Particle with internal angular momentum in a gravitational field

I. B. Khriplovich

Nuclear Physics Institute, Siberian Branch, USSR Academy of Sciences

(Submitted 1 January 1988)
Zh. Eksp. Teor. Fiz. **96**, 385–390 (August 1989)

We give a simple and general derivation of the equations of motion of a particle with internal angular momentum in an external gravitational field. The gravitational analog of the Lorentz force is an extra term, producing nongeodesic motion. We present considerations that fix the general form of the wave equation for particles of arbitrary spin in electromagnetic and gravitational fields.

1. The motion of a relativistic particle possessing internal angular momentum in an external gravitational field was first examined long ago by Papapetrou.[1] Using Fok's method[2] to derive the equations of motion, he demonstrated that a particle with angular momentum moves along a nongeodesic path. Similar results were subsequently obtained by Barducci, Casalbuoni, and Lusanna,[3] and by Ravndal,[4] using Grassmann variables for a spin-1/2 particle in a gravitational field.

In this paper, the equations of motion of a particle with angular momentum are derived in a simple and general manner. In the process, a remarkable analogy will become evident between the motion of a charged particle in an electromagnetic field and a rotating particle in a gravitational field. The equations of motion thus derived are applicable to arbitrary particle spin. For spin 1/2, they are identical with the equations obtained in Refs. 3 and 4.

The intimate connection between classical and quantum mechanical considerations in the present approach makes it possible to obtain a general form of the relativistic wave equation for a particle of arbitrary spin in an external field. For integer spin, the equation that we have found is the same as that previously proposed by Christensen and Duff.[5]

2. It is well known that for a particle in an external field, either electromagnetic or gravitational, the canonical momentum $p_\mu = i\hbar\partial_\mu$ enters into the relativistic wave equation through the combination of terms

$$\Pi_\mu = i\hbar\partial_\mu - eA_\mu - \hbar\Sigma^{ab}\Gamma_{\mu,ab}.$$

Here e is the charge on the particle, A_μ is the electromagnetic vector potential, $\Gamma_{\mu,ab} = -\Gamma_{\mu,ba}$ is the spin coupling of the gravitational field, and the Σ^{ab} are generators of the Lorentz group for the representation whereby the wave function ψ of the particle in question is transformed. Greek subscripts and superscripts refer to world lines, and Roman to tetrads. Bearing in mind that we shall eventually pass to the classical mechanics limit, we retain only terms up to first order in Planck's constant $\hbar$. Note that when the operator Π_μ is applied repeatedly, Christoffel symbols appear along with the ∂_μ. Because of the factor $\hbar$ however, these terms are always small, and we shall henceforth ignore them.

We can obtain the Heisenberg equations of motion using the covariant Hamiltonian

$$H_0 = -g^{\mu\nu}\Pi_\mu\Pi_\nu. \tag{1}$$

We work in a metric with signature $+---$. Below we shall discuss how to consistently incorporate into this Hamiltonian terms explicitly containing the electromagnetic field

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and the Riemann tensor

$$R_{\mu\nu\,ab} = -2\left[\partial_\mu\Gamma_{\nu,\,ab} - \partial_\nu\Gamma_{\mu,\,ab} + 2\left(\Gamma_{\mu,\,ca}\Gamma_{\nu,}{}^{c}{}_{b} - \Gamma_{\nu,\,ca}\Gamma_{\mu,}{}^{c}{}_{b}\right)\right].$$

