The relativistic kinetic equation for gravitationally interacting particles

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We derive the relativistic kinetic equation for the one-particle distribution function of gravitationally interacting particles to second order in the coupling constant. The interaction is described within the framework of general relativity. The kinetic equation for gravitationally interacting particles has a great deal in common with the Belyaev-Budker relativistic kinetic equation for plasma; the only difference consists of a change in a scalar factor that occurs in the kernel of the collision integral and depends on the momenta of the interacting particles. In the relativistic limit, the equation is the same as the equation derived by Bisnovatyĭ-Kogan and Shukhman [Soviet Phys. JETP **55**, 1 (1982)] for Newtonian gravitation. We discuss the application of the kinetic equation to a system containing two types of particles—nonrelativistic massive particles and ultrarelativistic massless particles.

INTRODUCTION

To deal with cosmological problems, it is necessary to have a proper understanding of the kinetic equation with gravitational interactions included. This equation was derived by Bisnovatyi-Kogan and Shukhman[1] for a nonrelativistic, uniformly expanding gas in the context of Newtonian gravitation. If a large number of collisions take place during the lifetime of the universe, so that

$$L=\ln(\langle v\rangle t/r_{min})\gg 1,$$

($\langle v\rangle$ is the mean thermal velocity of the particles, t is the cosmological time, and r_{min} is the separation at which the kinetic energy of the particles equals their interaction potential energy), then the collision integral for the gravitating particles[1] will take the same form as the Landau collision integral obtained for a plasma.[2] Taking the expansion of the universe into account has made it possible[1] to eliminate the divergence of the collision integral at large impact parameters.

In this paper, we derive the relativistic kinetic equation for a system of particles whose mutual interaction falls within the purview of general relativity. Introducing a set of random functions,[3] we construct a sequence of coupled equations for the one-particle, two-particle, ..., distribution functions. We go on to show that the derivation of the kinetic equation to second-order in the interaction requires only the linearized Einstein equations; three-particle correlations can be neglected.

In deriving the equation, we have assumed that the "average gravitational field" engendered by the particles[1] may (like the distribution function[3]) be regarded as constant over the region determined by the correlation length and the corresponding correlation time. In obtaining the right-hand side of the kinetic equation (the collision integral) in an expanding, spatially flat Friedmann space, the temporal dependence of the scale factor may therefore be neglected: the Einstein equations, whose solution is prerequisite to finding the "microscopic" gravitational fields created by the particles, has been linearized in the metric of flat Minkowski space, rather than in the Friedmann metric. As in plasma theory, this leads to divergence of the collision integral at large impact parameters, a situation that can be rectified by dealing with the averaged gravitational field in solving the Einstein equations. A similar situation is encountered in deriving the kinetic equation for gravitating nonrelativistic particles in the Newtonian gravitational theory.[1] If we ignore the expansion of the universe and replace e^2 with Gm^2, the collision integral is the same as the Landau collision integral. Introducing a cutoff at distances of order $\langle v\rangle t$ in the Coulomb logarithm yields the result obtained by Bisnovatyi-Kogan and Shukhman,[1] which was derived by taking account of the effect of the expansion of the universe on individual particle collisions.

The collision integral derived in the present paper also turns out to be proportional to an analog of the Coulomb logarithm, in which we assume a cutoff at distances of order $\langle v\rangle t$. After this procedure has been carried out, the kinetic equation can be used to describe a system of particles in an expanding Friedmann universe.

Not only is the kinetic equation that is obtained applicable to the case in which the average field represents an expanding Friedmann universe, but it can be used in other situations as well, where the average field can be treated as constant within the region defined by the correlation length and corresponding correlation time. It is only in the latter instance that it is necessary to investigate the whole question of introducing a cutoff at large impact parameters.

The relativistic kinetic equation for gravitationally interacting particles has much in common with the Belyaev-Budker relativistic kinetic equation for plasma.[4] The difference is that the factor $(ec)^4\,(u_i'u^i)^2$ in the Belyaev-Budker collision integral is replaced by the factor

$$G^2[2(p_i'p^i)(u_j'u^j)-(p_iu^i)(p_j'u^{j'})]^2,$$

where u_i and u_i' are the four-velocities of the colliding particles, p_i and p_i' are their momenta, e is the charge on a particle, and G is the gravitational constant.

We conclude by considering some applications of the theory, involving the interaction of the microwave background radiation with large-scale agglomerations of matter.

1. RANDOM FUNCTION IN THE LIOUVILLE EQUATION

Consider a system consisting of several types of particles (individual types will be denoted by Roman letters, a, b, c, ...). We also make use of the following notation: n_a is the number of particles of type a, the q^i are particle coordinates

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($q^0 = \eta$), and the $\tilde{p}_i$ are the covariant components of the momenta measured in the metric $\tilde{g}_{ij}$, which we represent as the sum of an "averaged" metric g_{ij} and a contribution δg_{ij} due to particle interactions ($i, j, k = 0,1,2,3$).

We now introduce a random function[1,5] for particles of type a:

$$\tilde{N}_a(q,\tilde{p}) = \sum_{l=1}^{n_a} \int d\tilde{s}\,\delta^4(q^i - q^i_{(l)}(\tilde{s}))\,\delta^4(\tilde{p}_j - \tilde{p}_j^{(l)}(\tilde{s})), \tag{1.1}$$

where $\tilde{s}$ is the canonical parameter along the trajectory, and $q^i_{(l)}, \tilde{p}_j^{(l)}$ are given by the equations of motion ($\tilde{p}^i = \tilde{g}^{ij}\tilde{p}_j$)

$$dq^i_{(l)}/d\tilde{s} = (m_a c)^{-1}\tilde{p}^i_{(l)}, \quad d\tilde{p}_i^{(l)}/d\tilde{s} = (m_a c)^{-1}\tilde{\Gamma}_{j,ik}\tilde{p}^j_{(l)}\tilde{p}^k_{(l)}. \tag{1.2}$$

As a consequence of Eq. (1.2), the function $\tilde{N}_a$ satisfies the equation

$$\tilde{p}^i \partial\tilde{N}_a/\partial q^i + \tilde{\Gamma}_{j,ik}\tilde{p}^j\tilde{p}^k \partial\tilde{N}_a/\partial\tilde{p}_i = 0. \tag{1.3}$$

Here $\tilde{\Gamma}_{j,ik}$ is a Christoffel symbol of the second kind, calculated in the metric $\tilde{g}_{ij}$.

