Sensitivity of laser generation with sub-Poisson photon statistics to external electromagnetic radiation

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The sensitivity threshold of a laser detector of electromagnetic radiation is analyzed. The system becomes more sensitive with increase of generation power, although the modulation of the external field is transferred more efficiently for low-power generation. The transition to quantum generation fields with sub-Poisson photon statistics also led to a lowering of the threshold. The effect is assessed quantitatively.

Lasers can be used as detectors of electromagnetic radiation. In one possible variant, generation takes place on the $a \to b$ transition of the operating medium, while the external field is in resonance with an adjacent transition $a \to c$ of the same medium (see the figure). Conditions can be created in which the information contained in the external field will be passed on to the generation. We shall consider below the question of the sensitivity threshold of a similar system (a laser detector) and, in particular, the case in which the laser generates light with sub-Poisson photon statistics.

The problem of the sensitivity threshold of light detectors is extremely important. In our case, it is important to know how it depends on the parameters of the laser and, in particular, on the generation power. At first glance, it seems obvious that it is easier to act on the laser generation the weaker the generate. And actually, as we shall see later, the modulation of the external field is transferred more effectively to the generation in this case. However, we cannot make a judgment on the possibility of its observation from this fact only, since the fact that the self-noise of the laser is great at weak generation can introduce essential corrections. In final analysis, it turns out that it is most suitable to use the laser detector in the case of high power generation.

It is clear that the transition to generation of quantum light promises lowering of the sensitivity threshold, since in this case the shot noise and the excess noise cancel one another, lowering the level of self noise of the laser to "forbidden," conditionally zero values. There are no obscurities here in principle, but the quantitative side of the question is not at all understood and needs clarification, since the transition to generation of quantum light is very difficult in practice for various reasons and it is necessary to investigate beforehand how it can be done effectively.

BASIC EQUATIONS OF GENERATION

Within the framework of quantum theory of radiation, the theory of the laser is most simply constructed in the language of the matrix density for the generation field. This was first done systematically and in the general framework of the theory of kinetic equations by Lamb and Scully.[1] After some generalizations, the generation equation can be rewritten in the form

$$\dot{\rho} = [-aa^+(\hat{\mathcal{R}}_a\rho) + a^+(\hat{\mathcal{R}}_a\rho)a - a^+a(\hat{\mathcal{R}}_b\rho) + a(\hat{\mathcal{R}}_b\rho)a^+ + \text{h.c.}] + (\dot{\rho})_Q \tag{1}$$

Here ρ is the density matrix for the generation field in the form of a single traveling wave, a^+ and a are the creation and annihilation operators for photons in the generation field, $[a,a^+] = 1$, $(\dot{\rho})_Q$ describes the damping of the field in the cavity because of its finite Q:

$$(\dot{\rho})_Q = -C/2(a^+a\rho - 2a\rho a^+ + \rho a^+ a),$$

C is the width of the resonance curve at the generation frequency, and the operators $\hat{\mathcal{R}}_a$ and $\hat{\mathcal{R}}_b$ determine the development of the field because of its interaction with the working medium. $\hat{\mathcal{R}}_a$ has the following representation:

$$\hat{\mathcal{R}}_a = \frac{r_a\beta_a}{2} \times \frac{1 + iv + {}^1/_2\beta_b[(aa^+)_{\to} - (aa^+)_{\leftarrow}]}{1 + v^2 + {}^1/_2\beta_+[(aa^+)_{\to} + (aa^+)_{\leftarrow}] + {}^1/_2 iv\beta_-[(aa^+)_{\to} - (aa^+)_{\leftarrow}]}$$

and $\hat{\mathcal{R}}_b$ is obtained from $\hat{\mathcal{R}}_a^*$ by interchange of the indices $a \leftrightarrow b$ and the operator combinations $aa^+ \leftrightarrow a^+a$, r_a and r_b are the mean rates of incoherent excitation of the operating levels, $v = (\omega - \omega_{ab})/\gamma_{ab}$ is the relative detuning of the generation frquency ω from the frequency of the working transition ω_{ab}, and γ_{ab} is the homogeneous width of the luminescence line of the working transition; the arrows of the operator combinations indicate on which side of the density matrix they should stand. The nonlinear parameters β_a, β_b, and $\beta_\pm$ have the form

$$\beta_a = \frac{2|g_{ab}|^2}{\gamma_a\gamma_{ab}} \frac{1}{1+v^2}, \quad \beta_a \to \beta_b \text{ for } \gamma_a \to \gamma_b, \quad \beta_\pm = \beta_a \pm \beta_b,$$

where γ_a and γ_b are the widths of the working levels,

$$g_{ab} = i(\omega/2LS)^{1/2} d_{ab} e^{ikz}$$

is the interaction constant of an atom with a plane laser wave in the dipole approximation, L is the length of the cavity, and S is its transverse cross section.

