What if a film conductivity exceeds the speed of light?

V. I. Fal'ko and D. E. Khmel'nitskiĭ

Institute of Solid State Physics, USSR Academy of Sciences

(Submitted 29 December 1988; resubmitted 21 February 1989)
Zh. Eksp. Teor. Fiz. **95**, 1988–1992 (June 1989)

We investigate the electrodynamics of a thin film with account taken of retardation effects. We show that for a two-dimensional conductivity $\sigma > c/2\pi$ there exist weakly damped plasma waves at the lowest frequencies. We calculate the reflection, transmission, and absorption coefficients of obliquely incident light.

1. The conductivity σ of a film has units of velocity. The question in the title is therefore meaningful and calls for an exhaustive answer. Conductivity governs the character and rate of Maxwellian relaxation of the excess charges. In bulky conductors the excess charge density $\rho(\mathbf{r},t)$ relaxes without changing the initial distribution $\rho(\mathbf{r},0)$ and with a decrement $4\pi\sigma_3$:

$$\rho(\mathbf{r}, t)=\rho(\mathbf{r}, 0)\exp(-4\pi\sigma_3 t).$$

According to electrostatics,[1,2] charges relax by spreading with an effective velocity $2\pi\sigma$. This velocity becomes comparable with that of light if the film sheet resistance is 188 Ω. An electrostatic approach to relaxation in films with so high a conductivity is inadequate, and account must be taken of retardation, with the field described by the complete set of Maxwell equations.

Maxwellian relaxation corresponds to dissipative dynamics of the charges at low frequencies, $\omega\tau \ll 1$, where τ is the carrier free-path time. Nondissipative dynamics pertains to plasma oscillations whose spectrum takes the form $\omega_p(k) = (2\pi e^2 n_s k/m)^{1/2}$ in the two-dimensional ($2D$) case (Ref. 3). We calculate in this paper the spectrum of the plasma oscillations in the dissipative and nondissipative regions, with allowance for retardation effects.

2. Consider a $2D$ layer with conductivity σ perpendicular to the z axis. The complete set of Maxwell equations for the vector and scalar potentials $\mathscr{A} = (\mathbf{A}, A_z)$ and φ and the material equations are

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\begin{Bmatrix}\varphi\\ A_z\\ \mathbf{A}\end{Bmatrix}=\begin{Bmatrix}4\pi\rho\\ 0\\ 4\pi\mathbf{j}/c\end{Bmatrix}\delta(z), \tag{1}$$

$$\operatorname{div}\mathbf{A}+\frac{\partial A_z}{\partial z}+\frac{1}{c}\frac{\partial\varphi}{\partial t}=0, \tag{2}$$

$$\mathbf{j}=-\sigma\left(\nabla\varphi+\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right). \tag{3}$$

The continuity equation

$$\partial\rho/\partial t+\operatorname{div}\mathbf{j}=0 \tag{4}$$

follows from Eqs. (1) and (2).

We seek the natural oscillations in the system described by Eqs. (1)–(3) in the form of a wave $\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$ (where $\mathbf{k}$ is a two-dimensional wave vector) propagating along the $2D$ layer of the wave and having a field localized near the layer. The potentials $\mathbf{A}$, A_z, and φ are therefore proportional to $\exp(-\varkappa z)$ where

$$\varkappa=(k^2-\omega^2/c^2)^{1/2}, \quad \operatorname{Re}\varkappa>0.$$

It follows from (1) that $A_z = 0$ and the vector potential can be sought in the form $\mathbf{A} = (\mathbf{k}/k)A_\parallel + [\mathbf{k}\times\mathbf{l}_z]A_\perp$. Expressing ρ, $\mathbf{j}$, and φ with the aid of (1), (2), and (3) in terms of $\mathbf{A}$, we obtain the dispersion relation. The transverse-wave spectrum is given by the equation

$$\left(k^2-\frac{\omega^2}{c^2}\right)^{1/2}-i\frac{2\pi\sigma_\omega}{c}\frac{\omega}{c}=0, \tag{5}$$

in which account is taken of the frequency dispersion of the conductivity

$$\sigma_\omega=[ne^2\tau/m(1-i\omega\tau)].$$

The dispersion equation for the longitudinal waves is

$$1+i\frac{2\pi\sigma_\omega}{\omega}\left(k^2-\frac{\omega^2}{c^2}\right)^{1/2}=0. \tag{6}$$

