Modification of the distribution of spin waves under parametric resonance conditions in ferrites

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A detailed analysis is made of the evolution of the distribution function of parametric magnons in ferrites on increase in a supercriticality parameter ξ. It is shown that, instead of creation of a second group of pairs with a singular angular distribution, a completely different modification may take place. In fact, at some value of $\xi = \xi_c$ a regular part may appear in $n(\mathbf{k})$ near the equator: $n(\mathbf{k}) \neq 0$ applies in a certain range of polar angles $\theta(\mathbf{k})$ $[\theta(\mathbf{k}) - \pi/2| \leqslant \delta]$ and the width of this distribution δ vanishes at $\xi = \xi$ and then increases with the supercriticality parameter. A theory of this effect is developed using the self-consistent-field approximation: the value of ξ_c is determined, the distribution function $n(\mathbf{k})$ is found in the range $\xi > \xi_c$, and calculations are made of the nonlinear susceptibilities χ' and χ'' of a system of parametric magnons. An experimental study is reported of the behavior of parametric spin waves in yttrium iron garnet at high parallel pumping powers. Determination was made of the threshold of formation of a packet localized far from the equator of the resonance surface in the **k** space. It is shown that both types of modification of the distribution function of parametric magnons occur in yttrium iron garnet, depending on the region of excitation of spin waves in **k** space.

It is well known that under conditions of a parametric instability of spin waves induced by an external microwave magnetic field (pump) a steady state is established as a result of the action of spin waves on the pump: the damping constant of the waves $\gamma(\mathbf{k})$ increases on increase in their number or because of a "phase" limitation mechanism associated with the interaction of wave pairs. If this limitation mechanism predominates, the dependences of the principal characteristics of parametric waves on the amplitude of the pump field h are universal for a small excess of h above the parametric instability threshold h_1. This is due to a fact, established within the framework of the S theory,[1,2] that in the case of small values of the supercriticality parameter the distribution of parametric waves $n(\mathbf{k})$ along the directions $\mathbf{k}$ is singular: $n(\mathbf{k}) \neq 0$ is true only on a resonance surface and only for those directions of $\mathbf{k}$ for which the ratio of the coefficient representing coupling with the pump to the damping is maximal. In the case of a spherically symmetric problem which provides a good description of antiferromagnets, parametric spin waves are excited over the whole resonance surface. Under axial symmetry conditions typical of ferrites, parametric spin waves are excited on one or two circles of the resonance surface. In the absence of symmetry, just one pair of parametric spin waves is excited. It therefore follows that in all cases when the supercriticality parameter is small, only one singular group of pairs is excited and the pairs in the group are coupled in the same way to the pump.

The dependences of the number of parametric waves and of the nonlinear susceptibilities on the pump power, universal in the S theory, are supported well by experiments on ferrites[3,4] and antiferromagnets[5] when the supercriticality parameter is small. Much less work has been done on the problem of evolution of the distribution function of parametric waves on increase in the supercriticality parameter. In accordance with the concept of multistage excitation of parametric waves, it is usual to assume[1] that on increase in h in a certain range $h = h_2 > h_1$ a second singular group of parametric waves is created, then a third singular group of pairs is formed for $h = h_3 > h_2$, and so on.

We shall analyze in more detail than hitherto the evolution of the distribution function of parametric spin waves in ferrites on increase in h and show that, instead of creation of a second group of pairs with a singular angular distribution, the modification may be quite different. In fact, at some value of $h = h_2$ a regular part appears in $n(\mathbf{k})$ near the equator: $n(\mathbf{k}) \neq 0$ in the range of polar angles $\theta(\mathbf{k})$ defined by the inequality $|\theta(\mathbf{k}) - \pi/2 \leqslant \delta$, and the width of this distribution δ vanishes at $h = h_2$ and then increases on increase in the supercriticality parameter. This behavior of the distribution function is related to a large nonanalytic contribution of the long-range magnetic-dipole interaction to the matrix element $S(\mathbf{k}, \mathbf{k}')$ representing the scattering of parametric spin waves by one another.

We shall develop a theory of this effect using the self-consistent field approximation: we shall determine h_2, find the distribution function $n(\mathbf{k})$ in the range $h > h_2$, and also calculate the nonlinear susceptibilities χ' and χ'' of a system of parametric spin waves.

We shall report an experimental investigation of the behavior of parametric spin waves in yttrium iron garnet (YIG) as a function of the intensity of a static magnetic field H at high parallel pumping powers. We shall determine the threshold of creation of a packet of parametric spin waves h_c, localized far from the equator of the resonance surface in the $\mathbf{k}$ space. We shall show that at low values of $|\mathbf{k}|$, when H is close to the critical value H_c corresponding to $\mathbf{k} = 0$, a second group of parametric spin waves is generated close to the equator of the resonance surface, but it is not detectable experimentally. Therefore, h_c is the threshold of creation of a third group of parametric spin waves. However, if $H_c - H \geqslant 300$ Oe (i.e., if $|\mathbf{k}|$ is sufficiently large), the second group of parametric spin waves is created far from the equator. A qualitative picture of the modification of the distribution function of parametric spin waves in ferrites formulated

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below enabled us to carry out quantitative computer calculations of the threshold powers h_c^2 of creation of parametric spin waves localized far from the equator and observed in YIG. We found that the experimental and theoretical dependences of h_c^2 on H agreed for YIG.

1. EQUATIONS OF MOTION FOR PARAMETRIC SPIN WAVES IN CUBIC FERROMAGNETS

1.1. Initial equations. In the axial symmetry case, which is encountered in isotropic and cubic ferromagnets, the steady-state equations of the S theory [Eqs. (18.17) and (18.20) in Ref. 2] describing the distribution of parametric spin waves in the **k** space can be simplified by replacing the variables $\Omega = \theta, \varphi \to x, f$ in accordance with the expressions

$$x=\cos\theta(k),\ f(\Omega)=(1-x^2)\exp 2i\varphi(\mathbf{k}),$$

where $\theta(\mathbf{k})$ and $\phi(\mathbf{k})$ are the polar and azimuthal angles in a coordinate system oriented along the magnetization M; **k** is the wave vector of parametric spin waves lying on the resonance surface $\omega_k = \omega_p/2$. After this substitution the coefficient V representing coupling with the pump becomes a constant and only the dependence on x remains in the equations obtained from the S theory:

$$P(x)=i\gamma(x)\exp(i\psi(x)),\ \text{if}\quad n(x)\neq 0, \tag{1.1}$$

$$|P(x)|<\gamma(x),\ \text{if}\quad n(x)=0, \tag{1.2}$$

$$P(x)=hV+\int_{-1}^{1}dx'S^0(x,x')n(x')\exp(i\psi(x')).$$

Here, $P(\Omega) = P(x)f(\Omega)$ is the renormalized pump; $V(\Omega) = Vf(\Omega)$ is the coefficient representing the coupling of parametric spin waves to the pump h; $S^0(\Omega,\Omega') = S^0(x,x')f(\Omega)f^*(\Omega')$ is the zeroth axial Fourier harmonic of the four-wave interaction of pairs of parametric spin waves:

$$S^m(x,x')=\frac{1}{2\pi}\int_0^{2\pi}d(\varphi-\varphi')S(x,x',\varphi-\varphi')\exp\{im(\varphi-\varphi')\}, \tag{1.3}$$

where $n(\Omega) = n(x)f(\Omega)$ is the amplitude of parametric spin waves; finally, $\Gamma(\Omega) = \gamma(x)|f(x)|$ is the damping of spin waves. The dependence of $\Gamma(\Omega)$ on the polar angle was calculated for YIG in Ref. 6. In the case of $\gamma(x)$ the following expression was obtained, which is accurate to within 10 %:

$$\gamma(x)=\gamma(0)(1+dx^2)+\gamma_1x^4/(1-x^2), \tag{1.4}$$

where $\gamma(0)$ is the damping of parametric spin waves characterized by $\theta_k = \pi/2$, $\gamma_1 \approx \gamma(0)$, $d = 5.2k/k_0 + 1)/(0.4k/k_0 + 1)$, $k/k_0 = [H - H_c)/100\ \text{Oe}]^{1/2}$.

