_2'=[n_0'\cdot n_3']^\times k/\rho_{(24)}.\\&n_3=(p_3-p_1)/(-2p_1p_3)^{1/2},\quad n_3'=(p_4-p_2)/(-2p_2p_4)^{1/2},\\&n_0=(p_3+p_1)/(-2p_1p_3)^{1/2},\quad n_0'=(p_4+p_2)/(-2p_2p_4)^{1/2}.\end{aligned}\qquad(30)$$

In addition to the operators τ_1^δ, τ_3^δ, $P_{31}^{\pm\delta,\delta}$ (19) and (20), we now construct the operators $\tau_2^{\delta'}$, $\tau_4^{\delta'}$, $P_{42}^{\pm\delta',\delta'}$ by introducing the replacements $p_1\to p_2$, $p_3\to p_4$, $n_A\to n'_A$, $\delta\to\delta'$, $\xi\to\xi'=(-p_2p_4/2)^{1/2}$ into (19) and (20). The contribution of direct diagrams for the process defined by (29) (they correspond to the $1\to 3$, $2\to 4$ transitions) then reduces to the product of traces

$$\begin{aligned}M_{\pm\delta,\delta}^{\pm\delta',\delta'}&=\bar{u}^{\pm\delta'}(p_4)Q_1u^{\delta'}(p_2)\cdot\bar{u}^{\pm\delta}(p_3)Q_2u^{\delta}(p_1)\\&=(P_{42}^{\pm\delta',\delta'}Q_1)_t(P_{31}^{\pm\delta,\delta}Q_2)_t,\end{aligned}$$

and the $(1\to 4, 2\to 3)$ contribution reduces to the trace of products

$$\begin{aligned}M_{\pm\delta,\delta}^{\pm\delta',\delta}&=\bar{u}^{\pm\delta'}(p_4)Q_3u^{\delta}(p_1)\cdot\bar{u}^{\pm\delta}(p_3)Q_4u^{\delta'}(p_2)\\&=(P_{42}^{\pm\delta',\delta'}Q_3P_{31}^{\pm\delta,\delta}Q_4)_t,\end{aligned}$$

where Q_A are the corresponding Dirac operators. Details of this calculation are given in the Appendix. The nonzero matrix elements $M_\lambda^{\delta'\delta',\delta\delta}$ and $M_\lambda^{-\delta'\delta',-\delta\delta}$ for the process defined by (29) have the following form:

$$M_\lambda^{\delta'\delta',\delta\delta}=\xi\xi'(1-\delta\delta')[(1+\delta\lambda)a_1+(1+\delta'\lambda)a_2],\qquad(31)$$

$$\begin{aligned}M_\lambda^{-\delta'\delta',-\delta\delta}=&\delta\delta'\xi\xi'[(1+\delta\lambda)b_1\\&+(1-\delta\lambda)b_2+(1+\delta'\lambda)b_3+(1-\delta'\lambda)b_4],\end{aligned}\qquad(32)$$

$$a_i=a_{i1}+a_{i2},\quad b_k=b_{k1}+(1+\delta\delta')b_{k2},\quad i=1,2;\quad k=1,2,3,4,$$

$$a_{11}=\left[2p_4(p_1-k)-\frac{(k_0-k_3)}{\xi}p_3p_4+kn_1\cdot p_4n_\delta^*\right]\frac{N_{14}}{\Delta_{23}}e^{-i\varphi_{14}},$$

$$a_{12}=\left[2p_2(p_3+k)+\frac{(k_0+k_3)}{\xi}p_1p_2-kn_1\cdot p_2n_\delta\right]\frac{N_{23}}{\Delta_{14}}e^{-i\varphi_{23}},$$

$$a_{21}=\left[2p_1(p_4+k)+\frac{(k_0'+k_3')}{\xi'}p_1p_2-kn_1'\cdot p_1n_{\delta'}'\right]\frac{N_{14}}{\Delta_{23}}e^{-i\varphi_{14}},$$