To proceed to the classical limit, we represent the wave function in the customary form $\psi \sim \exp(iS/\hbar)$. Then $p_\mu\psi = i\hbar\partial_\mu\psi \sim (-\partial_\mu S)\psi$, so that in p_μ the small quantum constant $\hbar$ no longer appears, nor does it appear in the electromagnetic term $-eA_\mu$ in Π_μ. This leaves the gravitational contribution $-\hbar\Sigma^{ab}\Gamma_{\mu,ab}$ to Π_μ, which in general vanishes in the classical limit $\hbar \to 0$. We shall assume, however, that the spin of the particle is so large that its internal angular momentum tensor $S^{ab} = \hbar\Sigma^{ab}$ has a classical limit. Thus, as before, we shall use (1) as the classical Hamiltonian for the time being, with $\Pi_\mu = p_\mu = eA_\mu = S^{ab}\Gamma_{\mu,ab}$. It then simply remains to specify the classical Poisson brackets. These have the standard form

$$\{p_\mu,\ x^\nu\} = -\delta_\mu{}^\nu, \tag{2}$$

$$\{S^{ab},\ S^{cd}\} = \eta^{ac}S^{bd} + \eta^{bd}S^{ac} - \eta^{ad}S^{bc} - \eta^{bc}S^{ad}. \tag{3}$$

Here $\eta^{ac} = \mathrm{diag}(1, -1, -1, -1)$ is the metric of flat space. The equations of motion can now be found without difficulty:

$$\frac{dx^\alpha}{ds} = \{H_0, x^\alpha\} = 2g^{\alpha\beta}\Pi_\beta, \tag{4}$$

$$\frac{d^2x^\alpha}{ds^2} = \left\{H_0, \frac{dx^\alpha}{ds}\right\} = -\Gamma_{\beta\gamma}{}^{\alpha}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} - 2eF_\beta{}^\alpha\frac{dx^\beta}{ds} + S^{ab}R^{\alpha}_{\beta ab}\frac{dx^\beta}{ds}. \tag{5}$$

If the term containing the Christoffel symbol $\Gamma^\alpha_{\beta\gamma}$ on the right-hand side of Eq. (5) arises as a result of the nonvanishing Poisson brackets $\{g^{\alpha\beta}, \Pi_\gamma\}$, the remaining two terms in the expression for the force are due to the nonvanishing Poisson bracket $\{\Pi_\alpha, \Pi_\beta\}$. Equation (5) may obviously be rewritten in covariant form:

$$\frac{D\dot{x}^\alpha}{Ds} = -2eF_\beta{}^\alpha\dot{x}^\beta + S^{ab}R_{\beta ab}\dot{x}^\beta. \tag{6}$$

Since the Hamiltonian (1) has dimensions of energy squared, the dimensionality of the conjugate variable s is somewhat unusual. Transforming to proper time via the relation $s = \tau/2m$, we may write out Eqs. (4) and (6) in a more familiar guise:

$$\dot{x}^{\alpha} = \frac{1}{m} g^{\alpha\beta}\Pi_{\beta}, \tag{4a}$$

$$\frac{D\dot{x}^{\alpha}}{D\tau} = -\frac{e}{m} F_{\beta}{}^{\alpha}\dot{x}^{\beta} + \frac{S^{ab}}{2m} R^{\ \alpha}_{\beta ab}\dot{x}^{\beta}. \tag{6a}$$

It must be emphasized that as before, the quantity $g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}$ remains an integral of the equations of motion (5), (6), (6a), which is fully consistent with the obvious condition

$$g_{\alpha\beta}\frac{dx^{\alpha}}{d\tau}\frac{dx^{\beta}}{d\tau} = 1.$$

Furthermore, we note that for massless particles, which follow trajectories with $d\tau = 0$, Eqs. (4)–(6) must be used (with $e = 0$, of course, since there are no charged massless particles), where s is a parameter that varies along the trajectory (see Ref. 6, for example).

In the special case of spin 1/2, the resulting equations of motion are the same as those derived in Refs. 3 and 4. But this is just the case in which the treatment of the terms in Eqs. (5), (6), and (6a) containing the Riemann tensor is somewhat dubious. The small magnitude of $\hbar$, which appears in S^{ab}, is totally uncompensated here, and the effect in question can scarcely be discriminated from the background of other quantum effects.