1.2. Properties of the function S(x, x′) for ferromagnets. The explicit form of the Hamiltonian describing the interaction of pairs yields the symmetry properties of the function $S(\mathbf{k}, \mathbf{k}')$:

$$S(\mathbf{k},\mathbf{k}')=S(-\mathbf{k},\mathbf{k}')=S^*(\mathbf{k}',\mathbf{k}). \tag{1.5}$$

Hence, and from the definition (1.3), it follows that $S^0(x, x')$ should be an even function of each argument:

$$S^0(x,x')=S^0(-x,x')=S^0(x,-x'), \tag{1.6}$$

whereas its real part Re $S(x, x')$ is symmetric under the transposition of x and x'. Specific calculations demonstrate that in the case of ferromagnets we have Im $S^0(x, x') = 0$. Therefore,

$$S^0(x,x')=\text{Re}\,S^0(x,x')=S^0(x',x). \tag{1.7}$$

We shall identify the asymptotes $S(x, x')$ for x and x' close to 1 and 0. It follows from the axial symmetry of the problem (invariance under rotations about the direction of the magnetization) that $S(\theta,\theta',\varphi-\varphi')$ is independent of $\varphi-\varphi'$ if θ or θ' vanishes. Therefore, the integral in Eq. (3) for $m = 0$ remains finite in the limit $x,x' \to 1$ in spite of the presence of a singular factor $(1-x^2)^{-1}(1-x'^2)^{-1}$. It therefore follows that

$$S^0(1,\ x)=S^0(x,\ 1) \tag{1.8}$$

is finite.

In identifying the behavior of $S(x, x')$ at low values of x and x' we shall use the explicit form of Eqs. (6.26)–(6.30) from Ref. 2 for $S(\mathbf{k}, \mathbf{k}')$. These equations contain terms proportional to $\cos^2\theta(\mathbf{k}+\mathbf{k}')$ and $\cos^2\theta(\mathbf{k}-\mathbf{k}')$. After integration with respect to φ, we find from Eq. (1.3) that each of these terms makes a nonanalytic contribution to $S^m(x,x')$, proportional to the powers of $|x \pm x'|$.

Therefore, the expansion of $S^0(x, x')$, accurate to within terms $\sim x^3$, is

$$S^0(x,x')=S[1+{}^1\!/_2a(|x+x'|+|x-x'|)+b(x^2+x'^2)+{}^1\!/_2c_1(x^2+x'^2)(|x+x'|+|x-x'|)+c_2xx'(|x+x'|-|x-x'|)]. \tag{1.9}$$

This expansion satisfies all the necessary requirements: it is even in respect of each argument [Eq. (1.6)], is symmetric [Eq. (1.7)], and is finite in the limit $x \to 1$ [Eq. (1.8)]. We can therefore expect that a suitable selection of the coefficients a, b, c_1, and c_2 should ensure a good description of the behavior of $S^0(x, x')$ throughout the full range of x and x' $|x|, |x|\leqslant 1$.

Using Eqs. (6.26)–(6.30) from Ref. 2, we carried out computer calculations of the function $S^0(x, x')$ for different values of the magnetic field typical of those used in experiments on YIG ferrite samples. This function was then approximated by model dependences of Eq. (1.9). The coefficients a, b, c_1, and c_2 were found by minimizing the mean-square value of the discrepancy in the interval $-1/2 \leqslant x \leqslant 1/2$. The results obtained are listed in Table I for $T = 300$ K, $\omega_r/2\pi = 10^{10}$ s^{-1}, and $\mathbf{M}\|\langle 100\rangle$. Here, $H_c = 1680$ Oe is the critical value of the field at which the wave vector **k** of a spin wave with a frequency $\omega_p/2$ and with $\mathbf{k}\perp\mathbf{M}$ vanishes, and Δ_1 and Δ_2 at the maximum deviations of the computer-calculated functions $S^0_{x,0}$ and $S^0_{x,x}$ from the approximation described by Eq. (1.9) with a set of coefficients given in the interval $|x|, |x'| \leqslant \frac{1}{2}$. Clearly, the coefficients a, c_1 and c_2 describing nonanalytic contributions to $S^0(x,x')$ are fairly large and the coefficient b for the analytic part of the dependence $S^0(x, x')$ shows the reversal of the sign on increase in the magnetic field.

2. MULTISTAGE EXCITATION OF PARAMETRIC SPIN WAVES

The solution of Eqs. (1.1) and (1.2) beyond the parametric instability threshold, i.e., when $h \geqslant h_1 = \min(\gamma(x)/$

TABLE I. Dependence of the coefficients of the model function $S^0(x, x')$ of Eq. (1.9) on the magnetic field applied to a spherical sample of yttrium iron garnet.

$H_c - H$, Oe	a	b	c_1	c_2	Δ_1, %	Δ_2, %
100	2,6	−1,8	4,1	2	4	3
200	2,5	0,3	3,2	1	2	4
300	2,4	0,6	4,0	0	0,5	5
400	2,2	1,0	4,0	0	1	6
500	2,1	1,3	3,8	−1	0,5	10

V), is

$$n(x)=N_1\delta(x),\ N_1=(|hV|^2-\gamma^2(0))^{1/2}/S^0(0,0),$$
$$\sin\psi(0)=\gamma(0)/hV, \tag{2.1}$$
$$|P(x)|^2=\gamma^2(0)+N_1^2(S^0(x,0)-S^0(0,0))^2. \tag{2.2}$$

Clearly this state is stable if $|P(x)| < \gamma(x)$ for all values $x\neq 0$ [condition of Eq. (1.2)]. We shall introduce a function $\xi(x) = (h(x)/h_1)^2$, where $h(x)$ is the value of the pump at which $|P(x)| = \gamma(x)$. Obviously, the second threshold $\xi_2 = (h_2/h_1)^2$ corresponds to the minimum value of this function, which appears at a certain point $x = x_2$. The value of ξ_2 is found from the condition $\xi_2 = \min\xi(x)$ where

$$\xi(x)=1+(\gamma^2(x)-\gamma^2(0))S^{02}(0,0)/\gamma^2(0)(S^0(x,0)-S^0(0,0))^2. \tag{2.3}$$

This condition had been studied earlier, on the assumption of analyticity of the function $S^0(x, x')$ selected to be the dependence (1.9) with $a = c_1 = c_2 = 0$. If we additionally require that $\Gamma(\Omega) = \text{const}$ [i.e., $\gamma(x) = \gamma(0)/(1-x^2)$], a minimum of $\xi(x)$ occurs at $x = x_2$, where

$$x_2^2=0,5(3-\sqrt{5})\approx 0,38,\quad \xi_2=1+11b^{-2}. \tag{2.4}$$

We can see from Table I and Eq. (1.9) that the assumption that $S^0(x,x')$ for YIG is analytic is far from reality. We shall therefore consider the question of the threshold of creation of the second group of pairs ξ_2 and the point of generation of such a group assuming a more realistic form of the amplitude $S^0(x\,x')$. We shall expand $\xi(x)$ as a series in x:

$$\xi(x)=\xi(0)+\xi'|x|+\xi''x^2/2,\quad \xi(0)=1+\gamma''(0)/a^2\gamma(0),$$
$$\xi'=-b\gamma''(0)/a^3\gamma(0), \tag{2.5}$$
$$\xi''=\frac{3\gamma''(0)^2+\gamma(0)\gamma''''(0)}{6a^2\gamma^2(0)}-\frac{2\gamma''(0)(b^2+2ac_1)}{\gamma(0)a^4}$$

Here, $\gamma''(0)$ and $\gamma''''(0)$ are the second and fourth derivatives of $\gamma(x)$ at $x = 0$. We can see that nonanalyticity of the function $S^0(x, x')$ $(a\neq 0)$ makes $\xi(0)$ finite.

