Three-dimensional wave collapse in the nonlinear Schrodinger equation model

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(Submitted 11 November 1988)
Zh. Eksp. Teor. Fiz. **95**, 1945–1950 (June 1989)

The nonlinear Schrodinger equation model is used to show that beyond the critical point ($D\sigma > 2$, where D is the dimensionality of space and σ the degree of nonlinearity), the three-dimensional wave collapse can be spread out in time. The energy of a wave packet is then absorbed by the collapsing cavity during the stage following the establishment of the singularity at the center of the packet.

1. INTRODUCTION

The appearance of singularities in the solutions of nonlinear equations of motion is typical for many physical problems in which nonlinear dissipation or high-frequency absorption is ignored. Examples of this include the self-focusing of wave beams and self-compression of wave packets in nonlinear media,[1–5] self-focusing of electromagnetic waves in plasmas,[6] and the collapse of Langmuir waves.[6,7] As a rule, collapse occurs at the conclusion of the nonlinear stage of modulational instability in conservative systems of high dimensionality. Singular behavior of dissipative structures is also possible. Chemotactic collapse, described by equations with mutual diffusion, is an example of this.

Collapse is the result of competition between the mechanisms of nonlinear focusing of space-time trajectories of quasiparticles and their wave dispersion when nonlinear refraction predominates over dispersion at all stages, right up to the formation, in finite time, of the field singularity. Other things being equal, the competition between the above mechanisms depends on the dimensionality of the system, so that the character of collapse is different for systems of different dimensionality.

The phenomenon of wave collapse is particularly well demonstrated for the solutions of the nonlinear Schrodinger equation

$$i\Psi_t'+\Delta_D\Psi+|\Psi|^{2\sigma}\Psi=0. \quad (1)$$

These solutions were first investigated in the theory of self-focusing of waves ($D=2$) in media with cubic nonlinearity ($\sigma=1$; Refs. 1-5). Equation (1) is a model[8,9] for Langmuir collapse of high dimensionality ($D=3$). The idea of collapsing cavities is introduced to explain the absorption of waves in plasmas.[6,7] The relevance of this problem is indicated by the number of publications that have appeared in the very recent past.[9–11]

Some of the general properties of (1) are determined exclusively by $D\sigma$ (e.g., the stability of solitons), and the authors of Ref. 11 used this as a basis for their analysis of collapses in the so-called postcritical case ($\sigma D>2$), in which they employed numerical simulation of the solutions of (1) with $D=1$ and a high degree of nonlinearity ($\sigma=3$). In this paper, we draw attention to the nonequivalent behavior of collapsing cavities of different dimensionality. In particular, in the three-dimensional case, there are cavities in which energy dissipation in the singularity is continuous in time. There are no such solutions of (1) in the one-dimensional case for any σ. It is therefore expected that, in the three-dimensional case, the preferential absorption of wave energy by collapsing cavities does not occur at the usual stage at which the singularity is formed ($t<t_0$), but at the next stage, after the singularity has appeared ($t>t_0$). We shall call the latter the focal stage.

2. FOCAL STAGE OF COLLAPSE—THE STATIONARY MODEL

To gain an idea about the field structure in the neighborhood of the singularity during the focal stage of collapse, let us consider a symmetric solution of (1) for a time-independent energy flux flowing into the singularity. Such solutions are described by

$$i\Psi_t'+(D-1)r^{-1}\Psi_r'+\Psi_{rr}''+|\Psi|^{2\sigma}\Psi=0. \quad (2)$$

For time-independent fields $\Psi=\Psi_0\exp(-i\Omega t)$, equation (2) reduces to

$$\Psi_{0rr}''+(D-1)r^{-1}\Psi_{0r}'+\Omega\Psi_0+|\Psi_0|^{2\sigma}\Psi_0=0. \quad (3)$$

