The fractal structure of turbulent vortices

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A relation is established between the fractal structure of turbulent vortices (their fractal dimensions) and the observable characteristics of turbulence (characteristic scales, spectral energy density) for different mechanism of the inverse cascade, both virtual and real. A correspondence with the experimental data is established.

Some turbulent motions are characterized by an inverse cascade, i.e., an energy transfer from small-scale motions to large-scale motions. These are two-dimensional turbulence, turbulence in a rotating fluid, and magnetohydrodynamic turbulence.[1] One also assumes that the general circulation of the atmosphere may be fed by the energy of cyclones on account of an inverse cascade. A virtual inverse cascade is utilized in the model of virtual static equilibrium in order to balance the direct cascade (fragmentation of vortices).[2]

The physical mechanism of the inverse cascade may be the fusion of small-scale vortices into larger-scale coherent structures—peculiar cluster-vortices.[3] The mean statistical structure of such cluster-vortices must depend both on the way in which they associate, and on the dimension of space in which this association occurs. Such vortex clusters have been repeatedly observed in natural and numerical experiments (see, e.g., Refs. 3–7).

The homogeneity property in a hierarchic picture of fusion of small vortices into larger ones may be replaced by the condition of scale self-similarity, similar to the one occurring in the case of fractal clusters.[8] In general, in spite of the dynamical (moving) character of vortex clusters, their fractal properties should, on the average, be analogous to the one of clusters with fixed elements. This refers to the fractal dimension and to the dynamics of growth of the average size of the cluster.

One must, of course, keep in mind that the fractal character of the cluster-vortices is a phenomenon radically different from the fractal character of the trajectories of the motion of the fluid particles in turbulent motion (Refs. 9, 10). However, fractal properties of the trajectories of the motion have an essential influence on the properties of the clusters created as a result of such a motion.[8] In particular, the fractal dimension of the trajectories of the motion determines the fractal dimension of the cluster-vortices.

A fractal cluster-vortex realizes an intermittent motion. In those cases when the inverse cascade is a real phenomenon the fractal vortices which are formed are also real. In the case of a virtual inverse cascade,[2] the virtual cluster-vortices may model intermittency phenomena in turbulence with a direct cascade. Indeed, a fractal cluster-vortex has a "porous" (rough[8]) structure, whose fractal dimension can be considerably lower than the dimension of the space in which this cluster is formed. Essentially, models of the type of infinite fractal clusters for intermittent turbulence have been encountered already in the work of Novikov, Stewart, Yaglom, and others, although explicitly the notion of fractal dimension was not used in these papers.[11]

1. The cluster-vortices may be formed by an element-wise aggregation of "elementary" vortices (i.e., vortices which have no cluster structure) to the cluster. Another result of the association is a vortex-cluster consisting of clusters of smaller sizes, which in turn have been formed through the aggregation of even smaller clusters.

We start from a situation when the vortex-clusters are aggregated element-wise, with the elementary vortices moving along trajectories with fractal dimension close to the Brownian (a motion of the diffusion type). In this case the fractal dimension of the cluster-vortices which are formed is $D \approx 5/3$ for aggregation in two-dimensional space and $D \approx 2.5$ for aggregation in three-dimensional space. For such a character of the aggregation one can easily find a scaling law of growth of the mean size of a cluster, L as function of the time t (Ref. 8):

$$L \propto t^{1/\alpha}, \quad \alpha = D - (d-2), \tag{1}$$

where d is the dimension of the space in which the aggregation process occurs. Thus, for two-dimensional space it follows from Eq. (1) that

$$L \propto t^{3/5}. \tag{2}$$

This dependence can be tested experimentally (see below).

If one considers that the energy P of the cluster-vortex is proportional to the number N of elementary vortices which comprise it, taking into account that in the scaling range[8]

$$N \propto L^D, \tag{3}$$

as well as the estimate for the energy of a vortex[12]

$$P \propto kE(k),$$

where k is the wave number and $E(k)$ is the spectral energy density, we obtain

$$P \propto N \propto k^{-D} \propto kE(k).$$

From this it follows that in the scaling range

$$E(k) \propto k^{-D-1}. \tag{4}$$

For the type of aggregation considered we obtain from Eq. (4) for two-dimensional space

$$E \propto k^{-8/3}. \tag{5}$$

Reference 7 describes a numerical experiment which models a two-dimensional turbulent motion by means of a system formed of 100 identical point vortices. The dynamics of the characteristic size was investigated and a spectral analysis

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was carried out. The following characteristic size was adopted

$$L(t)=\Big(\sum_{\alpha<\beta}\frac{1}{l_{\alpha\beta}(t)}\Big)^{-1},$$

where

$$l_{\alpha\beta}=(x_\alpha-x_\beta)^2+(y_\alpha-y_\beta)^2,$$

and x_α, y_α are the dimensionless coordinates of the vortex with label α.