We shall first consider the case when a maximum of the function $\xi(x)$ occurs at $x = 0$ and we shall assume that the nonanalyticity of $S^0(x, x')$ is strong, i.e., that $|b|^2 \ll ac$. It is clear from Table I that this case is realized in YIG in the range of fields $H_c - H \gtrsim 150$ Oe. Let us assume that $\xi'' > 0$. It then follows readily from Eq. (2.5) that

$$|x_2|=-\xi'/\xi'',\quad \xi_2=\xi(0)-2\xi'^2/\xi''. \tag{2.6}$$

However, if $\xi'' < 0$, a minimum of $\xi(x)$ occurs at high values of x_2 and we cannot use the expansion described by Eq. (2.5). We shall find the minimum of $\xi(x)$ defined by Eq. (2.3) in the simplest case when $b = 0$ and $\Gamma(\Omega) = \text{const}$. We then have $\gamma(x) = \gamma(0)/(1-x^2)$ and it follows from Eqs. (2.3) and (2.5) that

$$x_2^2=[9c_1-a\pm((9c_1-a)^2+6c_1a^3\xi'')^{1/2}]/6c_1, \tag{2.7}$$
$$\xi''=2(3a-4c_1)/a^3<0,\ \xi(0)=2/a^3,\ \xi_2=\xi(x_2).$$

Hence, it is clear that x_2 is small for low values of $|\xi''|$ and that ξ_2 is close to $\xi(0)$. An increase in $|\xi''|$ increases the value of $|x_2|$ and for $c_1 \gg a$ it reaches 0.735. We then have $\xi_2 \approx 1 + 24/c_1^2$.

In traditional experiments it is usual to investigate the integral characteristics of parametric spin waves, so that determination of the creation threshold of the second group of pairs is difficult if it does appear close to the first, i.e., at low values of $|x_2|$. In this case it is interesting to consider the threshold of creation of the third group of pairs localized far from the equator. At low values of $|x_2|$ we can obtain simple expressions for the integrated amplitude of the second pair and for the pump power when the third threshold field h_3 is reached:

$$S^0(x,x')\approx S^0(0,y),\quad y=\max\{x,x'\}, \tag{2.8}$$

which follows from Eq. (1.9) if we retain only the terms linear in x and x'. Substituting the distribution function of parametric spin waves into Eq. (1.1) beyond the second threshold and using the approximation described by Eq. (2.8), we obtain the following expressions for N_1 and N_2 in the limit $h \gg h_2$:

$$n(x)=N_1\delta(x)+N_2(\delta(x+x_2)+\delta(x-x_2))/2,$$
$$N_1=N_1^*=(|h_2V|^2-\gamma^2(0))^{1/2}/S^0(0,0), \tag{2.9}$$
$$N_2=-\gamma(0)N_1^*/\gamma(x_2)+[|hV|^2-|h_2V|^2+\gamma(0)S^0(0,x^2)N_1^*/(\gamma(x^2))^2]^{1/2}/S^0(0,x_2).$$

In the approximation described by Eq. (2.8) the angle of localization of the second group of parametric spin waves and the amplitude of the first group of such waves are independent of the pump power h^2. Equations (1.1) and (1.2) and the dependences given in the system (2.9) allow us to obtain the following expression for the power at the third threshold:

$$|h_3V|^2=\gamma^2(0)+\min\Big\{N_1^{*2}(S^{02}(0,0)-(\gamma(0)S^0(0,x_2)/\gamma(x_2))^2$$
$$+\frac{(S^0(0,x_2))^2}{(S^0(0,x)-S^0(0,x_2))^2}[\gamma^2(x)-(\gamma(x_2)+(S^0(x,0)$$
$$-S^0(x_2,0))N_1^*(\gamma^2(x_2)/\gamma^2(0)-1)^{1/2})^2]\Big\}. \tag{2.10}$$

The behavior of parametric spin waves beyond the third

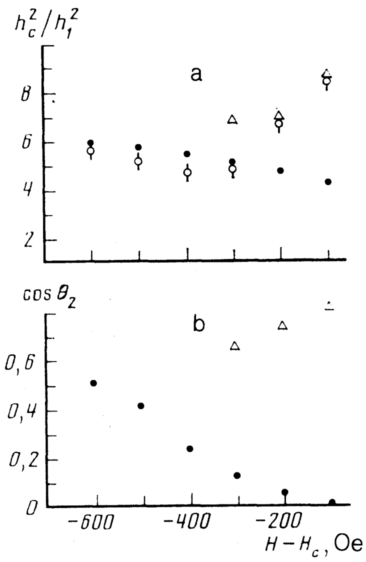

FIG. 1. a) Experimental values of the pump power at the threshold of excitation of parametric spin waves characterized by $\theta(\mathbf{k})\neq\pi/2$ (○); theoretical values of the pump power at the second threshold (●); theoretical values of the pump power at the third threshold (△). b) Theoretical values of the angle of creation of parametric spin waves: ●) second group of parametric spin waves; △) third group of parametric spin waves.

threshold is analogous to the behavior beyond the second threshold. The amplitude of the second group is frozen at the level

$$N_2\approx N_2^*=-\gamma(0)N_1^*/\gamma(x_2)+[|h_3V|^2-|h_2V|^2+(\gamma(0)S^0(0,x_2)N_1^*/\gamma(x_2))^2]^{1/2}/S^0(0,x_2). \qquad (2.11)$$

We can ignore here the difference between θ_2 and $\pi/2$ and we then have

$$N_1+N_2=N_{1,2}=[|h_3V|^2-\gamma^2(0)]^{1/2}/S^0(0,0),$$
$$N_3=-\gamma(0)N_{1,2}/\gamma(x_3)+[|hV|^2-|h_3V|^2+(\gamma(0)S^0(0,x_3)N_{1,2}^*/\gamma(x_3))^2]^{1/2}/S^0(0,x_3). \qquad (2.12)$$

We shall conclude this section by presenting (Fig. 1) the computer-calculated dependences of ξ_3, x_3, ξ_2, and x_2 on the magnetic field (apart from the values of ξ_3 and x_3 in a field $H=H_c-100$ Oe) for the values of the coefficients $S^0(x,x')$ and $\gamma(x)$ applicable to YIG.