$$a_{22}=\left[2p_3(p_2-k)-\frac{(k_0'-k_3')}{\xi'}p_3p_4+kn_1'\cdot p_3n_{\delta'}'^*\right]\frac{N_{23}}{\Delta_{14}}e^{-i\varphi_{23}},$$

$$b_{12}=p_3n_{\delta'}'\cdot p_4n_\delta\cdot\frac{N_{14}}{\Delta_{23}}e^{-i\varphi_{14}},\qquad b_{22}=p_1n_{\delta'}'\cdot p_2n_\delta\cdot\frac{N_{23}}{\Delta_{14}}e^{-i\varphi_{23}},$$

$$b_{32}=p_3n_{\delta'}'\cdot p_4n_\delta\cdot\frac{N_{23}}{\Delta_{14}}e^{-i\varphi_{23}},\qquad b_{42}=p_1n_{\delta'}'\cdot p_2n_\delta\cdot\frac{N_{14}}{\Delta_{23}}e^{-i\varphi_{14}},$$

$$b_{11}=[2p_3(p_1-k)n_\delta n_{\delta'}'+2kn_1\cdot p_3n_{\delta'}']\frac{N_{13}}{\Delta_{24}},$$

$$b_{21}=[2p_1(p_3+k)n_\delta n_{\delta'}'-2kn_1\cdot p_1n_{\delta'}']\frac{N_{13}}{\Delta_{24}},$$

$$b_{31}=[2p_4(p_2-k)n_\delta n_{\delta'}'+2kn_1'\cdot p_4n_\delta]\frac{N_{24}}{\Delta_{13}}e^{-i\varphi_{24}},$$

$$b_{41}=[2p_2(p_4+k)n_\delta n_{\delta'}'-2kn_1'\cdot p_2n_\delta]\frac{N_{24}}{\Delta_{13}}e^{-i\varphi_{24}},$$

where $\Delta_{AB}=(p_A-p_B)^2$, $N_{AB}^{-1}=4(-p_Ap_B\cdot p_Ak\cdot p_Bk)^{1/2}$. The sum of the squares of the moduli of the matrix elements (31) and (32), which determines the differential probability of the process, was evaluated by program SCHOONSCHIP.[15]

In terms of the invariant variables

$$s=-(p_1+p_2)^2,\ t=-(p_1-p_3)^2,\ u=-(p_1-p_4)^2,$$
$$s'=-(p_3+p_4)^2,\quad t'=-(p_2-p_4)^2,\quad u'=-(p_2-p_3)^2$$

the differential cross section for Møller bremsstrahlung scattering in the case of helicity-polarized initial electrons and photon can be written in the following form:

$$d\sigma_M=\frac{\alpha^3}{\pi^2 s}A_MW_M\,d\Gamma, \tag{33}$$

$$A_M=A_{MB}/tt'uu', \tag{34}$$

$$\begin{aligned}A_{MB}={}^1/_2\{&ss'(s^2+s'^2)+tt'(t^2+t'^2)+uu'(u^2+u'^2)\\&+\delta\delta'[ss'(s^2+s'^2)-tt'(t^2+t'^2)-uu'(u^2+u'^2)]\\&+\delta\lambda[-ss'(s^2-s'^2)-tt'(t^2-t'^2)-uu'(u^2-u'^2)]\\&+\delta'\lambda[-ss'(s^2-s'^2)+tt'(t^2-t'^2)+uu'(u^2-u'^2)]\},\end{aligned} \tag{35}$$

$$W_M=-\left(\frac{s}{x_1x_2}+\frac{s'}{x_3x_4}+\frac{t}{x_1x_3}+\frac{t'}{x_2x_4}+\frac{u}{x_1x_4}+\frac{u'}{x_2x_3}\right), \tag{36}$$

$$d\Gamma=\delta^4(p_1+p_2-p_3-p_4-k)\frac{d^3\mathbf{p}_3}{2p_{30}}\frac{d^3\mathbf{p}_4}{2p_{40}}\frac{d^3\mathbf{k}}{2\omega},$$

where $x_A=p_Ak$, and α is the fine-structure constant.

