Spinor inverse problems for a gravitational field

I. V. Bogdanov

Leningrad Institute of Exact Mechanics and Optics
(Submitted 11 November 1988)
Zh. Eksp. Teor. Fiz. **96**, 14–24 (July 1989)

Spherically symmetric inverse problems for the scattering of spin-$\frac{1}{2}$ fermions by a static gravitational field are considered within the framework of the general theory of relativity. Methods are developed for the reconstruction of the metric tensor from the scattering data for the Dirac equation in the Schwarzschild metric. The problem of finding the link between the *S*-matrix and the Hamiltonian operator in curved space for a neutrino with fixed zero orbital angular momentum and for a massless or massive field with fixed energy is discussed. The main links in the neutrino algorithms consist of two definite systems of two nonlinear ordinary differential equations constructed from the scattering data. It is established, first, that the inverse problems studied in this paper generalize the previously solved classical inverse problems for a gravitational field to the quantum case. Second, they generalize the familiar Marchenko and Regge-Newton methods in quantum theory of scattering to the case of a gravitational field. Third, these inverse problems are a logical extension of the inverse problems for the Klein-Gordon-Fock equation studied in a previous paper. The results may be of interest in atrophysics in connection with the direct determination of the inner structure of objects.

1. INTRODUCTION

In a number of recent papers[1–3] the author has studied in the framework of the theory of general relativity (TGR) the static inverse problems of scattering of classical particles by a spherically symmetric gravitational field. In these papers algorithms for the reconstruction of the curvature tensor of spacetime from the asymptotic characteristics of the geodesics of massless or massive particles were developed.

In the present paper the inverse problem in TGR for the scattering of quantum particles will be considered. Such problems will be solved here in the general spinor case for a spin-$\frac{1}{2}$ fermion, i.e., for the Dirac equation.

The generalization of nonrelativistic methods in inverse problems to relativistic spinor particles in flat space has been known for quite some time. The Gel'fand-Levitan method has been developed for the Dirac equation by Prats and Toll.[4] New results, as well as rigorous mathematical justification of previously known ones, were then presented by Gasymov and Levitan.[5–7] The Marchenko method for the Dirac equation was developed by Weiss, Stahel and Scharf.[8] Finally, the modification of the Regge-Newton (fixed energy) method for the Dirac equation was carried out by Coudray and Coz.[9] However, the gravitational field was not considered in the above papers.

The aim of the present paper is to determine the relation between the *S*-matrix, which can be introduced in asymptotically flat spaces, and the Dirac Hamiltonian operator in a gravitational field.

We shall use the relativistic units, where $\hbar = c = 1$.

The structure of the spherically-symmetric static gravitational field is given by the interior Schwarzschild metric[10]

$$ds^2=e^{\nu}dt^2-e^{\mu}dr^2-r^2(d\theta^2+\sin^2\theta d\varphi^2),\quad \nu<0,\quad \mu>0, \tag{1}$$

which we extend to the entire space. We assume that the metric functions $\nu(r)$ and $\mu(r)$ are regular at zero and that the space with the metric (1) is asymptotically flat, i.e., that ν and μ, together with all their derivatives, vanish sufficiently rapidly as $r\to\infty$.

The corresponding system of three Einstein equations within the framework of the hydrodynamic static model is given in Ref. 11. In such a model the gravitating matter is characterized by radial distributions of density $\rho(r)$ and pressure $p(r)$, which are connected by the Oppenheimer-Volkov integro-differential equation (the static condition). In the indicated approximations the Einstein system of equations can be integrated by quadratures; its solution has the form

$$\mu=-\ln\left(1-\varkappa r^{-1}\int_0^r d\xi\,\xi^2\rho\right),$$
$$\nu=\int_r^\infty d\xi\,\xi^{-1}e^{\mu}(e^{-\mu}-\varkappa\xi^2p-1), \tag{2}$$

where $\varkappa$ is the Einstein gravitational constant. The inverse formulas

$$\rho=(1-e^{-\mu}+r\mu'e^{-\mu})/\varkappa r^2,$$
$$p=(e^{-\mu}-1+r\nu'e^{-\mu})/\varkappa r^2 \tag{3}$$

permit the reconstruction of the matter density and pressure from the metric tensor.

From the Oppenheimer-Volkov equations, which follow from the field equations, one may deduce the differential equality

$$2r^2\nu''+r^2\nu'^2-r^2\mu'\nu'-2r\nu'-2r\mu'+4e^{\mu}-4=0, \tag{4}$$

which connects the functions ν and μ. In this manner, the components $g_{00}\equiv e^{\nu}$ and $g_{11}\equiv e^{\mu}$ of the metric tensor are mutually dependent in the hydrodynamic model under discussion. In the case of a static field the system of three Einstein equations contains only two unknown functions—μ and ν, and so turns out to be overdetermined.