3. NEW TYPES OF MODIFICATION OF THE DISTRIBUTION FUNCTION OF PARAMETRIC SPIN WAVES

In the preceding section we considered the case for the function $\xi(x)$ of Eq. (2.3) as a maximum at $x=0$. Then, an increase in the pump power results in multistage excitation of groups of parametric spin waves with a distribution function $n(x)$ singular in terms of $x=\cos\theta$. However, if $\xi(x)$ is minimal at $x=0$, it is natural to assume that $n(x)$ differs from zero in a certain range $|x|<\delta$ and we then face the problem of investigation of nonlinear behavior of parametric spin waves as a result of such modification of $n(x)$.

3.1. Nonlinear equations for the regular part of the distribution function n(x) and the phase ψ(x). We shall first calculate $P(x)$ using Eqs. (1.2) and (1.9). If $0\leqslant x\leqslant\delta$, we find that

$$P(x)=A+Bx^2+2S\Big\{x(a+c_1x^2)\int_0^x\sigma(x')\,dx'+c_1\int_x^\delta x'^3\sigma(x')\,dx'+x(c_1+2c_2)\int_0^x x'^2\sigma(x')\,dx'+(a+(c_1+2c_2)x^2)\int_x^\delta x'\sigma(x')\,dx'\Big\}, \qquad (3.1)$$

$$A=hV+S(\Sigma_0+b\Sigma_2),\quad B=bS\Sigma_0,\quad \Sigma_0=\int_{-\delta}^{\delta}\sigma(x)\,dx, \qquad (3.2)$$

$$\Sigma_2=\int_{-\delta}^{\delta}x^2\sigma(x)\,dx,\quad \sigma(x)=n(x)\exp-i\psi(x).$$

If $x\geqslant\delta$, then

$$P(x)=A+S[a\Sigma_0+(c_1+2c_2)\Sigma_2]x+Bx^2+Sc_1\Sigma_0x^3. \qquad (3.3)$$

In combination with Eq. (1.1), this gives a complete system of integral equations for the characteristics $n(x)$ and $\psi(x)$ of parametric spin waves. We shall derive from the above results more convenient integral differential equations for these quantities. It follows from Eqs. (3.1)–(3.3) that the functions $P(x)$ and $P'(x)$ are continuous at $x=\delta$ and also that

$$P(\delta)=hV+S\Sigma_0(1+a\delta+b\delta^2+c_1\delta^3)+S\Sigma_2(b+\delta(c_1+2c_2)),$$
$$P'(\delta)=S\Sigma_0(a+2b\delta+3c_1\delta^2)+S\Sigma_2(c_1+2c_2). \qquad (3.4)$$

Differentiating again, we obtain

$$P''(x)=2S\Sigma_0(b+3c_1x),\quad x\geqslant\delta, \qquad (3.5)$$

and in the range $0<x<\delta$, we have

$$P''(x)=2bS\Sigma_0+6c_1x\int_{-x}^{x}\sigma(x')\,dx'+2S\sigma(x)(a+2x^2(c_1-c_2))+4S(c_1+2c_2)\int_x^\delta x'\sigma(x')\,dx', \qquad (3.6)$$

$$P'''(x)=4S(3(c_1-c_2)\sigma(x)+3(c_1-2c_2)x\sigma'(x)+(c_1-c_2)x^2\sigma''(x)). \qquad (3.7)$$

If $|x|<\delta$ the S theory equations [Eq. (1.1)] together with Eq. (3.7) yield the required integral differential equation for $n(x)$ and $\psi(x)$, in which—in contrast to Eq. (3.1)—the limits of integration are independent of x. However, we shall be satisfied with a much simpler though approximate equation which can be derived from Eqs. (1.1), (1.2), and (1.3) replacing x in Eq. (3.6) with δ within the integration limits:

$$2S(a+2x^2(c_1-c_2))n(x)=-2S(b+3c_1x)\int_{-\delta}^{\delta}n(x')\times\exp[-i(\psi(x')-\psi(x))]\,dx'+(\gamma(x)\psi''(x)+\gamma'(x)\psi'(x))+i(\gamma''(x)-\gamma(x)\psi'^2(x)). \qquad (3.8)$$

The criterion of validity of these equations will be formulated later.

3.2. Simple solutions of the main equations (3.8) for the distribution function. It is easiest to obtain the solution of the system (3.8) when $b = c_1 = 0$ and there is no integral term:

$$n(x)=N_1\delta(x)+N_2(x),\quad N_1=N_1^*=[\gamma(x)\gamma''(x)]^{1/2}/aS,$$
$$\psi'(x)=[\gamma''(x)/\gamma(x)]^{1/2}\operatorname{sign}(x), \qquad (3.9)$$
$$N_2(x)=(\gamma'''(x)\gamma(x)+\gamma''(x)\gamma'(x))/4S(a-2c_2x^2)[\gamma''(x)\gamma(x)]^{1/2}.$$

This solution is obtained for $\xi>\xi(0)$. If $\xi<\xi(0)$, we have the usual singular solution from the S theory [Eq. (2.1)]. Substituting in this solution the value $\xi=\xi(0)$ from Eq. (2.5), we can readily see that $N(\xi(0)) = N_1^*$. Therefore, the number of parametric spin waves in the singular part of the distribution function increases when $\xi<\xi(0)$ in accordance with Eq. (2.1), whereas for $\xi>\xi(0)$, this number is frozen at the threshold level N_1^*. The regular part of the distribution $N_2(x)$ is proportional to $|x|$ at low values of the argument:

$$N_2(x)\approx|x|(\gamma(0)\gamma''''(0)+\gamma''^2(0))/4aS(\gamma''(0)\gamma(0))^{1/2}. \qquad (3.10)$$

Using the model dependence $\gamma(x)$ described by Eq. (1.4), we readily find from Eq. (3.9) that at low values of $|x|$, we have

$$N_2(x)\approx|x|d^2/2Sa(2d)^{1/2}. \qquad (3.11)$$

It is clear from the system (3.8) that if $c_1\neq0$, the property of the solution described by $N_2(x)\propto|x|$ is retained.

We shall first investigate the case when $b\neq0$ assuming, for the sake of simplicity, that $c_1=c_2=0$. If $b\neq0$, Eq. (3.8) becomes strongly nonlinear and it is difficult to analyze. However, in the case of small values of the supercriticality parameter $\xi-\xi(0)\ll1$, when the width δ of a packet of parametric spin waves is small, Eq. (3.8) can be investigated using perturbation theory in terms of δ.
Then, instead of Eq. (3.9), we obtain

$$\psi'(x)=\operatorname{sign}(x)(\gamma''(x)/\gamma(x))^{1/2}(1+b^2(2\delta|x|-\delta^2)/a^2),$$
$$N_1=N_1^*(1-\delta^2b^2/a^2), \qquad (3.12)$$
$$N_2(x)=N_2+N_2'|x|,\quad N_2=N_1^*(-b/a+3\delta b^2/a^2)-N_2'\delta^2 b,\quad N_2'=N_1^*\left(\frac{\gamma\gamma'''+\gamma''^2}{4\gamma\gamma''}+\delta b^2/a^2\right).$$

This solution has two important special features:. firstly, the singular part of the distribution N_1 decreases on increase in the supercriticality parameter and, obviously, at high values of ξ it should vanish; secondly, the regular part of the distribution at $x=0$ differs from zero ($N_2\neq0$) and increases on increase in ξ.