In the derivation of Eq. (1) it was assumed that the motion of the atoms can be neglected, and that $C \ll \gamma_b$ (or $C \ll \gamma_a$, or $C \ll \gamma_a, \gamma_b$). The problem is solved in the approximation of plane waves traveling along the axis, without account of diffraction phenomena.

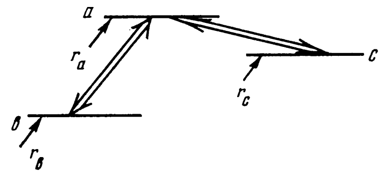

FIG. 1.

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The theory of Lamb and Scully can be generalized to our case, in which it is necessary to take into account the effect of the external field. Here we must replace in the indicated formulas the stationary population $N_a = r_a/\gamma_a$ of the upper working level in the absence of all fields by the effective population $N_a - N_1 x$, which arises under the action of the incoherent excitation and the external field, while the saturation parameters $\beta_\pm$ must be replaced by the effective $\beta_\pm - \beta_a x$. The parameter x is determined by the characteristics of the neighboring transition $a \to c$ and the power of the external field:

$$x = \frac{\gamma_c}{\gamma_a + \gamma_c} \frac{I_1}{1 + \nu_1^2 + I_1},$$

where γ_c is the width of the level c, $\nu_1 = (\omega_1 - \omega_{ac})/\gamma_{ac}$ is the relative detuning of the frequency of the external field from the frequency of the $a \to c$ transition, γ_{ac} is the homogeneous width of the luminescence line in the $a \to c$ transition, $N_1 = r_a/\gamma_a - r_c/\gamma_c$ is the stationary difference of populations of levels a and c in the absence of all fields. The dimensionless power of the external field I_1 is connected with the intensity S_1 by the formula

$$I_1 = \frac{3}{4} \frac{\gamma_{a\to c}}{\gamma_a} \frac{\gamma_a + \gamma_c}{\gamma_c \gamma_{ac}} \frac{|T_1|^2}{|1 - R_1|^2} S_1.$$

Here $\gamma_{a\to c}$ is the part of the quantity γ_a that is connected with the transition of the atom from level a to level c and, as is known, is expressed in terms of the matrix element d_{ac} of the dipole moment: $\gamma_{a\to c} = (4/3)\omega_{ac}^3 d_{ac}^2$ (here, everywhere except in the last formula, a system of units is employed in which $\hbar = c = 1$). The complex quantities R_1 and T_1 are coefficients of reflection and transmission of the laser mirror at the frequency of the external field.

ANTINORMAL DIAGONAL REPRESENTATION OF THE DENSITY MATRIX

We pass from Eq. (1) to the equation for the quantity

$$P_A(\alpha, t) = \langle \alpha | \rho | \alpha \rangle \quad (a|\alpha\rangle = \alpha|\alpha\rangle).$$

$P_A(\alpha, t)$ is the density matrix in the antinormal (index A) diagonal representation,[2] and is a smooth function of the variable α, thanks to which we can discard all derivatives with respect to α in the equation except the first and second (the diffusion approximation). The transition from the equation for ρ to the equation for $P_A(\alpha)$ is accomplished with the help of the substitution.

$$a\rho \to (\alpha + \partial/\partial\alpha^*) P_A, \quad a^+\rho \to \alpha^* P_A, \quad \rho a \to \alpha P_A, \quad \rho a^+ \to (\alpha^* + \partial/\partial\alpha) P_A.$$

As a result, we obtain

$$\dot{P}_A = (r_a \hat{L}_a + r_b \hat{L}_b + \hat{R} - {}^1/_2 r_a \hat{L}_a^2) P_A. \tag{2}$$