The wave inverse problems here considered consist in the reconstruction of the metric functions μ and ν, i.e., the curvature of spacetime, from spinor sattering data that de-

 0038-5646/89/070007-06$04.00

pend on the total angular momentum j for fixed energy E, and on E for fixed j.

In Sec. 2 the Dirac equation is discussed in the Schwarzschild metric. In Sec. 3 the inverse scattering problem is solved at fixed angular momentum $j = \frac{1}{2}$ for the neutrino. In Sec. 4 the inverse scattering problem is analyzed at fixed energy for massless and for massive particles.

2. THE DIRAC EQUATION

The interaction of classical spinor (spin $\frac{1}{2}$) and gravitational fields is governed by the generally-covariant Dirac equation. The properties of such an equation are discussed in detail in the book by Grib, Mamaev and Mostepanenko.[12] It has the form

$$i\gamma^n(x)\nabla_n\psi(x)=m\psi(x), \tag{5}$$

where $\psi(x)$ is a Dirac bispinor, ∇_n is the spinor covariant derivative in Riemann space, $\gamma^n(x)$ is a variable 4-vector (with respect to the label n) under general transformations of coordinates and is connected with the constant Dirac matrices γ^n; here and in the following summation over repeated indices is understood.

Let us introduce a quartet of frame 4-vectors $h^k_{(a)}$, numbered by the index $a = 0,1,2,3$. They are normalized according to $h^{(a)}_k h_{(b)k} = \eta_{ab}$, where η_{ab} is the metric tensor in the tangent Minkowski space, and they are called the tetrad. We also introduce the dual tetrad $h^{(a)}_k$, defined by the conditions $h^{(a)}_k h^k_{(b)} = \delta^a_b$, with $h^{(a)}_i h_{(a)k} = g_{ik}$. The variable fourth order matrices are given then by $\gamma^n(x) = h^n_{(a)}\gamma^a$, and the spinor covariant derivative has the form

$$\nabla_n=\partial_n+{}^1/_4 C_{abc}h^{(c)}_n\gamma^b\gamma^a. \tag{6}$$

Here C_{abc} are the Ricci rotation coefficients, which are connected to the tetrad vectors by the relations

$$C_{abc}=(\nabla_m h^n_{(a)})h_{(b)n}h^m_{(c)}, \tag{7}$$

with ∇_m the covariant derivative of the vector field. The Dirac equation in the Riemann space, Eq. (5), is conformally invariant for $m = 0$.

In terms of the tetrad the square of the interval is given by

$$ds^2=g_{ik}\,dx^i\,dx^k=\eta_{ab}(h^{(a)}_i\,dx^i)(h^{(b)}_k\,dx^k).$$

We may choose, in the case of the centrally-symmetric metric (1), for the basis of the tangent pseudo-euclidean space the vectors with components $h^0_{(0)} = e^{-\nu/2}$, $h^1_{(1)} = e^{-\mu/2}$, $h^2_{(2)} = r^{-1}$, $h^3_{(3)} = (r\sin\theta)^{-1}$, the remaining components being trivial. The components of the vectors of the dual tetrad are inverse in magnitude. The nontrivial covariant components are equal to

$$h_{(0)0}=e^{\nu/2},\ h_{(1)1}=-e^{\mu/2},\ h_{(2)2}=-r,\ h_{(3)3}=-r\sin\theta.$$

The variable "Dirac matrices" have the form

$$\gamma^0(r)=e^{-\nu/2}\gamma^0,\quad \gamma^1(r)=e^{-\mu/2}\gamma^1,$$
$$\gamma^2(r)=r^{-1}\gamma^2,\quad \gamma^3(r,\theta)=(r\sin\theta)^{-1}\gamma^3. \tag{8}$$

Let us calculate now the spinor covariant derivative operators. To this end it is necessary to know the Ricci rotation coefficients. The Christoffel symbols for a centrally-symmetric gravitational field are given in Ref. 10. Calculations according to formula (7) show that from among the 64 Ricci rotation coefficients only the following 9 are nontrivial:

$$C_{001}=\nu'(e^{-\mu/2}-e^{-\nu/2})/2,\quad C_{010}=-C_{100}=-\nu' e^{-\mu/2}/2,$$
$$C_{122}=C_{133}=-C_{212}=-C_{313}=-r^{-1}e^{-\mu/2},\ C_{233}=-C_{323}=-r^{-1}\operatorname{ctg}\theta.$$