The last term on the right-hand side of (5), (6), and (6a) is the new force, which compels a rotating body to deviate from a geodesic path. Our derivation makes its nature perfectly clear. Like the Lorentz force, it arises by virtue of the nonvanishing Poisson brackets $\{\Pi_{\alpha},\Pi_{\beta}\}$, or in the quantum case because of the fact that Π_{α} and Π_{β} do not commute. The correspondence here is obvious: $eF_{\alpha\beta}\leftrightarrow -\frac{1}{2}S^{ab}R_{\alpha\beta ab}$, the integral momentum S^{ab} is the analog of the charge e, and the Riemann tensor is the analog of the electromagnetic field. The new force under consideration might be designated the gravitational Lorentz force.

3. Making use of the Poisson brackets (3), it is straightforward to derive the equation of motion for the angular momentum tensor from the Hamiltonian (1). In terms of the variables s, we have

$$\dot{S}^{ab} = -2\dot{x}^{\mu}\Gamma_{\mu,ef}(\eta^{ae}S^{bf} - \eta^{be}S^{af}). \tag{7}$$

There is a small problem here, however. In the rest frame of a point particle, the angular momentum tensor has only spatial components. The covariant statement of this fact is

$$\dot{x}_a S^{ab} \equiv \dot{x}^{\mu} V_{\mu a} S^{ab} = 0, \tag{8}$$

$V_{\mu a}$ is a tetrad. But it is not difficult to show that Eqs. (5) and (7) in no way guarantee the required constancy of $\dot{x}_a S^{ab}$.

An escape from this quandary is suggested by the squared form of the Dirac equation in an external electromagnetic field—one adds the term $eF_{ab}S^{ab}$ to the Hamiltonian (1). Moreover, in conjunction with the symmetry $R_{abcd} = R_{cdab}$, the aforementioned analogy $eF_{ab}\leftrightarrow -\frac{1}{2}S^{ab}R_{abcd}$ suggests the form of yet another term, $-\frac{1}{4}R_{abcd}S^{ab}S^{cd}$. The resulting corrected Hamiltonian

$$H = -g^{\mu\nu}\Pi_{\mu}\Pi_{\nu} + eF_{ab}S^{ab} - {}^{1}/_{4}R_{abcd}S^{ab}S^{cd} \tag{9}$$

then yields for the equation of motion of the angular momentum

$$\dot{S}^{ab} = -\left(2\dot{x}^{\mu}\Gamma_{\mu,ef} + \frac{e}{m}F_{ef} - \frac{S^{cd}}{2m}R_{efcd}\right)(\eta^{ae}S^{bf} - \eta^{be}S^{af}). \tag{10}$$

At this point, it can readily be shown, using Eqs. (6a) and (10), that we indeed have

$$\frac{d}{d\tau}(\dot{x}_a S^{ab}) = 0. \tag{11}$$

In calculating the derivative

$$\frac{d}{d\tau}(\dot{x}_a S^{ab}) \equiv \frac{d}{d\tau}(\dot{x}^{\mu}V_{\mu a}S^{ab})$$

it is important not to forget to differentiate the tetrad $V_{\mu a}$ i.e., to calculate its Poisson bracket with the Hamiltonian. The definition of the spin coupling to the gravitational field must also be taken into account:

$$\Gamma_{\mu,ab} = {}^{1}/_{4}(V_{\nu a;\mu}V_{b}{}^{\nu} - V_{\nu b;\mu}V_{a}{}^{\nu}). \tag{12}$$

Finally, by virtue of (10), we obviously have $S_{ab}S^{ab} = \text{const}$.

Notice that at the same time the equation for the spin is modified, the Hamiltonian (9) leads to an additional term proportional to the product of the field strength and the Riemann tensor in the equations for $D\dot{x}/D\tau$. As for the newly appearing electromagnetic term, it has essentially been known for a long time. It represents the same force that gives rise to the Stern-Gerlach splitting of a polarized beam of neutral particles in a nonuniform magnetic field. We shall not consider such terms here, however, as they are high-order quantities in the small ratio of the body dimensions to the characteristic scale of variation of the fields.