In addition to the momenta $\tilde{p}^i_{(l)} = m_a c\,dq^i_{(l)}/d\tilde{s}$, we shall also make use of the momenta p^i in the metric g_{ij}:

$$p^i_{(l)} = \alpha^{-1}(q,p)\,\tilde{p}^i_{(l)}, \quad \alpha(q,p) = ds/d\tilde{s} = (g_{ij}p^i p^j)^{1/2}(\tilde{g}_{lk}p^l p^k)^{-1/2}. \tag{1.4}$$

Here s is the canonical parameter introduced with the aid of the metric g_{ij}. We transform from $\tilde{p}_i$ to p_i using the relation

$$\tilde{p}_j = \tilde{g}_{jk}\tilde{p}^k = \alpha\tilde{g}_{jk}g^{ki}p_i. \tag{1.5}$$

Next, we compute the Jacobian of the transformation (1.5), which is equal to the determinant of the matrix

$$\partial\tilde{p}_i/\partial p_j = \alpha\tilde{g}_{ik}(\delta_m{}^k + u^k v_m)\,g^{mj}, \tag{1.6}$$

where

$$u^k = (g^{ij}p_i p_j)^{-1/2}p^k, \quad v_m = u_m - \alpha^2\tilde{g}_{mj}u^j.$$

The vectors u^k and v_j are orthogonal ($u^i v_i = 0$), so the determinant of the matrix $\delta^k_m + u^k v_m$ is equal to unity, whereupon

$$|\partial\tilde{p}_i/\partial p_j| = \alpha^4\tilde{g}g^{-1}. \tag{1.7}$$

As a consequence of (1.7), the random function $\tilde{N}_a(q,\tilde{p})$ may be expressed in terms of the random function

$$N_a(q,p) = \sum_{l=1}^{n_a} \int ds\,\delta^4(q^i - q^i_{(l)}(s))\,\delta^4(p_j - p_j^{(l)}(s)) \tag{1.8}$$

as follows:

$$\tilde{N}_a(q,\tilde{p}) = g\tilde{g}^{-1}\alpha^{-5}N_a(q,p). \tag{1.9}$$

The functions $q^i_{(l)}$ and $p_j^{(l)}$ in (1.8) can be determined using the equations obtained by substituting (1.5) into Eq. (1.2) ($p^i = g^{ij}p_j$):

$$\begin{aligned} &dq^i_{(l)}/ds = (m_a c)^{-1}p^i_{(l)}, \\ &dp_i^{(l)}/ds = (m_a c)^{-1}(\Gamma_{j,ik} - \Omega_{jk}{}^m\Delta_{mi})\,p^j_{(l)}p^k_{(l)}. \end{aligned} \tag{1.10}$$

Here

$$\Delta_{ki} = g_{ki} - u_k u_i; \quad \Omega_{kj}{}^m = \tilde{\Gamma}_{kj}{}^m - \Gamma_{kj}{}^m$$

is the difference between the Christoffel symbols of the second kind for the metrics $\tilde{g}_{ij}$ and g_{ij}. By virtue of (1.10), the function $N_a(q,p)$ must satisfy the Liouville equation

$$\frac{\partial}{\partial q^i}(p^i N_a) + \frac{\partial}{\partial p_i}[(\Gamma_{j,ik} - \Omega_{jk}{}^m\Delta_{mi})\,p^j p^k N_a] = 0$$

or, making use of the identity

$$\frac{\partial}{\partial q^i}(p^i) + \frac{\partial}{\partial p_i}(\Gamma_{j,ik}p^j p^k) = 0,$$

the equation

$$p^i\frac{\partial N_a}{\partial q^i} + \Gamma_{j,ik}p^j p^k\frac{\partial N_a}{\partial p_i} = \frac{\partial}{\partial p_i}(\Omega_{jk}{}^m\Delta_{mi}p^j p^k N_a). \tag{1.11}$$

Equation (1.11) can also be obtained directly from (1.3) by making the substitutions (1.5) in (1.9).

The energy–momentum tensor of the particles may be written in terms of $\tilde{N}_a$ as

$$T^{ij} = \sum_b c\int \frac{d^4\tilde{p}_b}{(-\tilde{g})^{1/2}}\,\tilde{p}_b{}^i\tilde{u}_b{}^j\tilde{N}_b(q,\tilde{p}_b). \tag{1.12}$$

If we transform to momenta p_i and the function N_b, we obtain

$$T^{ij} = \sum_b c\int \frac{d^4 p_b}{(-g)^{1/2}}\,p_b{}^i u_b{}^j\alpha(q,p)\left(\frac{g}{\tilde{g}}\right)^{1/2} N_b(q,p_b). \tag{1.13}$$

2. THE EINSTEIN EQUATIONS

The quantities Ω^m_{jk} on the right-hand side of the Liouville equation (1.11) may be determined from the Einstein equations ($\varkappa = 8\pi G/c^4$)

$$\tilde{G}^{ij} = \varkappa T^{ij},$$

where $\tilde{G}^{ij}$ is the Einstein tensor, calculated in the metric $\tilde{g}_{ij}$, and T^{ij} may be expressed in terms of N_a through Eq. (1.13). If particle interactions are weak, then the Einstein equations can be linearized with respect to the "averaged" metric g_{ij} ($\tilde{g}_{ij} = g_{ij} + \delta g_{ij}$):

$$\delta G^{ij} + Q^{ij} = \sum_b \varkappa c\int \frac{d^4 p_b{}'}{(-g)^{1/2}}\,p_b{}^{i\prime}u_b{}^{j\prime}[N_b(q,q_b{}') - n_b f_b{}^0(q,p_b{}')]. \tag{2.1}$$

Here f^0_b is the "averaged" distribution function. The "averaged" field g_{ij} is determined by the energy-momentum tensor calculated with f^0_b, and the δG^{ij} are perturbations of the components of the Einstein tensor, up to terms linear in the δg_{ij}, while

$$Q^{ij} = \sum_b \frac{1}{2}\varkappa c\int \frac{d^4 p_b}{(-g)^{1/2}}\,p_b{}^i u_b{}^j(g^{lm} + u_b{}^l u_b{}^m)\,n_b f_b{}^0\delta g_{lm}.$$

We recognize that hereinafter we only require the quantities Ω^i_{jk} within the region delimited by the correlation length and corresponding correlation time. If g_{ij} can be considered constant within this region, then as we indicated in the Introduction, g_{ij} is effectively the Minkowski metric. Terms containing derivatives of g_{ij}, particularly the quantity Q^{ij}, which is a linear function of the Einstein tensor (in the metric g_{ij}), may be neglected. In that case, (2.1) takes the form of the Einstein equations linearized with respect to the Minkowski metric. With the notation ($\alpha,\beta,\gamma,... = 1,2,3$)

$$\delta g_{00}=\varphi, \quad \delta g_{0\alpha}=\psi_\alpha, \quad \delta g_{\alpha\beta}=-h_{\alpha\beta}$$

these equations become

$$-\frac{1}{2}(h_{,\alpha}^{,\alpha}-h_{\beta,\alpha}^{\alpha,\beta})$$
$$=\sum_b \varkappa m_b c^2 \int d^4p_b'(u_b^{0\prime})^2[N_b(q,p_b')-n_b f_b^0(p_b')], \quad (2.2a)$$

$$\frac{1}{2}(\psi_{,\gamma}^{\alpha,\gamma}-\psi_{,\gamma}^{\gamma,\alpha})+\frac{1}{2}(h_{,\beta}^{\alpha\beta}-h^{,\alpha})'$$
$$=-\sum_b \varkappa m_b c^2 \int d^4p_b' u_b^{0\prime} u_b^{\alpha\prime}(N_b-n_b f_t^0) \quad (2.2b)$$

$$\frac{1}{2}\varphi_{,\gamma}\delta^{\alpha\beta}-\frac{1}{2}\varphi^{,\alpha\beta}-\psi_{,\gamma}\delta^{\alpha\beta}+\frac{1}{2}(\psi^{\alpha,\beta}+\psi^{\beta,\alpha})'+\frac{1}{2}h^{\alpha\beta\prime\prime}$$
$$-\frac{1}{2}h''\delta^{\alpha\beta}+\frac{1}{2}(h_{,\gamma}^{,\gamma}-h_{\delta,\gamma}^{\gamma,\delta})\delta^{\alpha\beta}-\frac{1}{2}(h^{,\alpha\beta}+h_{,\gamma}^{\alpha\beta,\gamma}$$
$$-h_{,\gamma}^{\alpha\gamma,\beta}-h_{,\gamma}^{\beta\gamma,\alpha})$$
$$=\sum_b \varkappa m_b c^2 \int d^4p_b' u_b^{\alpha\prime} u_b^{\beta\prime}(N_b(q,p_b')-n_b f_b^0(p_b')). \quad (2.2c)$$