3.3. Equations for the width δ of the distribution and for the nonlinear susceptibilities χ' and χ''. We shall determine δ using

$$|P(\delta)|^2=\gamma^2(\delta),\quad \frac{d}{dx}|P(x)|^2_{x=\delta+0}=\frac{d\gamma^2(x)}{dx|_{x=\delta}}=\frac{d}{dx}|P(x)|^2_{x=\delta-0},$$
$$\frac{d^2}{dx^2}|P(x)|^2_{x=\delta+0}=\frac{d^2}{dx^2}|P(x)|^2_{x=\delta-0}=\frac{d^2}{dx^2}\gamma^2(x)_{x=\delta}. \qquad (3.13)$$

Substituting in the system (3.13) the expressions for $P(\delta)$, $P'(\delta)$, and $P''(\delta)$ from Eqs. (3.4) and (3.5), we find that simple transformations lead to

$$Xf^2+2Yf=\gamma^2-|hV|^2,\quad Xff_1+Yf_1=\gamma\gamma',$$
$$X(f_1^2+2ff_2)+2Yf_2=\gamma'^2+\gamma\gamma''. \qquad (3.14)$$

Here,

$$X=|S\Sigma_0|^2,\ Y=hV\operatorname{Re}S\Sigma_0,\ f(x)=S^0(0,x)/S^0(0,0),$$
$$f_1=df(x)/dx,\quad f_2=\tfrac12 d^2f(x)/dx^2.$$

The values of the functions —f, f_1, f_2, γ, γ', and γ''—are taken at the point $x=\delta$. In these equations we are ignoring the terms $b\Sigma_2$, $c_1\Sigma_2$, and $c_2\Sigma_2$ compared with Σ_0. Using the solution described by the system (3.12), we can estimate the last quantity: $\Sigma_2\approx-b\delta^3\Sigma_0/3a$. Therefore, our approximation is valid if $b\delta^3\ll3a$, and it can be used either for low values of b and any width δ, or for any b and small δ. This approximation simplifies the situation radically: the system of equations (3.14) becomes closed and the dependences of δ and Σ_0 on hV are specified. This makes it possible not only to find the width δ of a packet of parametric spin waves, but also the integrated characteristics such as the nonlinear susceptibilities χ' and χ'' without solving explicitly the initial integrodifferential equations in the system (3.8) and without subsequent integration in

$$\chi'=-\frac{2}{h}\int V(x)\operatorname{Re}\sigma(x)\,dx,\quad \chi''=\frac{2}{h}\int V(x)\operatorname{Im}\sigma(x)\,dx. \qquad (3.15)$$

In fact, using our notation we can transform the expressions in Eq. (3.15) to

$$\chi'=-2V\operatorname{Re}\Sigma_0/h,\quad \chi''=2V\operatorname{Im}\Sigma_0/h, \qquad (3.16)$$

i.e., we can describe the susceptibilities in terms of Σ_0 which depends on hV in accordance with the system (3.14).

3.4. Analysis of the solutions of the system (3.14) for integral characteristics of parametric spin waves. We readily obtain from the system (3.14) that

$$Xf^2f_1=f_1(|hV|^2-\gamma^2)+2f\gamma\gamma',\quad -Yff_1=f_1(|hV|^2-\gamma^2)+f\gamma\gamma', \qquad (3.17)$$
$$f_1^3(|hV|^2-\gamma^2)=f(ff_1(\gamma'^2+\gamma\gamma'')-2\gamma\gamma'(f_1^2+ff_2)). \qquad (3.18)$$

Equation (3.18) gives the dependence of δ on hV; after its substitution in Eq. (3.17) we obtain the dependences of X and Y (i.e., of Σ_0) on hV. We recall that in these equations the functions f, f_1, f_2, γ, γ', and γ'' depend on δ.

3.4.1. Solution for small values of the excess (supercriticality) above the intermediate threshold. To within terms not higher than δ^2, we obtain from Eqs. (3.17) and (3.18)

$$|S\Sigma_0|^2=\gamma^2(\xi-1-\xi'a\delta^2),$$
$$-hVS\operatorname{Re}\Sigma_0=\gamma^2(\xi-1-a\delta^2/3), \qquad (3.19)$$
$$\xi-\xi(0)=6(\delta\xi'(1+a\delta/3-5b\delta/a)+\delta^2\xi''/2). \qquad (3.20)$$

Here $\gamma=\gamma(0)$ and the coefficients $\xi(0)$, ξ' and ξ'' are given by Eq. (2.5), whereas a and b are the coefficients of the expansion of the function $S^0(x,x')$ of Eq. (1.9). If $b\neq0$, when $\xi'=0$, and we can limit Eq. (3.20) to the terms which are linear in δ. We then obtain

$$\delta=(\xi-\xi(0))/6\xi'. \qquad (3.21)$$

We recall that if $\xi<\xi(0)$, when $N(x)\propto\delta(x)$, the quantities $|\Sigma_0|^2$ and Re Σ_0 are described by simple formulas from the S theory:

$$|S\Sigma_0|^2=\gamma^2(\xi-1),\quad -ShV\,\mathrm{Re}\,\Sigma_0=\gamma^2(\xi-1). \tag{3.22}$$

Comparing these formulas with Eq. (3.19), we can easily see that a major qualitative modification of the distribution function of parametric spin waves $n(x)$ above the threshold [i.e., in the range where $\xi > \xi(0)$] has very little influence on the integral characteristics of the system of parametric spin waves (Σ_0, χ' and χ'') if $\xi - \xi(0)$ is small. In fact, if $b \neq 0$ the difference between Eqs. (3.19) and (3.22) is proportional to $(\xi - \xi(0))^2$. However, if $b = 0$, then $\xi' = 0$ and the difference between Eqs. (3.22) and (3.19) is proportional to $(\xi - \xi(0))^{3/2}$. Therefore, modification of the distribution function above the threshold may be undetectable in experiments which yield the integrated characteristics of a system of parametric spin waves $\chi'(\xi)$ and $\chi''(\xi)$.

3.4.2. Solution in the case of a slight nonanalyticity of the coefficients $S^0(x, x')$ and $hV \gg \gamma$. It is well known that in the case of an analytic dependence of the coefficients $S^0(x, x')$ on x and x' [when in terms of the notation of Eq. (1.9) we have $a = c_1 = c_2 = 0$] a packet of parametric spin waves is of zero width along x. Therefore, we may assume that at low values of a and c_1 such a packet remains narrow. In fact, it is clear from Eq. (3.18) that if the function

$$f_1(\delta)=a+2b\delta+3c_1\delta^2$$

vanishes at some value $\delta = \delta_m$, an increase in hV increases the width δ of a packet of parametric spin waves from 0 to δ_m. Then, at higher values of hV, we have

$$|S\Sigma_0|^2=(|hV|^2-\gamma^2(\delta_m))/f^2(\delta_m),$$
$$-hV\,\mathrm{Re}\,S\Sigma_0=(|hV|^2-\gamma^2(\delta_m))/f(\delta_m). \tag{3.23}$$

Here,

$$f(\delta)=1+a\delta+b\delta^2+c_1\delta^3,$$
$$\delta_m=(-b+(b^2-3ac_1)^{1/2})/3c_1. \tag{3.24}$$

This solution is obtained if

$$b^2>3ac_1,\quad \delta_m<1. \tag{3.25}$$

It is natural to call the above inequalities the conditions of slight nonanalyticity. The solutions given by the system (3.23) differ from simple S-theory solutions for one group of pairs by the substitutions $\gamma(0) \to \gamma(\delta_m)$ $S^0(0, 0) \to S^0(0, \delta_m)$. This has a simple physical meaning: above the intermediate threshold a packet of spin waves expands on increase in the supercriticality parameter until the "effective nonanalyticity" $dS^0(0,x)/dx$ vanishes in the limit $x \to \delta_m$.