Substitution of these values into Eq. (6) yields the following expressions for the spinor covariant derivatives

$$\nabla_0=\partial_0+{}^1/_4\gamma^0\gamma^1\nu' e^{(\nu-\mu)/2},$$
$$\nabla_1=\partial_1+{}^1/_8\nu'(1-e^{(\mu-\nu)/2}),$$
$$\nabla_2=\partial_2+{}^1/_2\gamma^1\gamma^2 e^{-\mu/2},$$
$$\nabla_3=\partial_3+{}^1/_2\gamma^2\gamma^3\cos\theta+{}^1/_2\gamma^1\gamma^3 e^{-\mu/2}\sin\theta,$$

which together with (8) lead to the Dirac equation in the Schwarzschild metric:

$$i[e^{-\nu/2}\gamma^0\partial_0+e^{-\mu/2}\gamma^1\partial_1+{}^3/_8\nu' e^{-\mu/2}\gamma^1-{}^1/_8\nu' e^{-\nu/2}\gamma^1+r^{-1}e^{-\mu/2}\gamma^1$$
$$+r^{-1}\gamma^2\partial_2+{}^1/_2 r^{-1}\operatorname{ctg}\theta\gamma^2+(r\sin\theta)^{-1}\gamma^3\partial_3]\psi(x)=m\psi(x). \tag{9}$$

In the standard representation

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

the stationary Dirac equation (9) breaks up into a system of two spinor equations:

$$(m-e^{-\nu/2}E)\psi_1$$
$$=i[e^{-\mu/2}\sigma_1\partial_r\psi_2+{}^3/_8\nu' e^{-\mu/2}\sigma_1\psi_2-{}^1/_8\nu' e^{-\nu/2}\sigma_1\psi_2+r^{-1}e^{-\mu/2}\sigma_1\psi_2$$
$$+r^{-1}\sigma_2\partial_\theta\psi_2+{}^1/_2 r^{-1}\operatorname{ctg}\theta\sigma_2\psi_2+(r\sin\theta)^{-1}\sigma_3\partial_\varphi\psi_2], \tag{10}$$

$$(m+e^{-\nu/2}E)\psi_2=-i[e^{-\mu/2}\sigma_1\partial_r\psi_1+{}^3/_8\nu' e^{-\mu/2}\sigma_1\psi_1-{}^1/_8\nu' e^{-\nu/2}\sigma_1\psi_1$$
$$+r^{-1}e^{-\mu/2}\sigma_1\psi_1+r^{-1}\sigma_2\partial_\theta\psi_1+{}^1/_2 r^{-1}\operatorname{ctg}\theta\sigma_2\psi_1+(r\sin\theta)^{-1}\sigma_3\partial_\varphi\psi_1], \tag{11}$$

containing the Pauli $\boldsymbol{\sigma}$ matrices, for the functions $\psi_1(r,\theta,\varphi)$ and $\psi_2(r,\theta,\varphi)$.

This system can be reduced, upon separation of the variables, to a system of two scalar ordinary equations. Separation of variables in the Dirac equation for the spherical de Sitter world has been considered by Chernikov and Shavokhina[13] in bispherical coordinates. Separation of variables in the Dirac equation for the Schwarzschild world in spherical coordinates can be carried out by analogy with the potential case.[14,15]

Indeed, the total angular momentum j and its component m are conserved for motion in an arbitrary central field. We shall look for spinor components of the wave function in the form

$$\psi_1=r^{-1}f(r)G\Omega_{jlm},\quad \psi_2=i^{l-l'+1}r^{-1}g(r)G\Omega_{jl'm}, \tag{12}$$

where f and g are scalar radial functions, the matrix $G(\theta,\varphi)$ realizes a rotation,[1] $\Omega_{jlm}(\theta,f)$ are spinor spherical harmonics, and $l = j \pm \frac{1}{2}$, $l' = j \pm \frac{1}{2}$. Let us introduce the number λ such that $\lambda = -j - \frac{1}{2} = -l - 1$ for $j = l + \frac{1}{2}$ and $\lambda = j + \frac{1}{2} = l$ for $j = l - \frac{1}{2}$. This number takes on the values $\lambda = ... -2, -1,1,2,...$, with $\lambda > 0$ for $j = l - \frac{1}{2}$ and $\lambda < 0$ for $j = l + \frac{1}{2}$. Then substitution of Eq. (12) into the spinor Eqs. (10) and (11) gives the following system for the radial wave functions:

$$f'+{}^1/_8(3-e^{(\mu-\nu)/2})\nu' f+\lambda r^{-1}e^{\mu/2}f-e^{\mu/2}(e^{-\nu/2}E+m)g=0, \tag{13}$$

$$g'+{}^1/_8(3-e^{(\mu-\nu)/2})\nu' g-\lambda r^{-1}e^{\mu/2}g+e^{\mu/2}(e^{-\nu/2}E-m)f=0. \quad (14)$$

In empty space $\nu\equiv\mu\equiv0$ and this system, naturally, goes over into the well-known system for free Dirac radial functions.