Here a prime denotes a derivative with respect to $\eta = ct$, and a superscript or subscript comma denotes a conventional derivative with respect to the spatial coordinate q^a. Raising and lowering of spatial indices is accomplished with the Kronecker delta: $h^{\alpha\beta}=\delta^{\alpha\gamma}\delta^{\beta\delta}h_{\gamma\delta}$. Repeated indices are summed over—$h = h_\alpha^\alpha$ and in addition

$$\Phi_b(q,p_b')=N_b(q,p_b')-n_b f_b^0(p_b').$$

We also make use of the identity

$$\Phi_b(\eta,\mathbf{q},p_b')$$
$$=(2\pi)^{-3}\int d^3\mathbf{q}'\int d^3\mathbf{k}\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}')]\Phi_b(\eta,\mathbf{q}',p_b'), \quad (2.3)$$

where $\mathbf{k}\cdot\mathbf{q}=\delta_{\alpha\beta}k^\alpha q^\beta$. We seek solutions for the unknowns φ, ψ_α, $h_{\alpha\beta}$ in the form

$$\varphi(\eta,\mathbf{q})=\frac{1}{(2\pi)^3}\sum_b\int d^4p_b'\int d^3\mathbf{q}'\int d^3\mathbf{k}\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}')]$$
$$\times\varphi^b(\eta,\mathbf{q}',p_b',\mathbf{k}), \quad (2.4a)$$

$$\psi_\alpha(\eta,\mathbf{q})=\frac{1}{(2\pi)^3}\sum_b\int d^4p_b'\int d^3\mathbf{q}'\int d^3\mathbf{k}\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}')]$$
$$\times\psi_\alpha^b(\eta,\mathbf{q}',p_b',\mathbf{k}), \quad (2.4b)$$

$$h_{\alpha\beta}(\eta,\mathbf{q})=\frac{1}{(2\pi)^3}\sum_b\int d^4p_b'\int d^3\mathbf{q}'\int d^3\mathbf{k}\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}')]$$
$$\times h_{\alpha\beta}^b(\eta,\mathbf{q}',p_b',\mathbf{k}). \quad (2.4c)$$

Substituting (2.3) and (2.4) into (2.2), we obtain the following equations for the Fourier transforms of the perturbations:

$$^1/_2(k^2h-k^\alpha k_\beta h_\alpha^\beta)=\varkappa m_b c^2(u_b^{0\prime})\Phi_b(\eta,\mathbf{q}',p_b'), \quad (2.5a)$$

$$^1/_2(k^2\psi^\alpha-k^\alpha k_\beta\psi^\beta)+{}^1/_2 i(k_\beta h^{\beta\alpha}-k^\alpha h)'$$
$$=\varkappa m_b c^2 u_b^{0\prime}u_b^{\alpha\prime}\Phi_b(\eta,\mathbf{q}',p_b'), \quad (2.5b)$$

$$-{}^1/_2k^2\varphi\delta^{\alpha\beta}+{}^1/_2k^\alpha k^\beta\varphi+ik_\gamma\psi^{\gamma\prime}\delta^{\alpha\beta}-{}^1/_2 i(k^\alpha\psi^\beta+k^\beta\psi^\alpha)'+{}^1/_2h^{\alpha\beta\prime\prime}$$
$$-{}^1/_2h''\delta^{\alpha\beta}-{}^1/_2(k^2h-k^\gamma k_\delta h_\gamma^\delta)\delta^{\alpha\beta}+{}^1/_2(k^\alpha k^\beta h+k^2h^{\alpha\beta}-k^\alpha k_\gamma h^{\gamma\beta}$$
$$-k^\beta k_\gamma h^{\gamma\alpha})=\varkappa m_b c^2 u_b^{\alpha\prime}u_b^{\beta\prime}\Phi_b(\eta,\mathbf{q}',p_b'). \quad (2.5c)$$

Here we have omitted the superscript b from the unknowns φ^b, ψ_α^b, and $h_{\alpha\beta}^b$. Where there is no danger of confusion, we shall continue below to omit the index b. Furthermore, we shall separate all perturbations into three types: scalar, vector, and tensor.[6] To do so, we represent $u^{\alpha\prime}$ in the form

$$u^{\alpha\prime}=(k^\alpha/k)u_\parallel'+u_\perp^{\alpha\prime}, \quad (2.6)$$

where $u_\parallel'=(k_\alpha u^{\alpha\prime})/k$, $k_\alpha u_\perp^{\alpha\prime}=0$. We introduce the notation $u_\perp'^2=\delta_{\alpha\beta}u_\perp^{\alpha\prime}u_\perp^{\beta\prime}=u'^2-u_\parallel'^2$, $u'^2=\delta_{\alpha\beta}u^{\alpha\prime}u^{\beta\prime}$ and the unit vector $S^{\alpha\prime}$ directed along $u_\perp^{\alpha\prime}$:

$$S^{\alpha\prime}=u_\perp^{\alpha\prime}/u_\perp'=(\delta_\beta^\alpha-k^\alpha k_\beta/k^2)u^{\beta\prime}/u_\perp'. \quad (2.7)$$

We also introduce the tensor

$$Q_{\alpha\beta}'=S_\alpha'S_\beta'-{}^1/_2(\delta_{\alpha\beta}-k_\alpha k_\beta/k^2), \quad (2.8)$$

with the properties

$$\delta^{\alpha\beta}Q_{\alpha\beta}'=0, \quad k^\alpha Q_{\alpha\beta}'=0, \quad Q^{\alpha\beta\prime}Q_{\alpha\beta}'={}^1/_2. \quad (2.9)$$

The tensor $u^{\alpha\prime}u^{\beta\prime}$ can be expanded in a set of linearly independent tensors:

$$u_b^{\alpha\prime}u_b^{\beta\prime}=\frac{1}{3}u_b'^2\delta^{\alpha\beta}+\frac{1}{2}(u_b'^2-3u_{\parallel b}'^2)\left(\frac{1}{3}\delta^{\alpha\beta}-\frac{k^\alpha k^\beta}{k^2}\right)$$
$$+u_{\parallel b}'u_{\perp b}'\left(\frac{k^\alpha}{k}S_b^{\beta\prime}+\frac{k^\beta}{k}S_b^{\alpha\prime}\right)+u_{\perp b}'^2Q_b^{\alpha\beta\prime}. \quad (2.10)$$