We now assume that $\Omega=\varkappa_0^2>0$, and consider solutions of (3) in the form of waves converging on the point $r=0$, which have the following asymptotic form as $r\to\infty$:

$$\Psi_0\propto e^{i\varkappa_0 r}r^{-(D-1)/2}. \quad (4)$$

Taking the required solution in the form

$$\Psi_0=U(r)e^{i\varphi(r)}r^{-(D-1)/2}, \quad (5)$$

and substituting it in (3), we obtain the following set of equations for U and φ:

$$U^{2\sigma}r^{-(D-1)\sigma}+\varkappa_0^2=\varphi_r'^2-U''/U+(D-1)(D-3)/4r^2,$$
$$(U^2\varphi_r')_r'=0, \quad (6)$$

where, according to (4), $\varphi_r'\to-\varkappa_0$ as $r\to\infty$. Assuming that $U^2\varphi_r'=C=\text{const}<0$, which correspond to a constant energy flux in the required solution, we find that the amplitude distribution $U(r)$ is the solution of the equation

$$U''+U[\varkappa_0^2+U^{2\sigma}r^{-(D-1)\sigma}-C^2/U^4-(D-1)(D-3)/4r^2]=0. \quad (7)$$

The last term in (7) vanishes for $D=1$ and $D=3$.

Equation (7) has no singular solutions in the one-dimensional case:

$$U|_{r\to0}\to\infty,\quad U,U''|_{r>0}>0,\quad U'|_{r>0}<0.$$

The collapse of one-dimensional fields for $\sigma>2$ does not

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therefore lead to the existence of a singularity spread in time in the post-focal region, and proceeds in accordance with the scenarios described in Refs. 11 and 12.

In the three-dimensional case ($D=3$), there is a singular solution corresponding to $U/r\to\infty$ as $r\to 0$. To find its asymptotic form in the neighborhood of $r=0$, we change the variables as follows:

$$\rho=|C|r,\quad \varkappa=\varkappa_0/|C|,\quad \xi=(-\ln\rho)^{-1}$$

and obtain the above equation in the form

$$\xi^4 U_{\xi\xi}''+(2\xi^3-\xi^2)U_\xi'+U(\varkappa^2\rho^2+U^2-\rho^2/U^4)=0. \qquad (8)$$

For $\rho=\exp(-1/\xi)\to 0$, we can neglect terms containing ρ^2. The solution of the remaining equation

$$\xi^4 U''+(2\xi^3-\xi^2)U_\xi'+U^3=0 \qquad (9)$$

in the neighborhood of the point $\xi=0(\rho=0)$ will be sought in the form

$$U=(\xi/2)^{1/2}(1+q). \qquad (10)$$

Substituting (10) in (9), and retaining the leading terms for $q\ll 1$, we obtain

$$\xi q'-q-{}^3/_4\xi=0, \qquad (11)$$

which has the solution $q=\frac{3}{4}\xi\ln\xi$. Thus, in the three-dimensional case, a singular solution of (5) exists and is described by the functions

$$U(r)=(-2\ln|C|r)^{-1/2}\left[1+\frac{3}{4}\,\frac{\ln(-\ln|C|r)}{\ln|C|r}+\dots\right], \qquad (12)$$

$$\varphi(r)=-\int\frac{|C|}{U^2}dr\approx 2^{1/2}|C|r(\ln|C|r-1). \qquad (13)$$

The constant $|C|$ is determined by the energy flux density $|U^2|\varphi_r'$ flowing into the field singularity.

3. DISTRIBUTED COLLAPSE

It is known that the self-contraction of a wave packet described by (1) is different in character for different D and σ. In the critical case ($D\sigma=2$), and at the time t_0 at which the singularity appears, it receives a finite energy flux from the initial wave packet, (i.e., the integral $N=\int|\Psi|^2dx_D$ is finite).[13,14] This is referred to as strong collapse.[8] Figure 1(a) illustrates the space-time trajectories (rays) of quasiparticles in a strong collapse. At the final stage ($t\sim t_0$), collapse is instantaneous ("nonaberrational"). In general, the picture is analogous to that described in Ref. 13, i.e., it is multifocal in character, and a photon of energy N is absorbed in each singularity.