Equations (5) and (6) contain a characteristic parameter $x = 2\pi\sigma/c$. Analysis of Eq. (5) shows that the transverse mode is purely relaxational

$$\omega_t(k)=-i\begin{cases}1/\tau, & k\to\infty\\ \left.\begin{matrix}(1-x)/\tau, & x<1\\ ck/(x^2-1)^{1/2}, & x>1\end{matrix}\right\}, & k\to 0.\end{cases} \tag{7}$$

The longitudinal mode can either be purely relaxational or correspond to weakly damped plasma oscillations. The real and imaginary parts of $\omega_l(k)$ are shown in Figs. 1a and 1b for $x<1$ and $x>1$, respectively. If $x<1$, then $\operatorname{Re}\omega_l(k)$ differs from zero at $k>k_c$, where

$$k_c=(c\tau)^{-1}\frac{1}{4x}\left[\frac{1-6x^2-4x^4+(1+2x^2)(1+8x^2)^{1/2}}{2}\right]^{1/2}. \tag{8}$$

If $k \ll k_c$, the spectrum is purely relaxational:

$$\omega_l=-i\frac{2\pi\sigma k}{(1-x^2)^{1/2}}, \tag{9}$$

and corresponds as $k\to\infty$ to ordinary plasma oscillations:

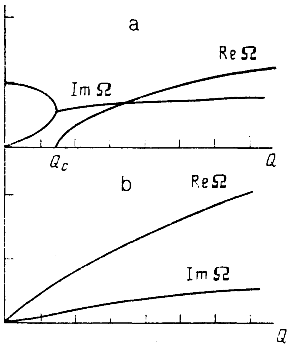

FIG. 1. Longitudinal-wave spectrum in a film with conductivities $\sigma<c/2\pi$ (a) and $\sigma>c/2\pi$ (b). The coordinates nondimensionalized with the collision time τ, are the $2D$ wave vector $Q=ck\tau$ and the plasma-oscillation frequency $\Omega=\omega\tau$. The threshold wave vector Q_c and Im $\Omega(0)$ vanish at the point $2\pi\sigma/c=1$.

 0038-5646/89/061150-03$04.00

$$\omega_l = \left(\frac{2\pi e^2 n}{m}k\right)^{1/2} - \frac{i}{2\tau}. \tag{10}$$

As $x \to 1$ the threshold value of $k_c \to 0$ and the plasma-oscillation spectrum at $x = 1$ and $k \ll 1/c\tau$ assume the form

$$\omega_l(k) = \left[\frac{c^2k^2}{2\tau}\right]^{1/3} e^{-i\pi/6}. \tag{11}$$

For $0 < x - 1 \ll 1$ the spectrum (11) is conserved in the region $(x-1)^{3/2}/c\tau \ll k \ll 1/c\tau$, and the longest-wavelength oscillations obey the dispersion law

$$\omega_l = xck/(x^2-1)^{1/2} - i\tau[xck/(x^2-1)]^2. \tag{12}$$

If $x - 1 \gtrsim 1$, the intermediate asymptotic value (11) is not realized and the plasmon dispersion law can be described by Eq. 12 for all $k < 1/c\tau$. It is seen from (12) that these long-wave oscillations attenuate weakly. The reason is that the wave field is concentrated in a region of thickness $\delta = 1/\mathrm{Re}\,\varkappa \sim k^{-2}$ much larger than the thickness of the k^{-1} region in which the dissipation takes place. The transition from the purely relaxational spectrum (9) to the plasma oscillations (12) can be tracked by examining the character of the field distribution near the layer. For $x < 1$ the damping rate $\varkappa$ is a real number and as $x \to 1$ we have $\delta \to 0$, i.e., the region in which the field is concentrated becomes infinitesimally thin. For $x > 1$ this region broadens, and the field manages to oscillate many times over the thickness δ, which is determined in this case by the frequency dispersion of σ_ω.

3. The velocity at which a signal can be propagated by a wave with a dispersion $\omega(k)$ can be determined by investigating the evolution of the wave packet. For example, the wave intensity $\psi(r,t)$ is determined by the integral

$$\psi(\mathbf{r},t) = \int (d\mathbf{k})\Delta_3(\mathbf{k}-\mathbf{k}_0)\exp[i\mathbf{kr} - i\omega(\mathbf{k})t],$$
$$\Delta_d(\mathbf{q}) = (2\pi\gamma^2)^{-d/2}\exp[-q^2/2\gamma^2].$$

Evaluating the integral, we find

$$\psi(\mathbf{r},t) \sim \exp[i\mathbf{k}_0\mathbf{r} - i\omega(\mathbf{k}_0)t]\exp\left[-\gamma^2\left(\mathbf{r} - \frac{\partial\omega}{\partial\mathbf{k}}t\right)^2 \Big/ 2\right],$$

from which it follows that $\mathbf{v} = \partial\omega/\partial\mathbf{k}$.