The first-order equations (13) and (14) are equivalent to one second-order equation for the function f:

$$\begin{aligned}&f''+[(3-u)\nu'/4-\mu'/2+(1+mE^{-1}e^{\nu/2})^{-1}\nu'/2]f' \quad (15)\\ &+[(3-u)\nu''/8+u^2\nu'^2/64+u\nu'^2/16-3\mu'\nu'/16\\ &+(1+mE^{-1}e^{\nu/2})^{-1}(u\nu'^2/16+\lambda r^{-1}e^{\mu/2}\nu'/2)\\ &-\lambda(\lambda e^{\mu/2}+1)r^{-2}e^{\mu/2}+e^{\mu}(e^{-\nu}E^2-m^2)]f=0,\end{aligned}$$

where $u(r)\equiv\exp[\frac{1}{2}(\mu-\nu)]$, and

$$g=e^{-\mu/2}(e^{-\nu/2}E+m)^{-1}[f'+{}^1/_8(3-u)\nu' f+\lambda r^{-1}e^{\mu/2}f]. \quad (16)$$

The equation (15) is fairly complicated: the coefficient of the first derivative depends on the energy, while energy and angular momentum enter the coefficient of f in a nontrivial fashion.

3. THE INVERSE PROBLEM AT FIXED ANGULAR MOMENTUM

We consider this inverse problem in the important case of a massless spinor field, i.e., the neutrino. Upon setting in the Dirac equation (15) the mass $m=0$ we have

$$\begin{aligned}&f''+(5\nu'/4-\mu'/2-u\nu'/4)f'\\ &+(u\nu''/8+u^2\nu'^2/64+3\nu'^2/16-3\mu'\nu'/16\\ &+\lambda r^{-1}e^{\mu/2}\nu'/2-\lambda^2r^{-2}\\ &(e^{\mu}-1)-\lambda r^{-2}(e^{\mu/2}-1)-\lambda(\lambda+1)r^{-2}+u^2\omega^2)f=0, \quad (17)\end{aligned}$$

where we have introduced the frequency $\omega=E$. Thus in the massless approximation the dependence of the coefficients of the wave equation on E and λ is substantially simplified. Let us note that the two terms preceding the centrifugal one in the coefficient of f are regular at zero, since it follows from Eq. (2) that $e^{\mu}-1\sim\varkappa r^2\rho\to0$ as $r\to0$.

We shall construct here a solution of the inverse problem only for the fixed value $\lambda=-1$, when the centrifugal term in Eq. (17) is absent and there is no singularity at zero. This case corresponds to a total angular momentum $j=\frac{1}{2}$ and to zero orbital angular momentum.

The space with the metric (1) is asymptotically flat, and the asymptotics for $r\to\infty$ of the radial wave functions for $l=0$ has the form

$$\begin{pmatrix} f\\ g\end{pmatrix}\infty\, 2^{1/2}\begin{pmatrix}\sin[\omega r+\delta(\omega)]\\ \cos[\omega r+\delta(\omega)]\end{pmatrix}.$$

The inverse scattering problem under study consists of the reconstruction of the metric tensor from the frequency dependence of the phase shift $\delta(\omega)$ of the neutrino wave.

Let us perform in the Dirac equation (17) for $\lambda=-1$ the standard change of the function

$$f=\chi\exp\left[\frac{1}{8}(2\mu-5\nu)\right]\exp\left(-\frac{1}{8}\int_r^{\infty}d\xi\, u\nu'\right). \quad (18)$$

which eliminates the first derivative. We then obtain the equation

$$\chi''-U\chi+\omega^2Q\chi=-\omega^2\chi \quad (19)$$

for the function χ, which approaches f asymptotically as $r\to\infty$. This equation is equivalent in form to the Schrödinger equation for an S-wave in a central field with potential $U-k^2Q$ that depends linearly on the square on the momentum $k=\omega$. Here

$$\begin{aligned}U&\equiv(\nu''-\mu'')/4+(\nu'-\mu')^2/16\\ &+r^{-1}e^{\mu/2}\nu'/2+r^{-2}e^{\mu/2}(e^{\mu/2}-1),\\ Q&\equiv u^2-1, \quad (20)\end{aligned}$$

and the "potential" $Q>0$ while U and $Q\to0$ as $r\to\infty$.

To solve the inverse problem we introduce the functions

$$\beta(r)\equiv r-\int_r^{\infty}d\xi(u-1),\quad \Phi(r)\equiv(\beta')^{-1/2}=u^{-1/2}. \quad (21)$$

The function β increases monotonically and has on the semiaxis $r>0$ only one zero at r_0, so that $\beta(r)>0$ for $r>r_0$.

If one first changes in the Schrödinger equation (19) the argument $r\to\beta(r)$, and then the function $\chi\to\Phi^{-1}\chi$, then the potential loses its dependence on energy. However, the boundary condition at $r=0$ then goes over into the boundary condition at $r=r_0$, and the regular wave function may have a node at the point r_0.

We shall further suppose that the gravitating matter is distributed in the exterior of an absolutely impenetrable sphere of finite radius r_0. In the inverse problem the value of r_0 is unknown *a priori* and is itself subject to determination. The regular solution satisfies $\chi(\omega,r_0)=0$, while its radial derivative satisfies $\chi'(\omega,r_0)=\Phi^{-1}(r_0)$. For empty space $r_0=0$.