The expressions for A_{MB} and W_M can be written in the form

$$\begin{aligned}A_{MB}={}^1/_2\{&(1+\delta\delta')[(1+\delta\lambda)ss's'^2+(1-\delta\lambda)ss's^2]\\&+(1-\delta\delta')[(1+\delta'\lambda)(tt't^2+uu'u^2)+(1-\delta'\lambda)(tt't'^2+uu'u'^2)]\},\end{aligned} \tag{37}$$

$$W_M=\left(\frac{p_1}{p_1k}+\frac{p_2}{p_2k}-\frac{p_3}{p_3k}-\frac{p_4}{p_4k}\right)^2. \tag{38}$$

The differential probability for Bhabha bremsstrahlung scattering

$$e^-(p_1)+e^+(p_2)\to e^-(p_3)+e^+(p_4)+\gamma(k) \tag{39}$$

can be readily obtained from the corresponding probability for (29) by means of the crossing transformation[13]

$$p_2\leftrightarrow -p_4,\quad \delta'\to-\delta'.$$

The invariant variables then transform as follows:

$$s\leftrightarrow u,\quad s'\leftrightarrow u',\quad x_2\leftrightarrow -x_4.$$

After the above replacements, the differential cross section for Bhabha bremsstrahlung scattering is given by the following expression when the helicity polarization states of the initial electron and positron and the photon are taken into account:

$$d\sigma_B=\frac{\alpha^3}{\pi^2 s}A_BW_B\,d\Gamma, \tag{40}$$

$$A_B=A_{MB}/ss'tt', \tag{41}$$

$$W_B=\frac{s}{x_1x_2}+\frac{s'}{x_3x_4}-\frac{t}{x_1x_3}-\frac{t'}{x_2x_4}+\frac{u}{x_1x_4}+\frac{u'}{x_2x_3}, \tag{42}$$

where A_{MB} is given by (35). By analogy with (38), we then have

$$W_B=\left(\frac{p_1}{p_1k}+\frac{p_4}{p_4k}-\frac{p_3}{p_3k}-\frac{p_2}{p_2k}\right)^2. \tag{43}$$

When soft photons are emitted ($s=s'$, $t=t'$, $u=u'$), the quantities A_M and A_B are given by

$$A_M=\frac{s^2+u^2}{t^2}+\frac{s^2+t^2}{u^2}+\frac{2s^2}{tu}+\delta\delta'\left(\frac{s^2-u^2}{t^2}+\frac{s^2-t^2}{u^2}+\frac{2s^2}{tu}\right), \tag{44}$$

$$A_B=\frac{u^2+s^2}{t^2}+\frac{u^2+t^2}{s^2}+\frac{2u^2}{st}-\delta\delta'\left(\frac{u^2-s^2}{t^2}+\frac{u^2+t^2}{s^2}+\frac{2u^2}{st}\right). \tag{45}$$

The last two expressions differ from the differential cross sections for the usual Møller and Bhabha scattering of longitudinally polarized initial particles (see Ref. 13) only by a factor. Summing over the photon polarizations, we obtain

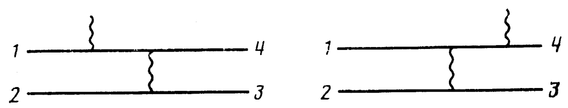
$$\begin{aligned}A_{MB}=&(1+\delta\delta')ss'(s^2+s'^2)\\&+(1-\delta\delta')[tt'(t^2+t'^2)+uu'(u^2+u'^2)].\end{aligned}$$