A surface-wave packet evolves not only along the plane but also in a transverse direction, and is determined by the dispersion of $\omega(k)$ and $\varkappa(k)$. In the case of interest to us we have $\varkappa = \varkappa_1 + i\varkappa_2$ with $\varkappa_1 \ll \varkappa_2$. Just as above

$$\psi(\mathbf{x},z,t) = \int (d\mathbf{k})\Delta_2(\mathbf{k}-\mathbf{k}_0)\exp[i\mathbf{kx} - i\omega(\mathbf{k})t - \varkappa(\mathbf{k})|z|]$$
$$\sim \exp[i\mathbf{k}_0\mathbf{x} - i\omega(\mathbf{k}_0)t - \varkappa(\mathbf{k}_0)|z|]\exp\left\{-\frac{\gamma^2}{2}\left[X^2 - 2i\frac{\partial\varkappa_1}{\partial\mathbf{k}}|z|\mathbf{X} + \left(\frac{\partial\varkappa_1}{\partial k}\right)^2 z^2\right]\right\}.$$

$$\mathbf{X} = \mathbf{x} - \frac{\partial\omega}{\partial\mathbf{k}}t - \frac{\partial\varkappa_2}{\partial\mathbf{k}}|z|, \quad \gamma \ll \varkappa_1. \tag{13}$$

We need not fear the last term, which increases as $|z| \to \infty$, in the argument of the exponential. It stems from the fact that far from the $z = 0$ plane the field is determined by the packet component that is least attenuated. No such growth occurs if $|\mathbf{k} - \mathbf{k}_0|$ is bounded. The second term in the exponent is of similar origin and causes the phase of the wave to be different for equal x and t but different z. This implies bending of the packet's wave front. The first term in the exponent describes the time evolution of the packet, viz., the packet contracts as it propagates along the surface. The energy density in the $(\mathbf{x},z)$ plane is then transported along a vector $\mathbf{v} = (1, -\partial\varkappa_2/\partial\mathbf{k})$ at an angle θ to the normal:

$$\cos\theta = [1 + (\partial\varkappa_2/\partial k)^2]^{-1/2}, \tag{14}$$

the signal propagation velocity along the plane is therefore

$$\mathbf{v} = \frac{\partial\omega}{\partial\mathbf{k}}\frac{1}{1+[\partial\varkappa_2/\partial k]^2}. \tag{15}$$

The field distribution in a surface plasma wave Eq. (12) is shown for $x > 1$ in Fig. 2. The Poynting vector $\mathbf{S}$ makes an angle θ with the normal,

$$\cos\theta = c/2\pi\sigma. \tag{14a}$$

The normal part of $\mathbf{S}$ is connected with the transport of the field energy w to the $2D$ layer in which the dissipation takes place. The tangential component determines the energy transported by a plasmon with velocity

$$v = \frac{S}{w}\sin\theta = c\frac{(x^2-1)^{1/2}}{x}. \tag{15a}$$

The phase velocity $u = \omega/k$ is determined by the velocity of the point where the phase front crosses the $2D$-layer plane:

$$u = \frac{c}{\sin\theta} = c\frac{x}{(x^2-1)^{1/2}}. \tag{15b}$$

The velocity (15b) is not observable as a signal-transport velocity, but the spectrum $\omega = uk$ can apparently manifest itself in Raman scattering of light. At short wavelengths $k \gg 1/c\tau$ the regime (12) is replaced by the usual dispersion law of plasma oscillations (10).

4. The surface-wave dispersion law determines the poles of the coefficients of reflection (R) and absorption (P), by a $2D$ layer, of a wave incident at an angle β to its normal. To calculate these coefficients we have solved the system (1)–(3) with boundary conditions corresponding to incident, reflected, and transmitted waves. The transmission and reflection amplitudes[1)] t and r, as well as the absorption coefficient P for the polarization corresponding to the intensity $\mathbf{E}$ of the electric field lying in the incidence plane, are given by

$$r_\parallel = x\cos\beta/(1+x\cos\beta), \quad t_\parallel = 1/(1+x\cos\beta),$$
$$P_\parallel = 1 - |r_\parallel|^2 - |t_\parallel|^2 = 2x\cos\beta/(1+x\cos\beta)^2. \tag{16}$$

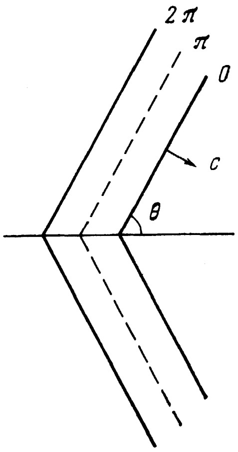

FIG. 2. Field distribution in a long-wave plasma oscillation in a film of conductivity $\sigma > c/2\pi$. The solid lines show the constant-phase surfaces.