The method of inversion, which coincides for $Q=0$ with the familiar Marchenko method, can be conveniently given the following form.

We introduce the linear integral equation—analog of the Marchenko equation

$$B(r,r')=\Phi(r)D(\beta(r)+r')+\int_{\beta(r)}^{\infty}d\xi\, B(r,\xi)D(\xi+r') \quad (22)$$

with symmetric kernel

$$D(r)=(2\pi i)^{-1}\int_{-\infty}^{\infty}d\omega[\exp 2i\delta(\omega)-1]e^{i\omega r}, \quad (23)$$

which (in the absence of bound states) is constructed from the phase shift.

The integral equation (22) is the principal link between the S-matrix, given for all ω, and the curvature. This equation determines the triangular integral operator of the transformation from the free to the perturbed solution. Let $y(\omega,r)$ be the Jost solution for the wave equation (19), i.e., $y\propto\exp(i\omega r), r\to\infty$, then in the upper half-plane of the complex frequency $\mathrm{Im}\,\omega\geqslant0$ we have

$$y(\omega,r)=\Phi(r)\exp[i\omega\beta(r)]+\int_{\beta(r)}^{\infty}d\xi\, B(r,\xi)\exp(i\omega\xi),\ r>r_0. \quad (24)$$

This is the analog of the Levin representation.[16] For the Jost solution we have the boundary condition

$$y(\omega,r_0)=Y(\omega)=|Y(\omega)|e^{-i\delta},$$

where $Y(\omega)$ is the Jost function. The region of triangularity of the kernel B is given by the determinant $g_{00}g_{11}$ of the radial-temporal part of the metric tensor.

The high-frequency asymptotics of the wave functions is, naturally, not at all free. It can be found from the corresponding asymptotics in the energy dependent problem by the replacement $r \to \beta$ and multiplication by Φ. For $\omega \to \infty$ and $r > r_0$ we have the rapidly oscillating functions

$$\chi(\omega, r) \propto \omega^{-1}\Phi(r)\sin[\omega\beta(r)], \quad y(\omega, r) \propto \Phi(r)\exp[i\omega\beta(r)],$$

and the high frequency limit of the phase shift is $\delta(\infty) = 0$.

For the kernel $B(r,r')$ we have the boundary condition at $r' = \beta$

$$\Phi U = \Phi'' - \beta'' B(r, \beta) - 2\beta'(d/dr)B(r, \beta). \tag{25}$$

The last formula relates the potentials U and Q through phase-shift information.

The gravitational analog of the Marchenko equation (22) is a linear integral Fredholm equation of the second kind, which can be inverted.[17] If $\widetilde{D}$ is the resolvent of the Fredholm kernel D (it depends parametrically on the integration limit β), then

$$B(r, r') = \Phi(r) \times \left[D(\beta(r)+r') + \int_{\beta(r)}^{\infty} d\xi D(\beta(r)+\xi)\widetilde{D}(\xi, r', \beta(r))\right] \tag{26}$$

and at the boundary $r' = \beta$ we have

$$B(r, \beta) = \Phi(r)H(\beta), \tag{27}$$

$$H(\beta) = D(2\beta) + \int_{\beta}^{\infty} d\xi\, D(\beta+\xi)\widetilde{D}(\xi, \beta, \beta). \tag{28}$$

For the metric tensor one may derive a closed system of two ordinary differential equations. Let us do so in terms of the functions ν and β. It is seen from the definition (21) that

$$\mu = \nu + 2\ln\beta', \tag{29}$$

and, upon substitution of μ and μ' in Eq. (4), we obtain the first equation of the system. The second equation is found from the boundary condition (25), keeping in mind Eqs. (20), (21), (27) and (29). The indicated system

$$r^2\beta'\nu'' - r(r\beta'' + 2\beta')\nu' + 2\beta'^3 e^{\nu} - 2r\beta'' - 2\beta' = 0, \tag{30}$$

$$re^{\nu/2}\nu' + 2\beta' e^{\nu} - 2e^{\nu/2} + 4r^2 H\beta' = 0 \tag{31}$$

contains the relativistic static condition (30) and the connection between the metric and the kernel of the transformation operator (31). In principle this system can be converted into a single closed nonlinear third-order differential equation for the function $\beta(r)$, although such a procedure is rather unwieldy. At the stage when the derivative ν' is eliminated from the system one obtains for it an algebraic equation of fourth degree.

The nonlinear system of equations (30) and (31) represents the final result in the algorithm for the reconstruction of the curvature from the phase shift $\delta(\omega)$. The second equation of the system is obtained from the S-matrix, as the latter determines the form of the function $H(\beta)$.