3.4.3. Solution for a strong analyticity when $hV \gg \gamma$. The conditions of Eq. (3.25) are no longer satisfied. Then, $f_1(\delta)$ remains positive in the interval $0 \leqslant \delta \leqslant 1$, and an increase in the supercriticality parameter causes δ to increase to unity. Bearing in mind that at x close to unity we have $\gamma(x) \approx \gamma_1 x^4/(1-x^2)$, we find from Eqs. (3.17) and (3.18) that

$$f^2(1)\,|S\Sigma_0|^2=|hV|^2,$$
$$-f(1)S\,\mathrm{Re}\,\Sigma_0=hV, \tag{3.26}$$
$$\delta^2=1-(2\sqrt{3}f(1)/hVf_1(1))^{1/2}. \tag{3.27}$$

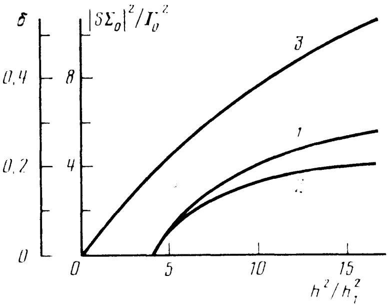

FIG. 2. Dependence of the width δ of a packet of parametric spin waves and of the integrated amplitude on the supercriticality parameter h^2/h_1^2 [curve 1 represents the solution of Eq. (3.29), curve 2 is the solution of Eq. (3.18), and curve 3 is the dependence of the integrated amplitude $|S\Sigma_0|$ of a packet on the supercriticality parameter].

The expressions in Eq. (3.26) differ from the asymptotes of the simple S-theory formulas for one group of pairs because $S^0(0, 0)$ is replaced with $S^0(0. 1)$.

3.4.4. Numerical solution for YIG. We shall now consider Eqs. (3.18) and (3.17) for a specific set of coefficients:

$$a=2.6;\quad b=-1.8;\quad c_1=4.1;\quad d_2=10.6, \tag{3.28}$$

which is typical of YIG when $H_c - H = 100$ Oe [Table I and Eq. (1.9)]. For this set of coefficients we have $\xi(0) = 4.4$, $\xi' = 2.2$, $\xi'' = 8$ and Eq. (3.20) becomes

$$\xi=4.4+13.2\delta+81\delta^2. \tag{3.29}$$

Figure 2 shows the dependences $\delta(\xi)$ obtained by solving the above equation, which is valid for small values of δ (curve 1) and those obtained from Eq. (3.18) (curve 2). This figure includes also the dependence of $|S\Sigma_0|^2/\Gamma^2(0)$ on ξ, obtained from Eq. (3.17) for the same set of coefficients given by Eq. (3.28). Clearly, the integral amplitude of parametric spin waves $|S\Sigma_0|^2$ is practically identical with the S-theory solution and the packet width $\delta(\xi)$ rises very slowly on increase in the supercriticality parameter ξ.

4. CRITERION OF THE LOSS OF STABILITY AS A RESULT OF A SECOND MODIFICATION OF THE DISTRIBUTION FUNCTION

In this section we shall show that the steady state of parametric spin waves, investigated in Sec. 3 and realized above the threshold $\xi > \xi(0)$, may become unstable on further increase in the supercriticality parameter. Consequently, a third group of pairs of parametric spin waves with a singular distribution $n(x) \propto \delta(x^2 - x_3^2)$, appears and it is concentrated at latitudes $x = \pm x_3$, $x_3 > \delta$. As in Sec. 2, the threshold of this instability is found from the condition

$$|P(x)|^2=\gamma^2(x)\quad \text{for}\quad x=\pm x_3, \tag{4.1}$$

but the self-consistent pumping $P(x)$ has to be calculated now for a steady state of the type described by Eq. (3.9) and realized when $\xi > \xi(0)$. This problem simplifies greatly if, as in the derivation of the system of equations (3.14), we ignore the second moment of the distribution function of parametric spin waves Σ_2 compared with the zeroth moment Σ_0. Then, if $x \gg \delta$, Eq. (3.3) for $P(x)$ simplifies to

$$P(x)=hV+S\Sigma_0 f(x). \tag{4.2}$$

Bearing in mind that the conditions of Eq. (3.13) should be satisfied when $|x| = \delta$, we shall represent this dependence as follows:

$$|P(x)|^2=\gamma^2(x)+(|x|-\delta)^3\varphi(x), \quad |x|>\delta. \tag{4.3}$$

The threshold condition of Eq. (4.1) is now equivalent to

$$\varphi(x_3)=0. \tag{4.4}$$

The function $\varphi(x)$ can be expanded as a Taylor series near the point $x=\delta$ and the coefficients of this series can be found from Eq. (4.3) by substituting there $P(x)$ from Eq. (4.2) and using the conditions of Eq. (3.17):

$$\varphi(x)=\sum_n A_n(|x|-\delta)^n, \tag{4.5}$$

$$A_n=\frac{1}{(n+3)!}\Big[-(\gamma^2(x))^{(n+3)}+X\{(f^2(x))^{(n+3)} -2f(\delta)f^{(n+3)}(x)\}+\frac{(\gamma^2(x))'f^{(n+3)}(x)}{f'(x)}\Big]_{x=\delta}. \tag{4.6}$$

The steady-state solution obtained in Sec. 3 should satisfy the condition of external stability:

$$A_0=-\frac{(\gamma^2(\delta))'''}{6}+2Xf_1(\delta)f_2(\delta)+(\gamma^2(\delta))'f_3(\delta)/f_1(\delta)<0. \tag{4.7}$$

This condition is obeyed at low values of δ since in this limit the only nonzero term is $2Xf_1f_2\to 2abX<0$ (the solution obtained in Sec. 3 is realized in the case when $a>0$ and $b<0$). An increase in the supercriticality parameter may result in breakdown of the condition (4.7) and then a new solution should appear, but if

$$A_1=-\frac{(\gamma^2(\delta))''''}{4!}+\frac{(\gamma^2(\delta))'f_4(\delta)}{f_1(\delta)} +X(2f_1(\delta)f_3(\delta)+f_2^2(\delta))>0, \tag{4.8}$$

then before breakdown of the condition (4.7) the equality of Eq. (4.4) is satisfied when

$$x_3-\delta\approx A_1/A_0+O((|x_3|-\delta)^2) \tag{4.9}$$

and the threshold of creation of the third group of pairs will be reached.

At the threshold value of the supercriticality parameter, we have

$$\varphi(x_3)=0, \tag{4.10}$$

$$d\varphi(x)/dx_{x=x_3}=0. \tag{4.11}$$

Eliminating x_3 from the above equations, we can find the threshold value of the supercriticality parameter ξ_3 and then use Eq. (4.10) to find x_3 and employ Eq. (3.18) to determine δ_3. For the set of coefficients in Eq. (3.28), typical of YIG when $H_c-H=100$ Oe, we carried out this procedure on a computer and obtained

$$\xi_3=8.7;\quad x_3=0.82;\quad \delta_3=0.12. \tag{4.12}$$

These values of ξ_3 and x_3 are marked on Fig. 1.

5. EXPERIMENTAL INVESTIGATION OF MULTISTAGE EXCITATION OF SPIN WAVES IN YTTRIUM IRON GARNET

5.1. Experimental method. An experimental investigation of multistage excitation was carried out on YIG single crystals. Parametric spin waves were excited by the method of parallel pumping at a resonance frequency $\omega_p/2\pi=9.37$ GHz. The threshold value of the pump field h_1 for the first group of parametric spin waves was determined by the traditional method from the appearance of a characteristic split in a pulse reflected from a cavity.