Figure 1 shows as dots (in a doubly logarithmic arbitrary scale) the data for $L(t)$ from Ref. 7. For a comparison with Eq. (2) Fig. 1 exhibits the straight line corresponding to a power law with exponent 3/5. Figure 2 shows the data (from the same paper, Ref. 7) for the spectral energy density [more precisely, for $kE(k)$]. The straight lines in the figure correspond to Eq. (5).

2. If the association of vortices is realized by an elementwise aggregation (as discussed in the preceding section), but the trajectories of the motion of elementary vortices have a fractal dimension different from the Brownian one, then the fractal dimension of the cluster-vortex will be different.

If the mean free path of an elementary vortex is large compared to the characteristic size of the aggregation region, one may assume that this vortex moves along a rectilinear trajectory. In this case the fractal dimension of the cluster-vortex coincides with the dimension of the space in which it is formed. The same happens for rectilinear trajectories of the motion of elementary vortices. From equations (1) and (4) we obtain for two-dimensional space

$$L\propto t^{1/2}, \tag{6}$$

$$E\propto k^{-3}. \tag{7}$$

One should note that a "-3" law for the energy spectral density in two-dimensional turbulence is widely known (see, for instance, Ref. 1).

In the experiment described in Ref. 13 two-dimensional turbulence was created behind a grid (perforated plate) by superimposing on the motion of mercury a strong transverse magnetic field ($B=0.68$ T). In this situation the role of the magnetic field reduces to the creation and maintenance of two-dimensional motion (in a plane perpendicular to the induction vector), but on the two-dimensional motion itself the magnetic field has no influence.[14] Figure 3 shows the experimental data from Ref. 13, showing the variation of the integral (correlation) scale of two-dimensional turbulence as a function of the distance from the grid (the x coordinate). According to the Taylor hypothesis (Ref. 11), $x\propto t$,

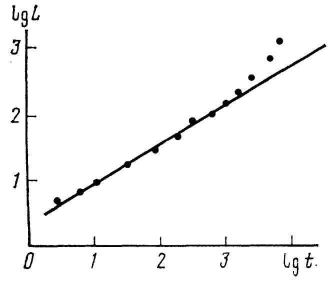

FIG. 1.

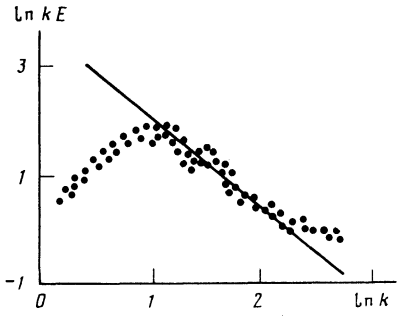

FIG. 2.

when the object is a verification of dependences of the type (6). The scales in Fig. 3 are logarithmic, but arbitrary. Figure 4 shows the experimental data of Ref. 13 for the verification of the spectral dependence (7).

3. We now consider cluster-vortices composed of clusters of smaller sizes, which in turn are formed from the fusion of smaller clusters. We first consider motion along trajectories of Brownian type.

The L as a function of t in this case is given by

$$L\propto t^{1/D}, \tag{8}$$

If the diffusion coefficient of the cluster-vortices does not depend on their sizes, i.e., for two-dimensional turbulence it coincides with Eq. (1). The fractal dimension of the cluster-vortices in this situation[8] is $D=10/7\pm0.4$ (two-dimensional case). Thus, in the case of diffusive cluster-vortex aggregation we have approximately for two-dimensional turbulence:

$$L\propto t^{7/10}, \tag{9}$$

$$E\propto k^{17/7}. \tag{10}$$

In Ref. 4 two-dimensional turbulence was created by moving a comb through a liquid film. The enlarging of vortex structures was observed visually (unfortunately, no spectral analysis was carried out). Figure 5 shows the data taken from Ref. 4 (the plots are logarithmic and the scales arbitrary). In order to verify the law (9) a straight line is drawn in the figure.