Uniqueness of the solution of the inverse problem is ensured by the unambiguous choice of the function $H(\beta)$ in Eq. (31).

The main steps in the process of reconstruction of the metric tensor from the neutrino scattering phase shift $\delta(\omega)$ are the following. It is first necessary to accomplish the Fourier transformation (23), i.e., obtain the kernel D of the analog of the Marchenko linear integral equation (22) and find the Fredholm resolvent $\widetilde{D}$ of this kernel. Then with the help of Eq. (28) it is necessary to construct the function $H(\beta)$, which specifies the structure of the nonlinear system of two differential equations of first and second order, Eqs. (30) and (31), for the functions $\beta(r)$ and $\nu(r)$. After integration of this system we obtain the value r_0, which gives the transcendental equation $\beta(r) = 0$. Now one may calculate from formula (29) the metric component g_{11}. From the field equations (3) one may determine the matter pressure and density and, further, the equation of state of the matter.

Once the reconstructed metric is known one may find the exact neutrino wave function by quadratures. Indeed, given the function $\Phi(r)$ we may calculate from formula (26) the kernel of the transformation operator $B(r,r')$, and with the help of the analog of the Levin representation (24)—the Jost solution $y(r)$. Then the radial component f for the bispinor is given by formula (18), and the component g by formula (16) for $m = 0$, $E = \omega$, and $\lambda = -1$. In this manner, in the massless case the Dirac bispinor in the gravitational field is completely determined by its proper asymptotics for $r \to \infty$.

4. THE INVERSE PROBLEM AT FIXED ENERGY

We shall consider this inverse problem for the general case of a massive spinor field. For its solution we shall proceed not from the one second-order equation (15), but from the system of two first-order equations (13) and (14).

The regular normalized solution of the Dirac equation has for $r \to 0$ the asymptotics

$$F_\lambda(r) \equiv \begin{pmatrix} g_\lambda \\ f_\lambda \end{pmatrix} \sim \begin{pmatrix} a_\lambda \\ 0 \end{pmatrix} r^\lambda,$$

and for $r \to \infty$ the asymptotics with the wave number $k = (E^2 - m^2)^{1/2}$:

$$F_\lambda(r) \sim A_\lambda k^{-\lambda} \begin{pmatrix} \cos(kr - \pi\lambda/2 + \delta_\lambda) \\ k(E-m)^{-1}\sin(kr - \pi\lambda/2 + \delta_\lambda) \end{pmatrix}$$

since the Schwarzschild world is asymptotically flat.

The inverse problem being analyzed consists in the reconstruction of the curvature of spacetime from an infinite sequence of scattering phases δ_λ, given for all λ and at fixed energy.

Let us make in the system of Dirac equations (13) and (14) the change of function

$$F_\lambda = \widetilde{F}_\lambda \exp\left(-\frac{3\nu}{8}\right)\exp\left(-\frac{1}{8}\int_r^{\infty} d\xi\, u\nu'\right), \tag{32}$$

then the second terms on the left sides drop out and the system takes on the form

$$\tilde{f}_\lambda' + \lambda r^{-1}e^{\mu/2}\tilde{f}_\lambda - e^{\mu/2}(e^{-\nu/2}E + m)\tilde{g}_\lambda = 0, \tag{33}$$

$$\tilde{g}_\lambda' - \lambda r^{-1}e^{\mu/2}\tilde{g}_\lambda + e^{\mu/2}(e^{-\nu/2}E - m)\tilde{f}_\lambda = 0. \tag{34}$$

In the limit $r \to \infty$ the radial spinor $\widetilde{F}_\lambda \to F_\lambda$.

The centrifugal term in the Dirac equation is modified in a gravitational field and acquires a factor $e^{\mu/2}$, but this difficulty may be overcome. Let us replace the independent variable r in the system (33) and (34) by

$$b(r) \equiv \alpha r \exp\left[\int_0^r d\xi\, \xi^{-1}(e^{\mu/2}-1)\right], \tag{35}$$

where we have introduced a constant unknown in the inverse problem

$$\alpha \equiv \exp\left[\int_0^\infty d\xi\, \xi^{-1}(1-e^{\mu/2})\right].$$

We note that $b(0) = 0$ and $b(r) \sim r, r\to\infty$. Let us introduce the diagonal potential matrix

$$\hat{V} \equiv \begin{pmatrix} V+W & 0 \\ 0 & V-W \end{pmatrix}, \tag{36}$$

here the central "potentials" have the form

$$V(b) \equiv E(1-rb^{-1}e^{-\nu/2}), \quad W(b) \equiv m(rb^{-1}-1), \tag{37}$$

with $V<0$, $W>0$ and $V,W\to 0$ as $b\to\infty$. After the indicated change of argument the Dirac system of equations (33) and (34) can be expressed in matrix form

$$[i\sigma_2 d/db - \lambda b^{-1}\sigma_1 - (W+m)\sigma_3]F_\lambda = (E-V)F_\lambda.$$

In this form the system is mathematically equivalent to the Dirac equation, in which $r\to b$, for a central field with the potential matrix (36) in flat space. Thus the inverse problem under discussion is modeled by a relativistic inverse problem but in flat space, which can be solved by the appropriately modified method of Coudray and Coz.[9] This generalized algorithm has the following structure.