We may express both $\psi_\alpha^b(\eta,\mathbf{q}',p_b',\mathbf{k})$ and $h_{\alpha\beta}^b(\eta,\mathbf{q}',p_b',\mathbf{k})$ as sums of scalar, vector, and tensor perturbations[6]:

$$\psi_\alpha^b=(k_\alpha/k)\psi_{\parallel\beta}+S_{\alpha b}'\psi_{\perp b}, \quad (2.11a)$$

$$h_{\alpha\beta}^b=\frac{1}{3}\mu_b\delta_{\alpha\beta}+\lambda_b\left(\frac{1}{3}\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)$$
$$+\sigma_b\frac{1}{2k}(k_\alpha S_{\beta b}'+k_\beta S_{\alpha b}')+\nu_b Q_{\alpha\beta}^b. \quad (2.11b)$$

Substituting (2.6), (2.10), and (2.11) into (2.5) and equating coefficients of the linearly independent spatial tensors $1/3\delta_{\alpha\beta}$, $1/3\delta_{\alpha\beta}-k_\alpha k_\beta/k^2$, $1/2(S_b^{\alpha\prime}k^\beta+S_b^{\beta\prime}k^\alpha)/k$, $Q_b^{\alpha\beta\prime}$ and of the independent vectors k^α, $S_b^{\alpha\prime}$ on both sides of Eq. (2.5), we obtain three independent systems of equations for the scalar, vector, and tensor perturbations.

1. Scalar perturbations:

$$k^2(\mu_b+\lambda_b)=3\varkappa m_b c^2(u_b^{0\prime})^2\Phi_b, \quad (2.12a)$$
$$ik(\mu_b'+\lambda_b')=-3\varkappa m_b c^2 u_b^{0\prime}u_{\parallel b}'\Phi_b, \quad (2.12b)$$
$$-k^2\varphi_b-{}^1/_3k^2(\mu_b+\lambda_b)-\mu_b''+2ik\psi_{\parallel b}'=\varkappa m_b c^2 u_b'^2\Phi_b, \quad (2.12c)$$
$$\lambda_b''-{}^1/_3k^2(\mu_b+\lambda_b)-k^2\varphi_b+2ik\psi_{\parallel b}'=\varkappa m_b c^2(u_b'^2-3u_{\parallel b}'^2)\Phi_b. \quad (2.12d)$$

2. Vector perturbations:

$$2k^2\psi_{\perp b}+ik\sigma_b'=4\varkappa m_b c^2 u_b^{0\prime}u_{\perp b}'\Phi_b, \quad (2.13a)$$
$$\sigma_b''-2ik\psi_{\perp b}'=4\varkappa m_b c^2 u_{\parallel b}'u_{\perp b}'\Phi_b. \quad (2.13b)$$

3. Tensor perturbations:

$$\nu_b''+k^2\nu_b=2\varkappa m_b c^2 u_{\perp b}'^2\Phi_b. \quad (2.14)$$

As corollaries of the system of equations (2.12) we have

$$u_b^{0\prime}\frac{\partial\Phi_b}{\partial\eta}-iku_b^{0\prime}u_{\parallel b}'\Phi_b=0, \qquad (2.15a)$$

$$u_b^{0\prime}u_{\parallel b}'\frac{\partial\Phi_b}{\partial\eta}-iku_{\parallel b}'^2\Phi_b=0. \qquad (2.15b)$$

These corollaries are none other than the conservation law for the energy-momentum tensor, $T^{ij}_{;j}=0$. As a result of (2.15), the solutions of the system (2.12) allow for two arbitrary functions.

If we assume that

$$\lambda_b=0,\quad \psi_{\parallel b}=0, \qquad (2.16)$$

then

$$\mu_b=\frac{3\varkappa m_bc^2}{k^2}u_b^{0\prime 2}\Phi_b$$

$$=\int_{-\infty}^{\eta}d\eta'\delta(\eta'+\varepsilon-\eta)\frac{3\varkappa m_bc^2}{k^2}u_b^{0\prime 2}\Phi_b(\eta',\mathbf{q}',p_b'), \qquad (2.17)$$

$$\varphi_b=-\int_{-\infty}^{\eta}d\eta'\delta(\eta'+\varepsilon-\eta)\frac{\varkappa m_bc^2}{k^2}(u_b^{0\prime 2}+u_b'^2-3u_{\parallel b}'^2)$$

$$\times\Phi_b(\eta',\mathbf{q}',p_b'). \qquad (2.18)$$

Here $\varepsilon\to\varepsilon''$, $\eta=ct$.

Likewise, the system (2.13) for the vector perturbations has the corollary

$$u_b^{0\prime}u_{\perp b}'\partial\Phi_b/\partial\eta-iku_{\parallel b}'u_{\perp b}'\Phi_b=0,$$

so the solution of the system of equations (2.13) contains one arbitrary function. Putting

$$\sigma_b=0, \qquad (2.19)$$

we obtain

$$\psi_{\perp b}=\int_{-\infty}^{\eta}d\eta'\delta(\eta'+\varepsilon-\eta)\frac{2\varkappa m_bc^2}{k^2}u_b^{0\prime}u_{\perp b}'\Phi_b(\eta',\mathbf{q}',p_b'). \qquad (2.20)$$

The solution of (2.14) may be written in the form

$$\nu_b=-\frac{2\varkappa m_bc^2}{k}\int_{-\infty}^{\eta}d\eta'\sin[k(\eta'-\eta)]u_{\perp b}'^2\Phi_b(\eta',\mathbf{q}',p_b'). \qquad (2.21)$$

Let us calculate the quantities Ω^i_{jk} on the right-hand side of Eq. (1.11):

$$\Omega_{00}{}^0={}^1/_2\varphi',\quad \Omega_{0\alpha}{}^0={}^1/_2\varphi_{,\alpha},\quad \Omega_{\alpha\beta}{}^0={}^1/_2(h_{\alpha\beta}'+\psi_{\alpha,\beta}+\psi_{\beta,\alpha}),$$
$$\Omega_{00}{}^\alpha={}^1/_2\varphi^{,\alpha}-\psi^{\alpha\prime},\quad \Omega_{0\beta}{}^\alpha={}^1/_2h_\beta{}^{\alpha\prime}+{}^1/_2(\psi_\beta{}^{,\alpha}-\psi_{,\beta}{}^\alpha),$$
$$\Omega_{\beta\gamma}{}^\alpha=-{}^1/_2(h_{\beta\gamma}{}^{,\alpha}-h_{\beta,\gamma}{}^\alpha-h_{\gamma,\beta}{}^\alpha). \qquad (2.22)$$