4. By means of a virtual inverse cascade[2] we shall model intermittency in turbulence in which the dominant process is the direct cascade, i.e., fragmentation of vortices. In the Richardson-Kolmogorov model a cascade energy transport from large scales to smaller ones and so on is represented as successive fragmentation of vortices down to scales when the smallest (elementary) vortices disappear rapidly under the influence of viscosity. One assumes that energy dissipation, which closes this inertial process, is automatically built un-

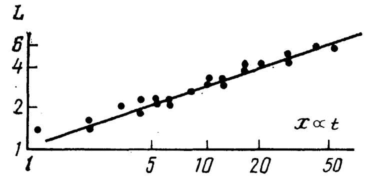

FIG. 3.

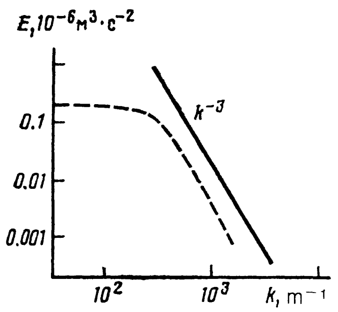

FIG. 4.

der any energy flow which this process transfers. If this were not so, then the scaling exponent in the spectral law

$$E \propto k^{-\gamma}$$

would be exactly equal to 5/3 (the Kolmogorov-Obukhov law[11]). The specific dissipation mechanism (viscosity) influences the possibility of energy transport according to the Richardson-Kolmogorov inertial channel, which causes γ to deviate from 5/3. Physically, this influence manifests itself through intermittency. Starting from the well-known remark of Landau[11] on the fluctuations of the rate of energy dissipation and the papers of Kolmogorov and Obukhov on intermittency,[11] many attempts were made to take into account the influence of intermittency on the scaling of the spectral energy density.[15] Since it is viscosity that is responsible for small-scale intermittency, we attempt to take into account the balance of the energy transport via the direct cascade with viscous dissipation by means of the inverse cascade model.[2]

In this model we propose to consider in addition to the real direct cascade (fragmentation), a virtual cascade which is statistically inverse to it, namely the fusion of vortices with fusion multiplicity inverse to the multiplicity of fragmentation in the direct cascade. In the virtual cascade the elementary (dissipative) vortices as a result of their diffusion (Brownian) motion will join into clusters (they will pair if the multiplicity of fragmentation in the direct cascade equals 1/2). Further, these clusters join with one another into even larger clusters, etc. (a hierarchical cascade).

Thus, the disappearance of elementary (dissipative) vortices under the influence of viscosity is replaced in the model under discussion by clustering of these vortices. It is clear that the energy dissipated in a volume of characteristic size $\sim l$ on vortices of size $\sim l$ must be proportional to the

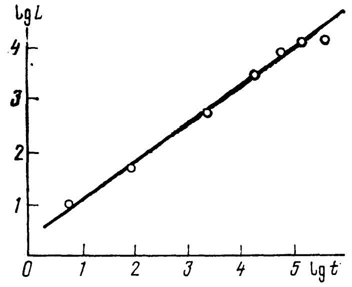

FIG. 5.

number of elementary (dissipative) vortices in the fractal cluster-vortex of characteristic size $\sim l$. Denoting this number by N_l and the dissipated power by W_l we express this assertion in the form

$$W_l \propto N_l. \tag{11}$$

If the energy is dissipated by the effect of viscosity, then for the isotropic case it can be estimated as follows[11]

$$W_l \sim M_l \int_{2^{-1/2}/l}^{2^{1/2}/l} k^2 E(k)\, dk, \tag{12}$$

where M_l denotes the fluid mass in a volume of characteristic size $\sim l$. Equation (12) can be rewritten in the form

$$W_l \sim l^d \int_{2^{-1/2}/l}^{2^{1/2}/l} k^2 E(k)\, dk, \tag{13}$$

where d is the dimension of the space in which the motion occurs. Assuming that the scale l^{-1} belongs to the scaling range (where the relation $E \sim k^{-\gamma}$ is satisfied) we obtain from Eq. (13)

$$W_l \propto l^{d+\gamma-3}. \tag{14}$$

For a fractal cluster[8]

$$N_l \propto l^D. \tag{15}$$

Thus, it follows from Eqs. (15), (14), and (11) that

$$\gamma = D + 3 - d. \tag{16}$$

For three-dimensional turbulence it follows from Eq. (16) that

$$\gamma = D, \tag{17}$$

i.e., the scaling exponent in the spectral law $E \sim k^{-\gamma}$ for three-dimensional isotropic turbulence is equal to the fractal dimension of the virtual cluster-vortex.