Let us introduce the matrix linear integral equation—the analog of the Coudray-Coz equation[2]

$$K(r,r') = P(b(r),r') - \int_0^{b(r)} d\xi\, \xi^{-1} K(r,\xi) P(\xi,r') \tag{38}$$

with kernel

$$P_{\alpha\beta}(r,r') = \sum_\lambda c_\lambda [F_\lambda^0(r)]_\alpha [\sigma_1 F_\lambda^0(r')]_\beta. \tag{39}$$

Here F_λ^0 are the free radial spinors, c_λ are constants calculated from the scattering phases δ_λ, and $_{\alpha,\beta}$ are spinor indices.

The integral equation (38) is the main link between the S-matrix, known for all λ, and the curvature. It determines the triangular integral operator, whose kernel $K(r,r')$ is a 2×2 matrix, of the transformation from the free to the perturbed solution. Consequently

$$F_\lambda(r) = F_\lambda^0(b(r)) - \int_0^{b(r)} d\xi\, \xi^{-1} K(r,\xi) F_\lambda^0(\xi), \tag{40}$$

and the region of triangularity of the kernel K is specified, as can be seen from Eq. (35), by the metric component g_{11}.

There are boundary conditions for the kernel $K(r,r')$ at $r' = b$:

$$V(r)+W(r) = -2b^{-1}K_{12}(r,b), \qquad V(r)-W(r) = 2b^{-1}K_{21}(r,b). \tag{41}$$

Obviously, all elements of the matrix K contribute to the potentials. The integral equation (38) is in fact equivalent to two systems of coupled equations, two each, for the functions K_{11}, K_{12} and K_{21}, K_{22}. The properties of such systems are analogous to the properties of the ordinary scalar Fredholm equation of the second kind.[17]

The neutrino has a mass $m=0$ and a "potential" $W=0$. In that case

$$V(r) = -2b^{-1}K_{12}(r,b) = 2b^{-1}K_{21}(r,b)$$

and one must verify the compatibility condition

$$K_{12}(r,b) + K_{21}(r,b) = 0. \tag{42}$$

If it is satisfied then the first of the identities (37), defining the potential V, gives a relation between the sought for functions $b(r)$ and $\nu(r)$. The second relation between these functions follows from the static condition (4) after passage from μ to b. Thereafter the whole problem reduces to the integration of a closed ordinary differential equation of second order with respect to, for example, the function $b(r)$. After this equation has been solved the metric function $\mu(r)$ can be found from formula (35). Then the constant α is also easily calculated.

For a massive spinor field $W\neq 0$ and the formulas (37) lead to unambiguous answers for the functions $b(r)$ and $\nu(r)$. However these functions do not necessarily satisfy the static condition (4), which acts here also as a compatibility condition.

Basically, the scheme for solving the inverse problem at fixed energy proceeds as follows. First, the coefficients c_λ in the expansion (39) for the kernel P have to be determined from the phase shifts δ_λ. This stage is analogous to the potential case[9] and consists of inversion of numerical matrices. Thereafter it is necessary to solve the analog of the Coudray-Coz matrix integral equation (38). The so-obtained transformation operator with kernel K permits the calculation of the potentials V and W from formula (41) and the radial spinor F_λ from formula (40). Once the metric has been reconstructed the exact radial wave function F_λ can be obtained with the help of Eq. (32).

Thus, for an arbitrary class of scattering data (coefficients c_λ) the inverse problem has in general no solution. For the neutrino field its solution is constrained by the compatibility condition (42), while for a massive field it is constrained by the condition (4) requiring the gravitational field to be static. However, for those classes of scattering data which satisfy the indicated conditions of compatibility, the inverse problems for massless and massive particles have unique solutions. To this end, in addition, one must require a definite rate of decrease of the phases δ_λ as $|\lambda|\to\infty$.

5. CONCLUSION

The main link in the inversion algorithm at fixed zero orbital angular momentum of the neutrino consists of a certain closed system of ordinary nonlinear differential equations for the metric tensor. This system is constructed from the scattering data and solves, in essence, the problem of relating the S-matrix and the Dirac Hamiltonian operator in the presence of gravitation.

The ultrarelativistic problem of the scattering of a spin-$\frac{1}{2}$ fermion by the curvature of spherical spacetime may be modeled by the nonrelativistic problem of the scattering of a spinless particle in a central field with a potential linearly dependent on the energy. The scattering of massive scalar particles by a gravitational field with spherical symmetry is also modeled by an analogous nonrelativistic problem.