Turning to the nonrelativistic limit, one can readily demonstrate that the interaction $eF_{ab}S^{ab}$ corresponds to a gyromagnetic ratio $g = 2$. If this term were chosen with an arbitrary coefficient, then for self-consistency, i.e., for (8) to hold, the Hamiltonian would have to incorporate one more term, so that the total electromagnetic correction to the Hamiltonian (1) would look like

$$\frac{g}{2}eF_{ab}S^{ab} - \frac{1}{2}(g-2)\frac{e}{m^{2}}\Pi^{a}F_{ab}\Pi_{c}S^{bc}. \tag{13}$$

The foregoing discussion is, in principle, a reformulation of a well-known derivation of the equations of motion for a particle with spin, given by Frenkel[7] and by Bargmann, Michel, and Telegdi.[8] A similar change in the Hamiltonian (9) would be necessary in the event of a change in the coefficient of $R_{abcd}S^{ab}S^{cd}$. It is thus clear that in any case the choice of (9) for the Hamiltonian of a charged particle with internal angular momentum is the simplest and most economical. We shall return to this question below.

4. Starting with the Hamiltonian (9), we can write down a single wave equation for a particle with spin in external fields:

$$\left\{g^{\mu\nu}(i\hbar D_{\mu} - eA_{\mu})(i\hbar D_{\nu} - eA_{\nu}) - m^{2} - e\hbar F_{ab}\Sigma^{ab} + \frac{\hbar^{2}}{4}R_{abcd}\Sigma^{ab}\Sigma^{cd}\right\}\psi = 0. \tag{14}$$

Here D_{μ} is the covariant derivative, incorporating both the spin coupling and, as necessary, a Christoffel symbol. We shall not discuss here any supplementary conditions placed upon the wave function ψ.

Let us compare (14) with other, more familiar wave equations. We start with the electromagnetic interaction. For spin 1/2, we obviously wind up with the usual squared

Dirac equation. If we wished to incorporate the anomalous magnetic moment into this equation, it would have to include a term corresponding to the second term in (13). We then have an obvious problem: such an interaction, with a mass in the denominator, grows with increasing energy and destroys the renormalizability of the theory.

For spin $s = 1$, the choice $g = 2$ corresponds to the Yang-Mills coupling of the electromagnetic interaction of charged vector bosons. With the Higgs mechanism for generating the mass of charged vector fields, such a theory is renormalizable. But even when mass is coupled rigorously into the nonrenormalizable electromagnetics of vector particles, $g = 2$ corresponds to the slowest growth of divergences. Note, by the way, that the minimal coupling of the electromagnetic interaction in the Proca formalism for massive vector particles corresponds to the choice $g = 1$. Then even the second term in (13) explicitly demonstrates the nonrenormalizability of this theory.

The electrodynamics of higher spins is nonrenormalizable, but here as well the choice $g = 2$ would correspond to the slowest growth of divergences. Notice that neither the Rarita-Schwinger equation for spin 3/2, with minimal coupling of the electromagnetic interaction, nor the Fierz-Pauli formalism for the electrodynamics of spin-2 particles[9] is consistent with Eq. (14), and neither yields $g = 2$.

The gravitational interaction is not renormalizable, no matter what the spin of the particle. In this latter case as well, however, similar considerations fix the coefficient of $R_{abcd}\Sigma^{ab}\Sigma^{cd}$. Changing it, as compared with (9) and (14), would require the introduction of additional terms which are singular in the mass, and which are like the second term in (13): the change would thereby encourage the additional growth of divergences in the theory. From that standpoint, the choice of the Hamiltonian (9) and the wave equation (14) is actually the best one available.

Let us begin now the $s = 1/2$. The squared Dirac equation in a gravitational field is

$$(-g^{\mu\nu}D_\mu D_\nu - m^2 + {}^1/_4 R)\psi = 0. \tag{15}$$

Henceforth we take $\hbar = 1$; we shall not consider the electromagnetic interaction any further. The equation proposed by Christensen and Duff[5] is consistent with (15)—our equation leads to a coefficient of 1/8 instead of 1/4. But in the present case, our line of argument based on (8) no longer works; the properties of the spin matrices

$$\Sigma^{ab} = \frac{i}{2}\sigma^{ab} = \frac{i}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a)$$

here are such that the term in question degenerates in one way or another into a scalar curvature R with absolutely no consequences whatsoever for the motion of the spin.