Consequently, the ratio of cross sections corresponding to parallel and antiparallel spins in the process $e^\pm e^-\to e^\pm e^-\gamma$ is

$$\frac{d\sigma_{\uparrow\uparrow}}{d\sigma_{\uparrow\downarrow}}=\frac{tt'(t^2+t'^2)+uu'(u^2+u'^2)}{ss'(s^2+s'^2)}.$$

APPENDIX

As an illustration, let us determine the contribution $M_{\delta\delta}^{\delta'\delta'}$ of the two diagrams



to the matrix element $M_\lambda^{\delta'\delta',\delta\delta}$

$$M_{\delta\delta}^{\delta'\delta'}=\bar{u}_{\delta'}(p_4)\left(\hat{e}_\lambda^*\frac{\hat{p}_4+\hat{k}}{2p_4k}\gamma_\mu+\gamma_\mu\frac{\hat{p}_1-\hat{k}}{-2p_1k}\hat{e}_\lambda^*\right)$$

$$\times u^\delta(p_1)\frac{\bar{u}^\delta(p_3)\gamma_\mu u^{\delta'}(p_2)}{\Delta_{23}}. \qquad (A1)$$

Applying the operator $\hat{e}_\lambda^* = \hat{e}_{\lambda 13}^* = \hat{e}_{\lambda 14}^* e^{-i\varphi_{14}}$ to the bispinsors $u^\delta(p_1)$ and $\bar{u}^{\delta'}(p_4)$ in accordance with (24), and using the commutation relations for the matrices γ^k and the Dirac equation, we obtain

$$M_{\delta\delta}^{\delta'\delta'}=\alpha_0\sum_{i=1}^{3}\alpha_i A_i$$

$$=\alpha_0\sum_{i=1}^{3}\alpha_i\bar{u}^{\delta'}(p_4)Q_\mu^{(i)}u^\delta(p_1)\cdot\bar{u}^\delta(p_3)\gamma_\mu u^{\delta'}(p_2), \quad (A2)$$

where

$$Q_\mu^{(1)}=\gamma_\mu, \quad Q_\mu^{(2)}=\gamma_\mu\hat{p}_4\hat{k},$$

$$Q_\mu^{(3)}=\hat{k}\hat{p}_1\gamma_\mu, \quad \alpha_0=-N_{14}e^{-i\varphi_{14}}/\Delta_{23},$$

$$\alpha_1=(1+\delta\lambda)2p_4(p_1-k)+(1+\delta'\lambda)2p_1(p_4+k),$$

$$\alpha_2=1+\delta\lambda, \quad \alpha_3=-(1+\delta'\lambda).$$

The first term in (A2) can be evaluated immediately:

$$A_1=(P_{42}^{\delta'\delta'}\gamma_\mu P_{31}^{\delta\delta}\gamma_\mu)_t=(P_{42}^{\delta'\delta'}\bar{P}_{31}^{\delta\delta})_t=\xi\xi'(1-\delta\delta').$$

To evaluate A_2, A_3, we apply the operators $\hat{k}$ and $u^\delta(p_1)$ and $\bar{u}^{\delta'}(p_4)$, and use the rules defined by (22). The results are:

$$A_2=-i[(k_0-k_3)A_{21}-\delta kn_1A_{22}],$$

$$A_3=-i[(k_0'+k_3')A_{31}+\delta'kn_1'A_{32}],$$

$$A_{21}=\bar{u}^{\delta'}(p_4)\gamma_\mu\hat{p}_4u^\delta(p_3)\cdot\bar{u}^\delta(p_3)\gamma_\mu u^{\delta'}(p_2),$$

$$A_{22}=\bar{u}^{\delta'}(p_4)\gamma_\mu\hat{p}_4u^{-\delta}(p_1)\cdot\bar{u}^\delta(p_3)\gamma_\mu u^{\delta'}(p_2), \qquad (A3)$$

$$A_{31}=\bar{u}^{\delta'}(p_2)\hat{p}_1\gamma_\mu u^\delta(p_1)\cdot\bar{u}^\delta(p_3)\gamma_\mu u^{\delta'}(p_2),$$

$$A_{32}=\bar{u}^{-\delta'}(p_4)\hat{p}_1\gamma_\mu u^\delta(p_1)\cdot\bar{u}^\delta(p_3)\gamma_\mu u^{\delta'}(p_2).$$