The inverse problem, discussed in Sec. 3, has only been solved for a massless field with zero orbital angular momentum. The questions of generalization of this solution to partial waves with $\lambda \neq -1$ and to massive fields remain open.

Also remaining open is the question of identifying the class of scattering data for which the inverse problem at fixed energy, discussed in Sec. 4, has a unique solution.

It is known that in the helicity representation the massless Dirac equation splits into two independent spinor equations of first order.[18] It may be that the problem of reconstructing the metric from the S-matrix in the helicity representation, i.e., the inverse problem fot the Weyl equation, also admits of a solution in closed form.

Let us say a few words about possible applications. The exact inversion algorithms developed here for the neutrino, with its wave properties and spin taken into account, may turn out to be useful in astrophysics for the direct determination of the internal configuration of relativistic objects of various nature, should the latter be in some real situation "translucent" to probing beams. The linear dimensions of such objects should be sufficiently small on an astronomical scale, and the neutrinos could be of such low frequency that their de Broglie wavelength could be comparable in size with the dimensions of the objects being probed. In contrast to the case of massive scalar particles, which was considered in an article by the author and Demkov,[19] the neutrinos readily penetrate and are hardly absorbed by the gravitating matter.

As far as practical considerations are concerned one would like to "activate" a neutrino source of natural origin, for example a neutron star. The emission spectrum of neutron stars has been actively investigated lately.[20] One can imagine favorable circumstances under which the detection of the scattered neutrino currents from such a source could be effectively achieved on Earth.

I consider it my pleasant duty to thank Y. N. Demkov for his usual interest, also the leader of the city seminar of gravitationists A. A. Gribov for invitation to appear.

[1)]The explicit expression for G is given by formula (13.43) of Ref. 12.

[2)]In Ref. 9 the radial spinor is denoted by $\binom{f_\varkappa}{g_\varkappa}$.

[1]I. V. Bogdanov, Zh. Eksp. Teor. Fiz. **84**, 1921 (1983) [Sov. Phys. JETP **57**, 1119 (1983)].

[2]I. V. Bogdanov, Teor. Mat. Fiz. **56**, 448 (1983) [Theor. Math. Phys. (USSR) **56**, 933 (1983)].

[3]I. V. Bogdanov, in Abstracts of Papers at Sixth Soviet Gravitational Conference [in Russian], Moscow, 1984, p. 223.

[4]F. Prats and J. S. Toll, Phys. Rev. **113**, 363 (1959).

[5]M. G. Gasymov and B. M. Levitan, Dokl. Akad. Nauk SSSR **167**, 967 (1966).

[6]M. G. Gasymov and B. M. Levitan, Dokl. Akad. Nauk SSSR **167**, 1219 (1966).

[7]M. G. Gasymov, Dokl. Akad. Nauk SSSR **169**, 1037 (1966) [Sov. Phys. Doklady **11**, 676 (1967)].

[8]R. Weiss, W. Stahel, and G. Scharf, Nucl. Phys. **A183**, 337 (1972).

[9]C. Coudray and M. Coz. J. Math. Phys. **12**, 1166 (1971).

[10]L. D. Landau and E. M. Lifshitz *The Classical Theory of Fields*, 4th ed., Pergamon, Oxford, 1975.

[11]Ya. B. Zel'dovich and I. D. Novikov, *Theory of Gravitation and Stellar Evolution* [in Russian] Nauka, Moscow, 1971.

[12]A. A. Grib, S. G. Mamaev, and V. M. Mostepanenko, *Quantum Effects in Strong External Fields* [in Russian], Atomizdat, Moscow, 1980.

[13]N. A. Chernikov and N. S. Shavokhina, Teor. Mat. Fiz. **16**, 77 (1973) [Theor. Math. Phys. (USSR) **16**, 684 (1973)].

[14]V. B. Berestetskiĭ, E. M. Lifshitz, and L. P. Pitaevskiĭ, *Quantum Electrodynamics*, 2nd ed., Pergamon, Oxford, 1982).

[15]M. L. Goldberger and K. M. Watson, *Collision Theory*, Wiley, New York, 1964.

[16]K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, Springer-Verlag, New York, 1977.

[17]V. I. Smirnov, *Course of Higher Mathematics* [in Russian] Nauka, Moscow, 1974, Vol. 4, Part 1.

[18]N. N. Bogolyubov and D. V. Shirkov, *Quantum Fields*, Benjamin/Cummings, Reading, Mass., 1983.

[19]I. V. Bogdanov and Yu. N. Demkov, Zh. Eksp. Teor. Fiz. **93**, 3 (1987) [Sov. Phys. JETP **66**, 1 (1987)].

[20]D. N. Voskresenskiĭ and A. V. Senatorov, Zh. Eksp. Teor. Fiz. **90**, 1505 (1986) [Sov. Phys. JETP **63**, 885 (1986)].

Translated by Adam M. Bincer