In the postcritical case ($D\sigma>2$), the energy flux is zero ($N=0$) at the singularity. The corresponding picture of space-time rays is shown in Fig. 1(b) for $t<t_0$. For $r\neq 0$ and $t=t_0$, the rays are oriented toward the center of the cavity. It may be expected that, in the three-dimensional case, the orientation of rays corresponding to the influx of energy into the singularity will ensure that the absorption of the field, especially at the focal stage of collapse, will be distributed in time, i.e., the singularity that arises at time $t=t_0$ can be maintained for a finite time by the energy flux flowing toward it from the periphery of the cavity. A "focal filament"

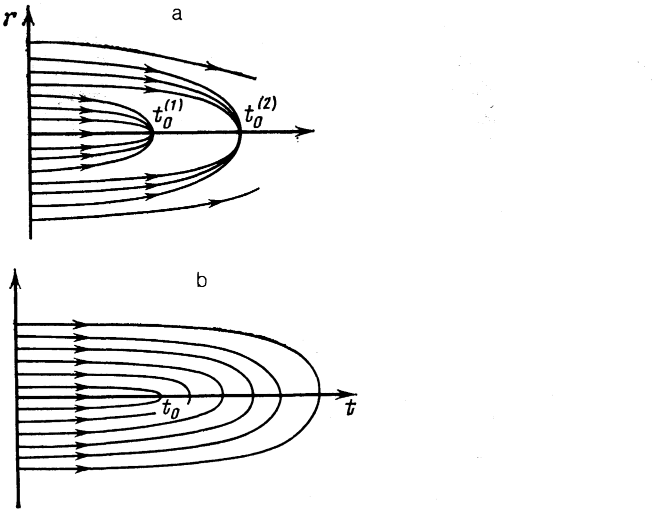

FIG. 1. Space-time focusing in the critical ($D\sigma=2$) and postcritical ($D\sigma>2$) regions.

(Fig. 1b) is formed as result. An analytic proof of this proposition does not seem to possible at present, but it has been confirmed by numerical experiments.

To avoid difficulties with integration of the conservation equation (2) for $t>t_0$, we assume that the medium exhibits multiphoton absorption that limits the intensity to a certain value. The parameters can then be chosen so that this absorption is confined to a small neighborhood of the point $r=0$, which simulates quite well the singularity as an energy sink. Let us now replace (2) with ($\sigma=1$)

$$i\Psi_t'+\Psi_{rr}''+(D-1)r^{-1}\Psi_r'+|\Psi|^2\Psi+i\alpha|\Psi|^n\Psi=0, \qquad (14)$$

which we shall integrate for sufficiently small values of the multiphoton absorption coefficient α. Figure 2 shows the field amplitude as a function of t for $r=0$, $D=3$, $n=8$, $\alpha=3.9\cdot 10^{-11}$, and initial distribution $\Psi(r,0)=7\exp(-1.125r^2)$. For comparison, the figure also shows the analogous results for two-dimensional collapse ($D=2$) and the same initial condition. It is clear that the three-dimensional collapse is distributed in time and that the two-dimensional collapse is multifocal. The total energy N of the three-dimensional cluster after $t=t_0$ decreases monotonically as a consequence of absorption near $r=0$, which is in contrast to the abrupt change found for the two-dimensional collapse (Fig. 3).

We note that the time-distributed collapse effect, obtained within the framework of the nonlinear Schrodinger

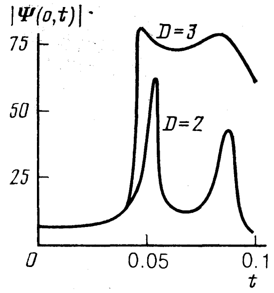

FIG. 2. Field amplitude at the center of the cavity as a function of time.

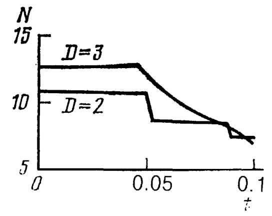

FIG. 3. Total energy in the cavity as a function of time.

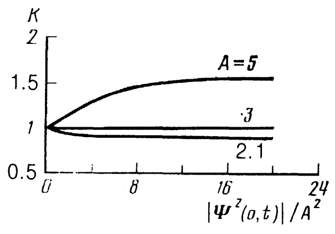

FIG. 5. Second harmonic gain as a function of field strength at the center of the cavity.

equation with a local nonlinearity, is only outwardly similar to the "funnel effect" described for the collapse of Langmuir waves in Ref. 15. The latter requires a singularity in the neighborhood of $r = 0$ that is stronger than r^{-2}, and (5) and (12) show that this does not occur in distributed three-dimensional collapses.