Substituting φ^b, ψ^b_α, and $h^b_{\alpha\beta}$ in the form (2.4) and making use of the expansion (2.11) and the solutions (2.16)–(2.21), we find

$$\Omega_{jk}{}^i=\sum_b\int d^4p_b'\int d^3\mathbf{q}'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}')]$$

$$\times\Omega_{jk}^{ib}(\eta,\eta',p_b',k)\Phi_b(\eta',\mathbf{q}',p_b'), \qquad (2.23)$$

where the $\Omega^{ib}_{jk}(\eta,\eta',p_b',\mathbf{k})$ take the form

$$\Omega_{00}{}^{0b}=\frac{\varkappa m_bc^2}{2(2\pi)^3k^2}(u_b^{0\prime 2}+u_b'^2-3u_{\parallel b}'^2)\delta'(\eta'+\varepsilon-\eta), \qquad (2.24a)$$

$$\Omega_{0\alpha}{}^{0b}=\frac{\varkappa m_bc^2}{2(2\pi)^3k^2}ik_\alpha(u_b^{0\prime 2}+u_b'^2-3u_{\parallel b}'^2)\delta(\eta'+\varepsilon-\eta), \qquad (2.24b)$$

$$\Omega_{\alpha\beta}{}^{0b}=\frac{\varkappa m_bc^2}{2(2\pi)^3k^2}\{-u_b^{0\prime 2}\delta_{\alpha\beta}\delta'-2i(S_{\alpha b}'k_\beta+S_{\beta b}'k_\alpha)u_b^{0\prime}u_{\perp b}'\delta$$
$$+2k^2\cos[k(\eta'-\eta)]u_{\perp b}'^2Q_{\alpha\beta}{}^{b\prime}\}, \qquad (2.24c)$$

$$\Omega_{00}{}^{\alpha b}=\frac{\varkappa m_bc^2}{2(2\pi)^3k^2}[ik^\alpha(u_b^{0\prime 2}+u_b'^2-3u_{\parallel b}'^2)\delta+4u_b^{0\prime}u_{\perp b}'S_b{}^{\alpha\prime}\delta'], \qquad (2.24d)$$

$$\Omega_{0\beta}{}^{\alpha b}=\frac{\varkappa m_bc^2}{2(2\pi)^3k^2}\{-u_b^{0\prime 2}\delta_\beta{}^\alpha\delta'-2i(k^\alpha S_{\beta b}'-k_\beta S_b{}^{\alpha\prime})u_b^{0\prime}u_{\perp b}'\delta$$
$$+2k^2\cos[k(\eta'-\eta)]u_{\perp b}'^2Q_\beta^{\alpha b\prime}\}, \qquad (2.24e)$$

$$\Omega_{\beta\gamma}{}^{\alpha b}=\frac{\varkappa m_bc^2}{2(2\pi)^3k^2}\{iu_b^{0\prime 2}(\delta_{\beta\gamma}k^\alpha-\delta_\beta{}^\alpha k_\gamma-\delta_\gamma{}^\alpha k_\beta)\delta-$$
$$-2ik\sin[k(\eta'-\eta)][Q_{\beta\gamma}{}^{b\prime}k^\alpha-Q_\beta{}^{\alpha b\prime}k_\gamma-Q_\gamma^{\alpha b\prime}k_\beta]u_{\perp b}'^2\}. \qquad (2.24f)$$

Here

$$\delta=\delta(\eta'+\varepsilon-\eta),\quad \delta'=\frac{d}{d\eta'}\delta(\eta'+\varepsilon-\eta),\quad \varepsilon\to 0.$$

3. THE KINETIC EQUATION

We now substitute (2.23) into (1.11):

$$p^i\frac{\partial N_a}{\partial q^i}+\Gamma_{j,ik}p^jp^k\frac{\partial N_a}{\partial p_i}=\frac{\partial}{\partial p_i}\sum_b\int d^4p_b'\int d^3\mathbf{q}'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'$$

$$\times\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}')]\Omega_{lm}{}^{jb}(\eta,\eta',p_b',\mathbf{k})p^lp^m\Delta_{ji}N_a(x)\Phi_b(x'). \qquad (3.1)$$

Here and henceforth we denote the set of all variables $(\eta,\mathbf{q},p)$ by x, and the set $(\eta',\mathbf{q}',p')$ by x'. We denote the momenta p_b' simply by p', and p_c'' by p''. We average (3.1) over the set of systems[3]:

$$p^i\frac{\partial}{\partial q^i}\langle N_a(x)\rangle+\Gamma_{j,ik}p^jp^k\frac{\partial}{\partial p_i}\langle N_a(x)\rangle$$

$$=\frac{\partial}{\partial p_i}\sum_b\int d^4p'\int d^3\mathbf{q}'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}')]$$

$$\times\Omega_{lm}{}^{jb}(\eta,\eta',p',\mathbf{k})p^lp^m\Delta_{ji}\langle N_a(x)\Phi_b(x')\rangle. \qquad (3.2)$$

Multiplying (3.1) by $\Phi_b(x')$ and averaging, we obtain

$$p^i\frac{\partial}{\partial q^i}\langle N_a(x)\Phi_b(x')\rangle+\Gamma_{j,ik}p^jp^k\frac{\partial}{\partial p_i}\langle N_a(x)\Phi_b(x')\rangle$$

$$=\frac{\partial}{\partial p_i}\sum_c\int d^4p''\int d^3\mathbf{q}''\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta''\Omega_{lm}{}^{jc}(\eta,\eta'',p'',\mathbf{k})$$

$$\times p^lp^m\Delta_{ji}\langle N_a(x)\Phi_b(x')\Phi_c(x'')\rangle\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}'')]. \qquad (3.3)$$

Equation (3.3) is the equation for the second moment $\langle N_a(x)N_b(x')\rangle$. Another equation for this same moment can be obtained from (3.3) with the replacement $a\rightleftarrows b$, $x\rightleftarrows x'$.

We now introduce one-, two-, and three-particle distribution functions:

$$\left\langle\int ds\,\delta(x-x_a(s))\right\rangle=f_a(x),$$

$$\left\langle\int ds\delta(x-x_a(s))\int ds'\delta(x'-x_b(s'))\right\rangle=f_{ab}(x,x'),$$

$$\left\langle \int ds\delta(x-x_a(s)) \int ds'\delta(x'-x_b(s')) \times \int ds''\delta(x''-x_c(s'')) \right\rangle = f_{abc}(x,x',x'').$$

Here, as before, we use the notation

$$\delta(x-x_a(s)) = \delta^4(q^i-q_a{}^i(s))\delta^4(p_j-p_j{}^a(s)).$$

For the moments of the random functions, we have[3]

$$\langle N_a(x)\rangle = n_a f_a(x), \tag{3.4a}$$

$$\langle N_a(x)N_b(x')\rangle = (n_a n_b - n_a\delta_{ab})f_{ab}(x,x') + n_a\delta_{ab}f_a(x)\int ds'\delta(x'-x_a(s'/x)), \tag{3.4b}$$

$$\begin{aligned}\langle N_a(x)N_b(x')N_c(x'')\rangle &= (n_a n_b n_c - n_a n_b\delta_{ac} - n_a n_b\delta_{bc} - n_a n_c\delta_{ab} + 2n_a\delta_{ab}\delta_{bc})f_{abc}(x,x',x'') \\ &+ (n_a n_c - n_a\delta_{ac})\delta_{ab}f_{ac}(x,x')\int ds' \times \delta(x'-x_a(s'/x)) + (n_a n_b - n_a\delta_{ab})\delta_{ac}f_{ab}(x,x') \\ &\times \int ds''\delta(x''-x_a(s''/x)) + (n_a n_b - n_b\delta_{ab})\delta_{bc}f_{ab}(x,x')\int ds''\delta(x''-x_b(s''/x')) \\ &+ n_a\delta_{ab}\delta_{bc}f_a(x)\int ds'\delta(x'-x_a(s'/x))\int ds''\delta(x''-x_a(s''/x)).\end{aligned} \tag{3.4c}$$

In these expressions $x_a(s/x)$ denotes a particle trajectory passing through a point x of phase space.