Here the evolution operators $\hat{L}_a$, $\hat{L}_b$, and $\hat{R}$ are expressed in terms of the differential operators

$$\hat{\lambda}_\pm = \frac{\partial}{\partial\alpha}\alpha + \text{c.c.}, \qquad \hat{\lambda}^2 = 2\frac{\partial^2}{\partial\alpha\partial\alpha^*}$$

in the following way

$$\hat{L}_a = (1 - N_1 x/N_a)\{{}^1/_2\beta_a(\hat{\lambda}_+ - i\nu\hat{\lambda}_-) \times [-1 + {}^1/_2(\beta_+ - \beta_a x)\hat{\lambda}_+\Lambda^{-1} - {}^1/_2 i\nu(\beta_- - \beta_a x)\hat{\lambda}_-\Lambda^{-1}] \times \Lambda^{-1} - {}^1/_4\beta_a\beta_b\hat{\lambda}_-^2\Lambda^{-1}\},$$

$$\hat{L}_b = {}^1/_2\beta_b(\hat{\lambda}_+ - i\nu\hat{\lambda}_-)[1 - {}^1/_2(\beta_+ - \beta_a x)\hat{\lambda}_+\Lambda^{-1} - {}^1/_2 i\nu(\beta_- - \beta_a x)\hat{\lambda}_-\Lambda^{-1}]\Lambda^{-1} - {}^1/_4\beta_a\beta_b(1-x)\cdot \times \hat{\lambda}_-^2\Lambda^{-1} + {}^1/_2\beta_b\hat{\lambda}^2\Lambda^{-1},$$
$$\hat{R} = {}^1/_2 C\hat{\lambda}^2, \quad \Lambda = 1 + \nu^2 + (\beta_+ - \beta_a x)|\alpha|^2. \tag{3}$$

For convenience, we transform from α and α^* to the variables u and φ: $\alpha = u^{1/2}e^{i\varphi}$. Then the $\hat{\lambda}$ operators take the form

$$\hat{\lambda}_+ = 2\frac{\partial}{\partial u}u, \quad \hat{\lambda}_- = -i\frac{\partial}{\partial\varphi},$$
$$\frac{1}{2}\hat{\lambda}^2 = \frac{\partial}{\partial u}u\frac{\partial}{\partial u} + \frac{1}{4u^2}\frac{\partial^2}{\partial\varphi^2}.$$

The last term on the right side of Eq. (2) does not follow from (1), since it arises only in those cases in which the excitation of the upper working level takes place without noise, while Eq. (1) is written under the assumption of Poisson statistics of the excitation.[3] It is precisely this term which makes it possible to formulate the generation field in a quantum state with sub-Poisson statistics of the photons.

For further mathematical simplification, one usually makes use of the fact that in the regime of stationary generation the number of photons in the cavity at the generation frequency can only fluctuate weakly about its mean value $\bar{n}$:

$$u = \bar{n} + \varepsilon, \quad \varepsilon \ll \bar{n}. \tag{4}$$

Moreover, we shall hereafter set $\nu = \nu_1 = 0$ and $N_b = N_c = 0$ everywhere. Then, in place of (2), (3), we obtain an equation of the Fokker–Planck type

$$\frac{\partial P_A}{\partial t} = \Gamma\frac{\partial}{\partial\varepsilon}(\varepsilon P_A) + \Gamma\bar{n}(\xi + 2)\frac{\partial^2 P_A}{\partial\varepsilon^2} + \frac{1}{2}D\frac{\partial^2 P_A}{\partial\varphi^2}, \tag{5}$$

where

$$\Gamma = CI(1+I)^{-1},$$
$$\xi = I^{-1} - \frac{1}{2}\frac{\gamma_b(1-x)}{\gamma_a + \gamma_b(1-x)},$$
$$D = \frac{C}{2\bar{n}}\left[1 + I\frac{\gamma_a}{\gamma_a + \gamma_b(1-x)}\right],$$
$$I = (\beta_+ - \beta_a x)\bar{n}.$$

Only the parameter ξ, which determines the fluctuations $\overline{\Delta n^2} = \overline{n^2} - \bar{n}^2 = \bar{n}(1+\xi)$ of the number of photons in the cavity depends on the character of the excitation (Poisson or regular) and is equal to I^{-1} for the Poisson excitation.

EFFECT OF EXTERNAL FIELD ON THE MEAN GENERATION POWER

If the second derivatives with respect to α are not taken into account in Eqs. (2) and (3), then we arrive at the semiclassical theory, for which the abridged self-consistent equation of the type

$$\dot{n} + \left[C - \frac{A(1-x)}{1 + (\beta_+ - \beta_a x)n}\right]n = 0 \tag{6}$$

is characteristic. Here $A = r_a\beta_a - r_b\beta_b$ is the linear (unsaturated) amplification coefficient of the medium at the frequency of generation without an external field.