We investigated various groups of parametric spin waves characterized by $\theta(\mathbf{k})\neq\pi/2$ by a new method involving investigation of their collective excitations, which under certain conditions are independent and are observed separately from oscillations of the main group of waves excited on the equator. Collective oscillations of the second, third, and higher groups are readily excited by a transverse weak signal h_1 which interacts only with the waves characterized by $\theta(\mathbf{k})\neq\pi/2$ because the matrix element of the interaction [see, for example, Eqs. (6.23) and (14.14) in Ref. 2] is

$$V_\perp\propto\sin 2\theta(\mathbf{k})\exp i\varphi(\mathbf{k}). \tag{5.1}$$

Collective oscillations were excited by a combination resonance method developed in Ref. 7. The apparatus included not only a magnetron, used as a source of pulsed microwave pumping of spin waves, but also a source of a weak microwave signal in the form of a klystron operating at a frequency $\omega_s(t)=\omega_p+\Omega(t)$ which varied periodically within a narrow range $|\Omega|\lesssim 1$ MHz near the magnetron frequency. The signal at the combination frequency Ω, separated out in the nonlinear system of parametric spin waves, excited resonantly collective oscillations of parametric spin waves. The special feature of such a method of excitation of collective oscillations, distinguishing it from a method described in Ref. 7, was the use of a bimodal cavity. Two degenerate orthogonal TE_{112} modes in a cylindrical cavity were used as a channel for parallel pumping of parametric spin waves and as a weak signal channel, respectively. The required polarization of the modes and the coupling between them were ensured by rotation of the supply waveguides at the ends of the cavity about its cylindrical axis. An alternating field of the first mode $\mathbf{h}$ at the point of the location of a sample was oriented parallel to a static magnetization field $\mathbf{H}$ and in this case the field of the small signal $\mathbf{h}_\perp$ was perpendicular to $\mathbf{H}$. Since the matrix element of the interaction of the transverse signal $\mathbf{h}_\perp$ (and of the associated uniform precession) with spin waves was proportional to $\sin 2\theta(\mathbf{k})$ [Eq. (5.1)], parametric spin waves excited near the equator of the resonance surface did not react to the small signal $h_\perp$. Therefore, the response of the system of parametric spin waves to a small signal should be observed only at a certain critical value of the pump field $h=h_c$ corresponding to creation of parametric spin waves far from the equator. Elimination of the influence of an intense group of parametric spin waves characterized by $\theta(\mathbf{k})=\pi/2$, which appeared immediately after the first threshold, on the process of interaction with a weak transfer signal was the basic principle of the method employed by us.

A simplified variant of this principle was already used in Ref. 8 where the field in the transverse channel $h_\perp$ was induced by the sample itself on appearance of parametric spin waves with a polar angle far from $\pi/2$. The "active" variant of the method, utilizing an external transverse field $h_\perp$, made it possible to record with high precision the threshold excitation power of groups of parametric spin waves characterized by $\theta(\mathbf{k})\neq\pi/2$ and to measure the parameters of their interaction with a small signal $h_\perp$, which carry information on the state of these groups of parametric spin waves. These parameters are primarily the resonance frequency of collective oscillations and the resistivity of the system of parametric spin waves to the field $h_\perp$. These parameters could be determined by direct observation of oscillograms of a signal reflected from the resonator on the klystron side. Under subthreshold conditions the resonator was matched to the waveguide system of the magnetron and klystron in such a way that there was no reflected signal at the natural frequency of the resonator which was $\omega_0/2\pi = 9.37$ GHz. The klystron signal frequency was modulated linearly by altering the voltage on its reflector using a sawtooth pulse. A complete oscillogram of the reflected signal was in the form of a resonance curve of the resonator. At the center of the curve there were low-frequency zeroth beats between the klystron signal and the attenuated magnetron signal, which passed from the parallel to the transverse channel because of imperfect decoupling between the modes which in our experiments amounted to ~ 50 dB.

If $h_\parallel > h_c$, collective excitations were excited in a sample at a frequency Ω_{res} and additional absorption of the small-signal energy appeared in the resonator at a frequency separated from ω_p by Ω_{res}. The resonance curve in the oscillogram of the reflected signal became distorted at the relevant point and a resonance peak appeared at this point due to collective oscillations of parametric spin waves characterized by $\theta(\mathbf{k})\neq\pi/2$. An increase in the pump power increased the amplitude of the peak and shifted it away from the central frequency.

5.2. Experimental results and discussion. The threshold h_c for the excitation of parametric spin waves characterized by $\theta(\mathbf{k})\neq\pi/2$ was determined to within ~ 0.5 dB from the distortion of the resonance curve in the oscillogram of the reflected small signal, i.e., by the method described above. Figure 3 shows the dependences of these thresholds on the static magnetic field obtained for a YIG sphere with the $\mathbf{M}\|\langle 100\rangle$ orientation. The values of h_c/h_1 corresponded to data obtained at a fixed value of the field $H = H_c - 100$ Oe in an earlier investigation of emission of radiation along the transverse channel.[8]

The absence of absorption of the small signal in the range $h_1 < h < h_c$ was a direct proof that only parametric spin waves $\theta(\mathbf{k})$ close to $\pi/2$ were excited in the system. Broadening to the range $H > H_c$ showed that in the interaction with the transverse small signal that indeed occur only for the waves with $\theta(\mathbf{k})\neq\pi/2$. In fact, in the range $H \gtrsim H_c$ the spectrum of spin waves was lifted relative to the point $H \gtrsim H_c$ by such a large amount that only the long-wavelength part of the spectrum with $\theta(\mathbf{k})\neq\pi/2$ was in a parametric resonance. This lifted the forbiddeness of the interaction of parametric spin waves with the field of a transverse small signal and as a result an absorption peak of the small signal appeared practically directly after the first threshold. The steep reduction in the value of h_c/h_1 on transition across H_c left no doubts about the fact that this was due to a change

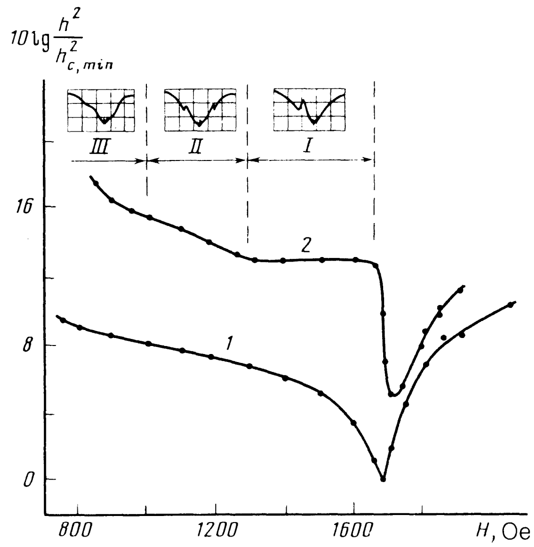

FIG. 3. Experimental dependence of the threshold power on the magnetic field: 1) first threshold ($h = h_1$); 2) threshold of excitation of parametric spin waves characterized by $\theta(\mathbf{k})\neq\pi/2$ ($h = h_c$).

in the distribution of parametric spin waves over the angles $\theta(\mathbf{k})$.