Taking into consideration the relationship between N_a and Φ_a ($\Phi_a = N_a - n_a f_a^0$), and bearing in mind that f_a^0 is not a random function, it is straightforward to obtain expressions for the mean values

$$\langle N_a(x)\Phi_b(x')\rangle, \quad \langle N_a(x)\Phi_b(x')\Phi_c(x'')\rangle.$$

Substituting (3.4) into (3.2), (3.3) and analogous equations, we obtain an infinite succession of kinetic equations for the distribution functions f_a, f_{ab}, f_{abc}, etc.

To derive the kinetic equation for the one-particle distribution function f_a to second order in the interaction, we terminate this chain, taking

$$\begin{aligned} f_{ab}(x,x') &= f_a(x)f_b(x') + g_{ab}(x,x'), \\ f_{abc}(x,x',x'') &\approx f_a(x)f_b(x')f_c(x''). \end{aligned} \tag{3.5}$$

As a result, we have an approximate system for $f_a(x)$ and $g_{ab}(x,x')$ with $n_a \gg 1$:

$$p^i\frac{\partial f_a}{\partial q^i} + \Gamma_{j,ik}p^j p^k\frac{\partial f_a}{\partial p_i} - \frac{\partial}{\partial p_i}(\Omega_{lm}{}^j p^l p^m \Delta_{ji} f_a) = \frac{\partial}{\partial p_i}\sum_b n_b \int d^4p' \int d^3\mathbf{q}' \int d^3\mathbf{k} \int_{-\infty}^{\eta} d\eta' \exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}')] \times \Omega_{lm}{}^{jb}(\eta,\eta',p',\mathbf{k})p^l p^m \Delta_{ji} g_{ab}(x,x'), \tag{3.6}$$

$$p^i\frac{\partial}{\partial q^i}g_{ab}(x,x') = \frac{\partial}{\partial p_i}\int d^4p'' \int d^3\mathbf{q}'' \int d^3\mathbf{k} \int_{-\infty}^{\eta} d\eta'' \exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}'')] \times \Omega_{lm}{}^{jb}(\eta,\eta'',p'',\mathbf{k})p^l p^m \Delta_{ji} f_a(x)f_b(x') \int ds''\delta(x''-x_b(s''/x')). \tag{3.7}$$

In deriving (3.6) and (3.7), we have assumed that $x' \neq x_a(s/x)$—that is, there is no trajectory followed by particles of type a that passes through the phase-space point x. The expression $\overline{\Omega}{}^j_{lm}$ on the left-hand side of (3.6) describes the self-consistent gravitational field:

$$\Omega_{lm}{}^j = \sum_b n_b \int d^4p_b' \int d^3\mathbf{q}' \int d^3\mathbf{k} \int_{-\infty}^{\eta} d\eta' \Omega_{lm}^{jb}(\eta,\eta',p_b',\mathbf{k}) \times [f_b(x') - f_b{}^0(x')]\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}')].$$

In view of the weakness of the interactions, we can consider the trajectory of a type-b particle in the integral over s'' in (3.7) to be a geodesic in Minkowski space:

$$p_i{}^b(s''/x') = p_i' = \text{const},$$
$$\mathbf{q}_b(\eta''/x') = \mathbf{q}' + (\mathbf{v}'/c)(\eta''-\eta'), \quad q_b{}^0(\eta''/x') = \eta'',$$

where

$$\mathbf{v}' = c\mathbf{u}_b'/u_b{}^{0'}, \quad \mathbf{u}_b' = (u_b{}^{1'}, u_b{}^{2'}, u_b{}^{3'}).$$

Integrating over s'', $\mathbf{q}''$, p'' in (3.7), we obtain

$$p^0\frac{\partial g_{ab}}{\partial \eta} + p^\alpha\frac{\partial g_{ab}}{\partial q^\alpha} = \frac{\partial}{\partial p_i}\int d^3\mathbf{k}\int_{-\infty}^{\eta} d\eta'' \times \Omega_{lm}^{jb}(\eta,\eta'',p',\mathbf{k})p^l p^m \Delta_{ji}(u^{0'})^{-1} f_a(x)f_b(x') \times \exp\left[-i\mathbf{k}(\mathbf{q}-\mathbf{q}') + \frac{i}{c}(\mathbf{k}\mathbf{v}')(\eta''-\eta')\right]. \tag{3.8}$$

In Eq. (3.8) for $g_{ab}(x,x')$ we have put $\Gamma_{i,jk} = 0$ on the left-hand side, since at the very outset we stipulated that inside the correlation radius the metric coefficients g_{ij} be constant.

The solution of this equation takes the form

$$g_{ab}(x,x') = \int d^3\mathbf{k}' \int_{-\infty}^{\eta} \frac{d\tau}{p^0}\left[\frac{\partial}{\partial p_i}(p^l p^m \Delta_{ji} f_a(x))\right]_\tau \times \int_{-\infty}^{\tau} \frac{d\tau'}{u^{0'}} f_b(x')\Omega_{lm}{}^{jb}(\tau,\tau',p',\mathbf{k}')\exp[-i\mathbf{k}(\mathbf{q}-\mathbf{q}') + \frac{i}{c}(\mathbf{k}'\mathbf{v})(\eta-\tau) + \frac{i}{c}(\mathbf{k}'\mathbf{v}')(\tau'-\eta'). \tag{3.9}$$

The subscript τ here means that after taking the derivative with respect to p_i, the arguments η and $\mathbf{q}$ must be replaced by τ and $\mathbf{q} + \mathbf{v}(\tau - \eta)/c$.

The solution (3.9) only takes into account the effect that the trajectory of particle b has on particle a; the inverse effect is described by the solution to the equation obtained from (3.7) by the change of variables $a \rightleftarrows b$, $x \rightleftarrows x'$. The solution itself is obtained from (3.9) by making the same substitutions. The sum of these solutions then goes on the right-hand side of Eq. (3.6). The net result is the desired relativistic kinetic equation to second-order accuracy in the interaction.

To obtain the analog of the well-known Belyaev-Budker equation[4] for relativistic plasma, we consider a distribution function that varies so slowly in space and time that it can be considered constant within the region defined by the correlation length and the corresponding correlation time. In calculating the integrals over $\mathbf{q}'$, η', τ, and τ' in (3.6) and (3.9), we can neglect the space-time dependence of f_a

and f_b. After integrating over $\mathbf{q}'$ and $\mathbf{k}'$, we arrive at the following equation for f_a ($f_a = f_a(q^i,p_j)$, $f'_b = f_b(q^i,p'_j)$):

$$p^i\frac{\partial f_a}{\partial q^i}+\Gamma_{j,ik}p^jp^k\frac{\partial f_a}{\partial p_i}-\frac{\partial}{\partial p_i}[\Omega_{lm}{}^jp^lp^m\Delta_{ji}f_a]=\frac{\partial}{\partial p_i}\sum_b J_i{}^{ab}, \tag{3.10}$$

where

$$J_i{}^{ab}=(2\pi)^3n_b\int d^4p'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'\Omega_{lm}{}^{jb}(\eta,\eta',p',\mathbf{k})p^lp^m$$