For the fractal dimensions of the clusters considered in the present section one may use the value derived theoretically in Ref. 16 for a similar process of hierarchical clusterization, $D = 1.72 \pm 0.01$. We then obtain for three-dimensional isotropic turbulence

$$\gamma = 1.72 \pm 0.01. \tag{18}$$

According to current experimental data the scaling exponent in the spectral law (taking intermittency into account) is $\gamma = 1.69$–1.72 (Ref. 15). Thus, viscous dissipation is satisfactorily balanced by the hierarchical energy cascade.

5. In two-dimensional (quasi-two-dimensional) turbulence in addition to (or in place of) the real fragmentation of vortices an active role may be played by their real fusion (see above). In various ranges of the scale the relative contribution of these processes may be different. Thus, for example, in the quasi-two-dimensional turbulence of oceans, on account of the large value of the Reynolds number and the constant presence of hard-to-control perturbing factors, the direct cascade (fragmentation of vortices) must coexist with the (real) inverse cascade. In this case, if a real fusion of vortices is accomplished according to the cluster-cluster (hierarchical) mechanism (in the same way as for the virtual inverse cascade discussed above) then the geometry of the cluster-vortices is defined as the intersection of two fractal systems, the virtual one and the real one.

The fractal dimension of such an intersection is given by the equation[17]

$$D=D_1+D_2-d,$$

where d is the Euclidean dimension of space (in this case $d = 2$) and D_1 and D_2 are the fractal dimensions of the intersecting objects (in this case $D_1 = D_2 \approx 10/7$). Consequently, in the case of the intersection of the direct and inverse cascades in two-dimensional space

$$D\approx {}^6/_7, \quad \gamma=D+1\approx {}^{13}/_7.$$

6. If one introduces a passive admixture to turbulence, the growth in time of the fractal cluster-vortex also determines the growth of the characteristic dimension of the cloud of admixture. If the motion of dissipative vortices has a diffusion character and the diffusion coefficients of the vortices does not depend on their sizes, then the growth of the characteristic size of the fractal cluster-vortex is given by the formula[8]

$$l\propto t^{\alpha}, \quad \alpha=(D+2-d)^{-1}. \tag{19}$$

Making use of the results of Sec. 4, one can rewrite Eq. (19) in a form which does not involve d:

$$l\propto t^{\alpha}, \quad \alpha=(\gamma-1)^{-1}. \tag{20}$$

If we consider the effective diffusion coefficient of an admixture[11]

$$K={}^1/_6 dl^2(t)/dt,$$

we obtain from Eq. (19)

$$K\propto l^{d-D}, \tag{21}$$

or (in a form which does not contain d)

$$K\propto l^{3-\gamma}. \tag{22}$$

If one replaces γ by 5/3 (the Kolmogorov-Obukhov spectral law), one obtains from Eq. (22)

$$K\propto l^{4/3} \tag{23}$$

i.e., the well-known Richardson-Obukhov law.[11]

Since $D = 1.72 \pm 0.01$ (see Sec. 4) differs little from 5/3, then the law

$$K\propto l^{1.28\pm 0.01} \tag{24}$$

differs insignificantly from Eq. (23) (current experiments seem unable to tell the difference).

In oceanic quasi-two-dimensional turbulence one has to take $d = 2$, $D\approx 6/7$ (see Sec. 5), i.e.,

$$K\propto l^{8/7}. \tag{25}$$

It is interesting to compare Eq. (25) with the experimental data from Ref. 18 (and also Ref. 19). In that paper one finds data obtained in the range $l = 10^4$–10^7 cm in different regions of the ocean. This yields $K\sim l^{1.1}$, in remarkable agreement with Eq. (25).

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