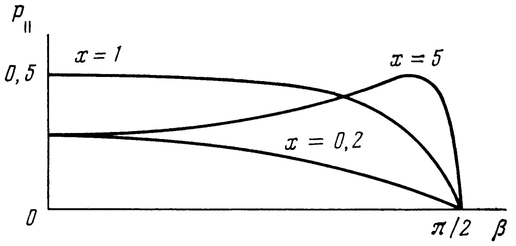

FIG. 3. Absorption coefficient of a wave polarized in the incidence plane $P_{\|}$ vs the angle of incidence or different values of the parameter x.

Figure 3 shows the $P_{\|}(\beta)$ dependence for $\omega\tau = 0.1$ at various values of x. If a wave of frequency ω and wave vector $(\mathbf{k},q)$ is incident on the film, the amplitudes $r_{\|}$ and $t_{\|}$ have a pole at the unphysical values $q = i\varkappa(\omega,k)$, where ω and k are related by Eq. (6). For $x > 1$ the angular dependence $P(\beta)$ has at $\cos\beta = 1/x$ a maximum that meets the condition that the incident wave be in resonance with the surface plasmon

$$\omega = uk_{\|} = (u/c)\omega\sin\beta.$$

In high-mobility samples, when $x \gg 1$, these resonance conditions are met for almost grazing incidence.

For another polarization of the incident wave, when $\mathbf{E}$ is perpendicular to the plane of incidence, the quantities $r_{\perp}$, $t_{\perp}$, and $P_{\perp}$ can be obtained from (16) by following the rule

$$r_{\perp}(x) = t_{\|}(1/x), \quad t_{\perp}(x) = r_{\|}(1/x),$$
$$P_{\perp}(x) = P_{\|}(1/x).$$

Therefore when natural light is incident on the film the transmitted light is predominantly polarized in the incidence plane. For normal incidence of the wave we have $r_{\|} = r_{\perp}$ and $t_{\|} = t_{\perp}$. The results obtained from (16) with $\cos\beta = 1$ agree with the answer to problem 5 in §86 of Ref. 4.

5. It is natural to compare the electrodynamics of a thin film with the electrodynamics of a thin wire. The wire conductivity has the dimension of the diffusion coefficient, and at low frequency the relaxation spectrum is of the form $\omega(k) = -2i\sigma_1 k^2$ (Ref. 1). At high frequencies, without allowance for retardation, we have $\omega_p = (2ne^2/m)^{1/2}k$. If $\xi = 2ne^2/mc^2 > 1$, however, the complete set of Maxwell equations must be solved. Such scattering leads to a dispersion law

$$\omega(k) = -\frac{i}{2\tau}\frac{1}{1+\xi}\{1\pm[1-4\xi(1+\xi)(ck\tau)^2]^{1/2}\}. \tag{17}$$

In the low-frequency region we obtain for $k \ll k_1 = (1/2c\tau)[\xi(1+\xi)]^{-1/2}$ the known result $\omega = -2i\sigma_1 k^2$, while for $k \gg k_1$ we have

$$\omega_p(k) = ck[\xi/(1+\xi)]^{1/2}. \tag{18}$$

The authors thank I. V. Kukushkin and V. B. Timofeev for posing the question in the title of the article, as well as Yu. S. Barash and M. I. Kaganov for helpful discussions.

[1)]We have defined r and t as the ratios of the electric field intensity in the reflected (transmitted) wave to that in the incident wave.

[1]M. I. D'yakonov and A. S. Furman, Zh. Eksp. Teor. Fiz. **92**, 1012 (1987) [Soviet Phys. JETP **65**, 574 (1987)].
[2]A. O. Govorov and A. V. Chaplik, Poverknost' **12**, 5 (1987)
[3]T. Ando, A. B. Fowler, and F. Stern, Rev. Mod. Phys. **54**, 437 (1982).
[4]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon, 1984, §86.

Translated by J. G. Adashko