This was the reason for special interest in the kink of the $h_c(H)$ curve at the point $H \approx H_c - 300$ Oe. The attribution of this kink to a change in the distribution of parametric spin waves was confirmed by a change in the nature of the interaction of these waves with the small-signal field. In the region labeled I, where $H > H_c - 300$ Oe, there was one strong wide absorption peak, whereas in the region II corresponding to fields $H < H_c - 300$ Oe there were two symmetric narrow low-intensity peaks at frequencies $\omega_p \pm \Omega_{res}$. The insets in Fig. 3 show oscillograms of the reflected small signal typical of the three regions. In fields $H < H_c - 700$ Oe (region III) the peaks were strongly broadened and one peak was observed clearly at a frequency $\omega_p - \Omega_{res}$. The broadening was expected since a nonlinear correction to the damping of parametric spin waves appeared in this range of fields because of three magnon coalescence processes.[9]

We were particularly interested in regions I and II where parametric spin waves have $\mathbf{k}\neq 0$ ($H < H_c$) and there is no three-magnon nonlinear damping ($H > H_{3m} \approx H_c - 700$ Oe). In the region II the experimental and theoretical values of the threshold powers for the creation of the second group of pairs of parametric spin waves h_2^2 were practically the same (Fig. 1). This suggests that, in this range of fields, creation of a singular group of parametric spin waves with angles $\theta(\mathbf{k})$ far from $\pi/2$ occurs in agreement with the theory.

A theory of collective oscillations, which makes it possible to understand the nature of the response of the system of parametric spin waves to a weak transverse signal $h_\perp$ in the region II, can be developed quite simply. The matrix element of the coupling between $h_\perp$ and the parametric spin waves via uniform precession is given by Eq. (5.1). Therefore, the transverse field $h_\perp$ excites collective oscillations of parametric spin waves with the azimuthal mode number $m = -1$, which do not interact with collective oscillations of the first group of waves localized on the equator of the resonance surface. In fact, the matrix element relating collective oscillations of two groups of parametric spin waves has the property $S(\mathbf{k},\mathbf{k}') = S(-\mathbf{k},\mathbf{k}')$ [for a Fourier transform we have $S^m(x,x') = (-1)^m S^m(x,-x')$], so that at low values of $x = \cos\theta(\mathbf{k})$ the matrix element $S^{\pm 1}(x,x')$ is proportional to $|x|$. Similarly, by analogy with Eq. (1.9), we can show that for $x, x' \to 0$, we have

$$S^{\pm 1}(x,x') \sim |x+x'| - |x-x'|$$

and this element vanishes for $x = 0$ or $x' = 0$. Then the frequency Ω_2 of the resonance of the response second group of pairs to $h_\perp$ can be calculated exactly in the same way as the frequency of the resonance of the response of the first group of pairs to a weak longitudinal signal Ω_1 (Refs. 1 and 10). We then obtain

$$\Omega_2^{\pm} = i\Gamma_2 - 2S_2^- N_2 \pm (4S_2^+(2T_2^+ + S_2^+)N_2^2 - \Gamma_2^2)^{1/2}, \quad (5.2)$$

where

$$S_2^{\pm} = (S^1(\theta_2,\theta_2) \pm S^{-1}(\theta_2,\theta_2))/2,$$
$$T_2^+ = (T^1(\theta_2,\theta_2) + T^1(\theta_2,-\theta_2))/2.$$

Here, $\Gamma_2 = \Gamma(\theta_2)$, N_2 is the number of parametric spin waves in the second group of pairs, and $S^{\pm 1}(\theta,\theta')$ and $T'(\theta,\theta')$ are the axial Fourier harmonics of the matrix elements $S(\mathbf{k},\mathbf{k}')$ and $T(\mathbf{k},\mathbf{k}')$ with the numbers $m = {}^{+}1$. It follows from Eq. (5.2) that the resonant response to a transverse field appears at two frequencies, which corresponds to an oscillogram of the region II in Fig. 3. Thus, the usual multistage excitation of singular groups of parametric spin waves appears in the region II; there is a quantitative agreement between the experiments on YIG and our calculations (without recourse to fitting parameters) of the threshold of creation of the second group. Moreover, the nature of the response of this system to a weak transverse signal is understood qualitatively.

The pattern of the phenomena observed in the region I is more complex. It is clear from Fig. 1 that an increase in H brings the angle of creation of the second group of pairs $\theta(\mathbf{k})$ closer to the equator and in the region I we have $|\cos\theta_2| < 0{,}1$. Consequently, the response of parametric spin waves becomes much weaker and the method is insensitive to the second threshold, so that only the third threshold can be detected when parametric spin waves are excited far from the equator (Fig. 1b). It is clear from Fig. 1a that the experimental values of the threshold power for the excitation of parametric spin waves characterized by $\theta(\mathbf{k}) \neq \pi/2$ in fields $H_c - H = 200$ Oe and 100 Oe agree with the theoretical values of the threshold of creation of the third group of pairs. We recall that, according to the theory, the modification of the distribution function beyond the second threshold should be very different in fields $H_c - H = 200$ Oe and 100 Oe. When $H = H_c - 200$ Oe and $h > h_2$ the second singular group of waves characterized by $|\cos\theta_2| \approx 1/20$ should be created, i.e., it should be created close to the first group of pairs characterized by $\cos\theta_1 = 0$. However, if $H = H_c - 100$ Oe, then in the fields $h > h_2$ we can expect gradual broadening of a packet of parametric spin waves and, for the supercriticality parameter corresponding to the threshold of creation of the third group of pairs, a packet of parametric spin waves remains fairly narrow: $|\cos\theta(\mathbf{k})| \lesssim 0.12$. Our experimental method was not sufficiently sensitive to detect the difference between the two types of modification of the distribution of pairs of parametric spin waves close to the equator. As pointed out already, this method can be used only to detect the creation threshold of the third group of pairs characterized by $|\cos\theta_3| \approx 0.7$–$0.8$. Indirect evidence in support of our description of modifications of the distribution function of parametric spin waves at the equator is provided by the quantitative agreement between the experimental and theoretical values of the threshold field h_3.

The nature of the response of the system of parametric spin waves to a weak transverse signal $h_\perp$ in the region I can be understood qualitatively. In fact, it follows from Eq. (1.4) that in the case of YIG we have $\Gamma(\pi/4) \approx 4\Gamma(\pi/2)$. Therefore, in the expression for the resonance frequency Ω_3, which is similar to Eq. (5.2) for Ω_2, the square of the damping factor $\Gamma_3^2 \approx \Gamma^2(\pi/2)$ under the radical is 16 times greater than $\Gamma^2 \approx \Gamma^2(\pi/2)$. Consequently, for moderate values of the excess above the threshold h_3 the radical becomes purely imaginary and the expressions for $\Omega_3^{\pm}$ can be represented in the form

$$\Omega_3^+ \approx -2S_3^- N_3, \quad \Omega_3^- \approx -2i\Gamma_3. \quad (5.3)$$

Because of the large value of Γ_3 no response is observed at the frequency Ω_3^-. This corresponds to an oscillogram with one peak at the frequency Ω_3^+ (Fig. 3). A quantitative comparison of the theoretical and experimental values of the resonance frequency and susceptibility $\chi_\perp''(\Omega_3^+)$ in the field $H = H_c - 200$ Oe was made in Ref. 11. Therefore, in the range of fields I and in the range II there is a quantitative agreement between the experiments and theory in respect of the threshold of creation of parametric spin waves far from the equator and a qualitative understanding is gained of the nature of the response of the system to a weak transverse signal. All this allows us to assume that the observed pattern of modification of the distribution function of parametric spin waves (due to an increase in the intensity of the pump field) does occur in reality.

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