4. STABILITY OF THREE-DIMENSIONAL COLLAPSE

Direct numerical simulation shows that asymmetric perturbations are not trapped by a field singularity in a three-dimensional collapse in a cubic medium. This can be demonstrated by expanding the collapsing solution of (1) for $D = 3$, $\sigma = 1$ at the focal stage in terms of spherical harmonics (using the coordinates r,θ,φ)

$$\Psi = \sum_s \Psi^{(s)}(r,t)\,Y^{(s)}(\theta,\varphi), \qquad (15)$$

where $\Psi^{(s)}(r,t)$ is the radial field function,

$$Y^{(s)}(\theta,\varphi) = P_l^m(\cos\theta)\cos m\varphi$$

are spherical harmonics, and s represents the set of azimuthal (m) and polar (l) indices.

We shall suppose that the field structure near the singularity is nearly spherically symmetric, so that the amplitudes of all the harmonics other than $Y^{(0,0)}$ are small. This means that, in the equations for the harmonics that are obtained after substituting (15) in (1), we can neglect the squares of all amplitudes other than the fundamental. The amplitudes $\Psi^{(s)}(r,t)$ are found to be related by a set of equations that describes the effect of a powerful harmonic on all the others. The amplitudes of the fundamental are described by a nonlinear equation, whereas those of the weak harmonics are the solutions of linear equations with variable coefficients. These equations involve only the polar index l because of the symmetry of the fundamental and the fact that we have chosen Ψ in the form of (15). It is interesting to consider the stability of the symmetric solution with respect to the lowest harmonics $l = 1$ and $l = 2$. The $l = 1$ harmonic can be disposed of by suitably choosing the origin of coordinates. Whenever it differs from zero, this means that the collapse is not located at the origin $r = 0$.

Figures 4 and 5 show the calculated amplitudes of the harmonics for the initial distribution

$$\Psi^{(0,0)}(r,0) = Ae^{-r^2/2}, \qquad \Psi^{(2,0)}(r,0) = r^2 e^{-r^2/2}.$$

Collapse of the fundamental occurs for $A > 2.1$ (Fig. 4). At the beginning of the collapse development, the amplitude of the mode $\Psi^{(2,0)}(r,t)$ is found to vary, but then, after a certain time has elapsed, it becomes constant. The energy $N_2 = \int |\Psi^{(2,0)}|^2 dx_D$ does not increase for $A \leqslant 3$; for large A it increases but the total gain $K = N_2(t)/N_2(0)$ remains finite (Fig. 5). This shows that the symmetric three-dimensional collapse is stable against perturbations that violate its symmetry.

We may therefore conclude that, for $D\sigma > 2$, three-dimensional collapse differs from one-dimensional collapse by the essential fact that it is spread out in time. Numerical experiments with one-dimensional collapse cannot serve as a basis for conclusions about the structure of collapsing cavities in the three-dimensional case. For $D = 3$, the energy of the initial wave packet is dissipated preferentially not at the focal stage, but after the establishment of the singularity which acts as a sink for the energy of the cluster. In real media, the absorption of energy at the center of the packet is due to nonlinear or high-frequency dissipation. We may expect that, under asymmetric initial conditions, the resulting collapsing wave packets will be symmetrized (symmetric perturbations have higher growth rates) and decay preferentially at the focal stage $t > t_0$.

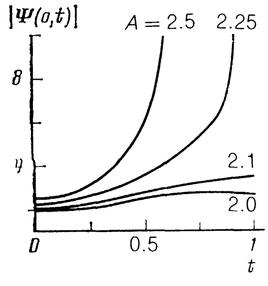

FIG. 4. Field amplitude at the center of the cavity as a function of time for different initial conditions.

NOTE added in proof (16 April 1989). Distributed collapse was first described by the present authors in a lecture presented at the Eighth All-Union on Nonlinear Waves (Gor'kiĭ, March 1987) and in Proceedings of the Third International Group on Nonlinear and Turbulent Processes in Physics (Kiev, 13–26 April, 1987, Vol. 2, p. 210). In a recent paper, V. M. Malkin has shown [Pis'ma Zh. Eksp. Teor. Fiz. **48**, 603 (1988)] that the singular self-similar solution of (2) can serve as a model of distributed collapse.

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Translated by S. Chomet