$$\Delta_{ji}\left\{\int_{-\infty}^{\eta}\frac{d\tau}{p^0}\left[\frac{\partial}{\partial p_k}(p^sp^t\Delta_{rk}f_a)\right]_\tau f_b'\int_{-\infty}^{\tau}\frac{d\tau'}{u^{0\prime}}\Omega_{st}{}^{rb}(\tau,\tau',p',-\mathbf{k})\right.$$

$$\times\exp\left[-\frac{i}{c}(\mathbf{kv})(\eta-\tau)-\frac{i}{c}(\mathbf{kv}')(\tau'-\eta')\right]$$

$$+\int_{-\infty}^{\eta'}\frac{d\tau'}{p^{0\prime}}\left[\frac{\partial}{\partial p_k'}(p^{s\prime}p^{t\prime}\Delta_{rk}'f_b')\right]_\tau$$

$$\times f_a\int_{-\infty}^{\tau'}\frac{d\tau}{u^0}\Omega_{st}{}^{ra}(\tau',\tau,p,\mathbf{k})$$

$$\left.\times\exp\left[-\frac{i}{c}(\mathbf{kv})(\eta-\tau)-\frac{i}{c}(\mathbf{kv}')(\tau'-\eta')\right]\right\}. \tag{3.11}$$

We now transform to a seven-dimensional distribution function, which depends on the coordinates and the spatial components of the momentum p_α:

$$n_af_a(x)=F_a(q^i,p_\alpha)\delta((g^{ij}p_ip_j)^{1/2}-m_ac). \tag{3.12}$$

The equation for F_a is obtained from (3.10) by integrating both sides over p_0 (we must also carry out the integration over p_0' in (3.11)).

Next, we make use of the identity

$$\left[\frac{\partial}{\partial p^k}(p^sp^t\Delta_{rk}\Omega_{st}{}^rF_a)\right]_\tau=p^0\left[\frac{\partial}{\partial p_\alpha}\left(\frac{1}{p^0}p^sp^t\Delta_{r\alpha}\Omega_{st}{}^rF_a\right)\right]_\tau. \tag{3.13}$$

In calculating the derivative with respect to p_k on the left-hand side of this equation, all components of p_i are treated as being independent, and only then do we allow for the fact that $p_0=(m^2c^2+p^2)^{1/2}$. On the right-hand side of (3.13), this dependence is taken into account prior to calculating the derivative with respect to the spatial components of the momentum p_α.

Taking advantage of (3.13), then, we obtain the kinetic equation for F_a:

$$p^i\frac{\partial F_a}{\partial q^i}+\Gamma_{j,\alpha k}p^jp^k\frac{\partial F_a}{\partial p_\alpha}$$

$$-u^0\frac{\partial}{\partial p_\alpha}\left[\Omega_{lm}{}^jp^lp^m\Delta_{j\alpha}\frac{1}{u^0}F_a\right]=u^0\frac{\partial}{\partial p_\alpha}\sum_b J_\alpha{}^{ab}, \tag{3.14}$$

where

$$J_\alpha{}^{ab}=(2\pi)^3\int d^3p'\int d^3\mathbf{k}\int_{-\infty}^{\eta}d\eta'\Omega_{lm}{}^{jb}(\eta,\eta',p',\mathbf{k})p^lp^m$$

$$\times\Delta_{ji}(u_0u^{0\prime})^{-1}\left\{\int_{-\infty}^{\eta}d\tau\left[\frac{\partial}{\partial p_\beta}\left(\frac{p^sp^t}{p^0}\Delta_{r\beta}F_a\right)\right]_\tau\right.$$

$$\times F_b'\int_{-\infty}^{\tau}\frac{d\tau'}{u^{0\prime}}\Omega_{st}{}^{rb}(\tau,\tau'p',-\mathbf{k})\cdot$$

$$\times\exp\left[\frac{i}{c}(\mathbf{kv})(\tau-\eta)+\frac{i}{c}(\mathbf{kv}')(\eta'-\tau')\right]$$

$$+\int_{-\infty}^{\eta'}d\tau'\left[\frac{\partial}{\partial p_\beta'}\left(\frac{p^{s\prime}p^{t\prime}}{p_0'}\Delta_{r\beta}'F_b'\right)\right]_{\tau'}$$

$$\times F_a\int_{-\infty}^{\tau'}\frac{d\tau}{u^0}\Omega_{st}{}^{ra}(\tau',\tau,p,\mathbf{k})$$

$$\left.\times\exp\left[\frac{i}{c}(\mathbf{kv})(\tau-\eta)+\frac{i}{c}(\mathbf{kv}')(\eta'-\tau')\right]\right\}. \tag{3.15}$$

It only remains here to substitute the explicit expressions (2.24) for the $\Omega_{ik}^{jb}(\eta,\eta',p',\mathbf{k})$ into (3.15) and carry out the integration over τ, τ', and η'.

The kinetic equation takes the form

$$\frac{c}{p^0}\left(p^i\frac{\partial F_a}{\partial q^i}+\Gamma_{j,\alpha k}p^jp^k\frac{\partial F_a}{\partial p_\alpha}\right)-c\frac{\partial}{\partial p_\alpha}\left[\Omega_{lm}{}^jp^lp^m\Delta_{j\alpha}\frac{1}{p^0}F_a\right]$$

$$=\frac{\partial}{\partial p_\alpha}\sum_b\int d^3p'\mathscr{E}_{\alpha\beta}\left(\frac{\partial F_a}{\partial p_\beta}F_b'-\frac{\partial F_b'}{\partial p_\beta'}F_a\right), \tag{3.16}$$

where

$$\mathscr{E}_{\alpha\beta}=2G^2(p^0p^{0\prime})^2\left[1+\frac{v^2}{c^2}+\frac{v'^2}{c^2}-4\frac{(\mathbf{vv}')}{c^2}-\frac{v^2v'^2}{c^4}\right.$$

$$\left.+2\frac{(\mathbf{vv}')^2}{c^4}\right]^2\int d^3\mathbf{k}k_\alpha k_\beta[k^2c^2-(\mathbf{kv})^2]^{-2}\delta(\mathbf{kv}-\mathbf{kv}'). \tag{3.17}$$

Equation (3.16), with the kernel (3.17), is quite similar to the Belyaev-Budker equation[4] as given in the representation derived in Ref. 3. The only difference is that the factor $(ec)^4(u_i'u^i)^2$ (see Eq. (22) of Ref. 3) in the kernel $\mathscr{E}_{\alpha\beta}$ is replaced by

$$G^2(p^0p^{0\prime})^2(u^0u^{0\prime})^2\left[1+\frac{v^2}{c^2}+\frac{v'^2}{c^2}-4\frac{(\mathbf{vv}')}{c^2}\right.$$

$$\left.-\frac{v^2v'^2}{c^4}+2\frac{(\mathbf{vv}')^2}{c^4}\right]^2,$$

which can be rewritten in the form

$$G^2[2(u^iu_i')(p^jp_j')-(u^ip_i)(u^{j\prime}p_j')]^2. \tag{3.18}$$

The reason for the difference is not particularly hard to understand: electo-magnetic fields are produced by a current four-vector associated with particles, which is proportional to the integral of the distribution function over momenta, multiplied by the four-velocity u^i. In general relativity theory, gravitational fields are produced by the energy-momentum tensor, which is proportional to the analogous integral of the distribution function multiplied by u^iu^j. The upshot is that in a term that is second-order in the interaction (such as the collision integral), a quadratic function of the velocities $(u_iu^{i\prime})$ of the colliding particles in the collision integral is replaced by a fourth-order polynomial in the variables u^i and $u^{i\prime}$. Furthermore, it is quite natural that the square of the particle's electric charge, e^2, is replaced by $Gp^0p^{0\prime}/c^2$, inasmuch as p^0/c is the particle's relativistic mass.