If we now substitute

$$u^\delta(p_3)\cdot\bar{u}^\delta(p_3)\to\tau_3^\delta,$$

$$u^{\delta'}(p_2)\cdot\bar{u}^{\delta'}(p_2)\to\tau_2^{\delta'}, \quad u^{-\delta}(p_1)\cdot\bar{u}^\delta(p_3)\to P_{31}^{\delta,-\delta},$$

$$u^\delta(p_1)\cdot\bar{u}^\delta(p_3)\to P_{31}^{\delta\delta},$$

$$u^{\delta'}(p_2)\cdot\bar{u}^{\delta'}(p_4)\to P_{42}^{\delta'\delta'}, \quad u^{\delta'}(p_2)\cdot\bar{u}^{-\delta'}(p_4)\to P_{42}^{-\delta',\delta'},$$

$$\gamma_\mu P_{31}^{\delta\delta}\gamma_\mu\to\bar{P}_{31}^{\delta\delta}, \quad \gamma_\mu P_{42}^{\delta'\delta'}\gamma_\mu\to\bar{P}_{42}^{\delta'\delta'},$$

in (A3), we obtain

$$A_{21}=(\bar{P}_{42}^{\delta'\delta'}\hat{p}_4\tau_3^\delta)_t=-i\xi'(1-\delta\delta')p_3p_4,$$

$$A_{22}=(\bar{P}_{42}^{\delta'\delta'}\hat{p}_4P_{31}^{\delta,-\delta})_t=-i\xi(1-\delta\delta')p_4n_\delta^*,$$

$$A_{31}=(\tau_2^{\delta'}\hat{p}_1\bar{P}_{31}^{\delta,\delta})_t=-i\xi(1-\delta\delta')p_1p_2,$$

$$A_{32}=(P_{42}^{-\delta',\delta'}\hat{p}_1\bar{P}_{31}^{\delta\delta})_t=-i\xi'(1-\delta\delta')p_1n_{\delta'}'.$$

Finally, substituting these expressions in (A1), we obtain

$$M_{\delta\delta}^{\delta'\delta'}=\alpha_0\xi\xi'(1-\delta\delta')$$

$$\times\left\{(1+\delta\lambda)\left(2p_4(p_1-k)-\frac{(k_0-k_3)}{\xi}p_3p_4+kn_1\cdot p_4n_\delta^*\right)\right.$$

$$\left.+(1+\delta'\lambda)\left(2p_1(p_4+k)+\frac{(k_0'+k_3')}{\xi'}p_1p_2-kn_1'\cdot p_1n_{\delta'}'\right)\right\}.$$

Thus, the evaluation of the contribution of the two exchange diagrams $M_{\delta\delta}^{\delta'\delta}$ to the matrix elements $M_\lambda^{\delta'\delta,\delta\delta}$ has been reduced to the evaluation of the trace of the product of only two Dirac matrices. We note that the contribution of direct diagrams to the matrix element $M_\lambda^{\delta'\delta,\delta\delta}$ vanishes because the operators $P_{31}^{\delta\delta}$ and $P_{42}^{\delta'\delta'}$ in the massless case contain an even number of Dirac matrices. Both direct and exchange diagrams contribute to the matrix element $M_\lambda^{-\delta'\delta',-\delta\delta}$, and (22) then enables us to reduce the determination of $M_\lambda^{-\delta'\delta',-\delta\delta}$ to the evaluation of the trace of only for Dirac matrices.

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Translated by S. Chomet