For $s = 3/2$, our equation (14) is consistent with the equation proposed in Ref. 5, and the coefficient of $R_{abcd}\Sigma^{ab}\Sigma^{cd}$ fortuitously agree. As for the squared form of the Rarita-Schwinger equation in a gravitational field,

$$(-g^{\mu\nu}D_\mu D_\nu - m^2 + {}^1/_4 R)\psi^\lambda - {}^1/_2\gamma^a\gamma^b R^{\lambda}_{\varkappa ab}\psi^\varkappa = 0, \tag{16}$$

the principal term in the Riemann tensor affecting the motion of the spin is reproduced in either approach. The two approaches yield different coefficients of R, however: once again, it becomes 1/8 instead of 1/4. An additional term $R^{\lambda}_{\varkappa}\psi^\varkappa$ containing the Ricci tensor also makes its appearance.

The important thing, however, is that in the most interesting case—that of an Einstein space with $R^{\lambda}_{\varkappa} = 0$ and $R = 0$—Eq. (14) is trivially identical with Eq. (15), and entirely nontrivially identical with (16).

It is easy to avoid the departure of (15) from the Dirac equation which occurs for $R \neq 0$ by modifying Eq. (14) for half-odd-integer spins:

$$(-g^{\mu\nu}D_\mu D_\nu - m^2 + {}^1/_4 R_{abcd}\Sigma^{ab}\Sigma^{cd} + {}^1/_8 R)\psi = 0. \tag{17}$$

We have thereby also partially removed the discrepancy with the Rarita-Schwinger equation (16). The additional term $\frac{1}{8}R$ has no effect at all on the actual motion of the spin.

As for higher half-odd-integer spins, Eq. (17), by virtue of the foregoing considerations, is preferable to the equation put forth in Ref. 5.

A curious situation arises for integer spins: Eq. (14) (which is identical to the corresponding equation of Ref. 5) exactly reproduces the equations in the Feynman gauge for the photon and graviton in an external gravitational field:

$$-g^{\mu\nu}D_\mu D_\nu A_\lambda + R_\lambda{}^\varkappa A_\varkappa = 0, \tag{18}$$

$$-g^{\mu\nu}D_\mu D_\nu f_{\varkappa\lambda} + R_\varkappa{}^\rho f_{\rho\lambda} + R_\lambda{}^\rho f_{\varkappa\rho} - 2R_\varkappa{}^\rho{}_\lambda{}^\sigma f_{\rho\sigma} = 0. \tag{19}$$

Not only is the most important term with the Riemann tensor reproduced in Eq. (19), but so are all terms containing the Ricci tensor in the other equations.

I am grateful to A. I. Vainshtein for discussions and to N. A. Voronov, who, in particular, first drew my attention to the papers by Papapetrou[1] and Ravndal.[4] I thank the staff of the Banach Mathematics Center in Warsaw, where this work was begun in October, 1988 during the school on gauge theories, for their warm hospitality.

[1] A. Papapetrou, Proc. Roy. Soc. **A209**, 248 (1951).
[2] V. A. Fok, Zh. Eksp. Teor. Fiz. **9**, 375 (1939).
[3] A. Barducci, R. Casalbuoni, and L. Lusanna, Nucl. Phys. **B124**, 521 (1977).
[4] F. Ravndal, Phys. Rev. **D21**, 2823 (1980).
[5] M. Christensen and M. J. Duff, Nucl. Phys. **B154**, 301 (1979).
[6] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields*, 3rd Edition, Pergamon Press, New York (1971), §87.
[7] J. Frenkel, Z. Phys. **37**, 243 (1926).
[8] V. Bargmann, L. Michel, and V. Telegdi, Phys. Rev. Lett. **2**, 435 (1959).
[9] M. Fierz and W. Pauli, Proc. Roy. Soc. **A173**, 211 (1939).

Translated by Marc Damashek