Integrating over $\mathbf{k}$ in (3.17), we obtain the following expression for $\mathscr{E}_{\alpha\beta}$:

$$\mathscr{E}_{\alpha\beta}=\frac{2\pi G^2L(p^0p^{0\prime})^2}{c^4}\mathscr{K}^{-3/2}\Big[1+\frac{v^2}{c^2}+\frac{v'^2}{c^2}-4\frac{(\mathbf{vv}')}{c^2}-\frac{v^2v'^2}{c^4}+2\frac{(\mathbf{vv}')^2}{c^4}\Big]^2\Big\{\mathscr{K}^2\delta_{\alpha\beta}-\Big(1-\frac{v'^2}{c^2}\Big)v_\alpha v_\beta-\Big(1-\frac{v^2}{c^2}\Big)v_\alpha'v_\beta'+\Big(1-\frac{\mathbf{vv}'}{c^2}\Big)(v_\alpha v_\beta'+v_\alpha'v_\beta)\Big\}, \qquad (3.19a)$$

$$\mathscr{K}^2\equiv v^2+v'^2-2(\mathbf{vv}')-c^{-2}(v^2v'^2-(\mathbf{vv}')^2).$$

In addition, $\mathscr{E}_{\alpha\beta}$: can also be expressed in terms of the spatial components u_α of the four-velocity:

$$\mathscr{E}_{\alpha\beta}=\frac{2\pi G^2L}{c^5u^0u^{0\prime}}[(u_iu^{i\prime})^2-1]^{-3/2}[2(u^ku_k')(p^jp_j')-(u^kp_k)(u^{j\prime}p_j')]^2\times\{-g_{\alpha\beta}[(u^iu_i')^2-1]-u_\alpha u_\beta-u_\alpha'u_\beta'+(u^iu_i')(u_\alpha u_\beta'+u_\alpha'u_\beta)\}, \qquad (3.19b)$$

where $L=\int k^{-1}dk$ is the counterpart of the Coulomb logarithm.

It is also straightforward to derive the covariant kinetic equation for the function f_a of the eight variables q^i and p_j. This equation likewise differs from the analogous Belyaev-Budker equation in that $(ec)^4(u^iu_i')^2$ is replaced by (3.18):

$$u^i\frac{\partial f_a}{\partial q^i}+\Gamma_{j,ik}p^ju^k\frac{\partial f_a}{\partial p_i}-\frac{\partial}{\partial p_i}(\Omega_{lm}{}^ju^lp^m\Delta_{ji}f_a)=\frac{\partial}{\partial p_i}\sum_b\int d^4p'\mathscr{E}_{ij}\Big(\frac{\partial f_a}{\partial p_i}f_b'-\frac{\partial f_b'}{\partial p_i'}f_a\Big), \qquad (3.20)$$

where

$$\mathscr{E}_{ij}=2\pi c^{-6}G^2Ln_b[(u^ku_k')^2-1]^{-3/2}[2(u^ku_k')(p^lp_l')-(u^kp_k)(u^{l\prime}p_l')]^2\{-g_{ij}[(u^ku_k')^2-1]-u_iu_j-u_i'u_j'+(u^ku_k')(u_iu_j'+u_i'u_j)\}. \qquad (3.21)$$

The collision integral obtained in this way diverges logarithmically. Just as in plasma theory, it is possible to avoid this problem by introducing a cutoff in the expression for L.

In $\int k^{-1}dk$, we assume an upper limit k_∞ equal to $1/r_{\min}$, where $r_{\min}$ is the distance at which the kinetic energy of the colliding particles equals their potential energy. The lower limit k_0 is $1/R$, where $R=\langle v\rangle t$, $\langle v\rangle$ is the mean thermal velocity of the particles, and t is the age of the universe, since taking the expansion of the universe into account (see Refs. 1, 5, and 7) eliminates the divergences as $k\to0$, the contribution to the integral for L from the region $k<1/R$ being negligible.

Note that, as expected, the right-hand side of the kinetic equation (3.21) that we have obtained vanishes when we substitute for f_a the relativistic Maxwellian distribution

$$f_a(q^i,p_j)=A_a\exp(-c\bar{u}_ip^i/k_BT)\delta((g^{ij}p_ip_j)^{1/2}-m_ac).$$

Here A_a is a normalization factor, $\bar{u}_i$ is the mean four-velocity in the equilibrium state, T is the temperature, and k_B is Boltzmann's constant.

4. PHYSICAL APPLICATIONS

We now use Eq. (3.16) with the kernel (3.19) to investigate the interaction of the microwave background radiation with large-scale agglomerations of matter.[8] In this application, F_a becomes the photon distribution function, F_γ, and F_b is the distribution function appropriate to the large-scale agglomerations. Obviously, $v\sim c$ and $v'\ll c$. Equation (3.19) therefore takes the form

$$\mathscr{E}_{\alpha\beta}=8\pi G^2Lm_b{}^2(p^2\delta_{\alpha\beta}-p_\alpha p_\beta)/c^3. \qquad (4.1)$$

Substituting (4.1) into (3.16), ignoring the coordinate-dependence of F_γ, and transforming to spherical coordinates in momentum space, we obtain

$$\frac{\partial F_\gamma}{\partial t}=\frac{8\pi G^2m_b\rho_bL}{c^3}\Delta_{\theta,\varphi}F_\gamma. \qquad (4.2)$$

Here $\rho_b=m_b\int d^3p'F_b'$ is the matter density, and $\Delta_{\theta,\varphi}$ is the angular part of the Laplacian:

$$\Delta_{\theta,\varphi}=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial}{\partial\theta}\Big)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}.$$

We seek a solution of (4.2) expressed as an expansion in spherical harmonics:

$$F_\gamma=\sum f_{l,m}(t,p)Y_{l,m}(\theta,\varphi). \qquad (4.3)$$

Substituting (4.3) into (4.2), it is not hard to show that all harmonics with $l\neq0$ die out with time—that is, a homogeneous, anisotropic distribution of ultrarelativistic particles becomes isotropic by virtue of their interactions with the large-scale agglomerations:

$$f_{l,m}(t,p)=f_{l,m}^0(p)\exp\Big(-\int_{t_0}^{t}\frac{dt'}{\tau(t')}\Big), \qquad (4.4)$$

where $\tau(t)=c^3(8\pi G^2m_b\rho_bl(l+1))^{-1}$.

This effect is most significant as it relates to very small-scale fluctuations. Ignat'ev and Popov,[8] who first examined the interaction of ultrarelativistic particles with nonrelativistic massive particles and derived Eq. (4.2), reported that for $m_b\sim10^{16}M_\odot$, $\rho\sim10^{-30}$ g/cm^3 and $t\sim2\cdot10^{10}$ yr. They also showed that the microwave background should be highly uniform on scales $\Delta\theta<10$ arcmin.

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