If the external-field power of I_1 (meaning also the parameter x) does not depend on the time or if the characteristic time of its change is much longer than the time C^{-1}, then the solution of (6) can be represented in the form

$$n=n_0(1-\Delta),\quad I=I_0(1-\Delta)\,[1-\gamma_b x/(\gamma_a+\gamma_b)],\tag{7}$$

where n_0 is the number of photons accumulated in the cavity in the stationary generation regime generation in the absence of an external field ($\beta_+ n_0 = A/C - 1 = I_0$). The quantity Δ determines the change associated with the external field:

$$\Delta=\frac{1+\mathcal{I}_0}{\mathcal{I}_0}\frac{(\lambda-1)\mathcal{I}_1}{1+\lambda\mathcal{I}_1},\quad \lambda=\frac{\gamma_a+\gamma_b+\gamma_c}{\gamma_a+\gamma_b}.$$

Here

$$\mathcal{I}_0=I_0\frac{\gamma_a}{\gamma_a+\gamma_b},\quad \mathcal{I}_1=I_1\frac{\gamma_a}{\gamma_a+\gamma_c}.$$

The case $\lambda T_1 \ll 1$ is most interesting from the practical viewpoint, since it corresponds to the operation of the detector on the linear portion of its characteristic, when the information contained in the external field is transferred in similar fashion to the generation. Actually, if we assume that the intensity of the external field I_1 is modulated according to the law $I_1 = [1+\mu_1(t)]I_1'$, then we find from (7), in the case $\lambda T_1 \ll 1$, that the generation is also modulated according to the law $\bar{n} = (1-\mu(t))\bar{n}'$, where

$$\mu(t)=\frac{\Delta}{1-\Delta}\mu_1(t),\quad \Delta=\frac{1+\mathcal{I}_0}{I_0}\frac{\gamma_c}{\gamma_a+\gamma_c}I_1'$$

(the prime everywhere denotes the power of the carrier, which does not depend on the time). In the general case, the requirement $\lambda\tilde{I} \ll 1$, does not necessarily lead to the requirement $I_1 \ll 1$, i.e., the external field can saturate the transition.

PHOTORECORDING OF LASER RADIATION

Using Eq. (5), and carrying out standard calculations (see, for example, Ref. 3), we can obtain an explicit expression for the spectrum of the photocurrent

$$i_\omega^{(2)}=\lim_{T\to\infty}\frac{1}{T}\iint_{-T/2}^{T/2} dt_1\,dt_2\,\overline{i(t_1)i(t_2)}\exp[i\omega(t_1-t_2)],$$

which, for $\omega > 0$, will have the form

$$i_\omega^{(2)}=i_{\rm shot}^{(2)}\left[1+2q\xi\frac{C\Gamma}{\Gamma^2+\omega^2}+qn_0(1-\Delta)C\lim_{T\to\infty}\frac{|\mu_\omega|^2}{T}\right],\tag{8}$$

where q is the quantum yield of the photocathode. The first term here is the level of the shot noise of photorecording, the second is the level of the excess noise of the laser, which at $\xi < 0$ for quantum field generation will compensate for the shot noise at $\omega < C$. The third term is the information and is determined by the Fourier components

$$\mu_\omega=\int_{-T/2}^{T/2} dt\,\mu(t)\,e^{i\omega t}.$$

We note that in the solution of the abridged equation (6), the condition of adiabaticity was used, as a consequence of which the spectral width of the modulation μ should be less than the spectral width C. Thus, for the quantum field, the noise turns out to be compensated for in all ranges of frequency of interest to us.

SENSITIVITY THRESHOLD OF THE LASER DETECTOR

Integrating (8) over a band of frequencies $\Delta\omega$ that is less than the width C but greater than the characteristic frequency interval μ_ω, we obtain

$$\int_0^{\Delta\omega} i_\omega^{(2)}d\omega=i_Q^{(2)}\left[\Delta\omega\left(1+2q\xi\frac{C\Gamma}{\Gamma^2+\omega^2}\right)+qn_0(1-\Delta)C\overline{\mu^2}\right].\tag{9}$$

Here $\overline{\mu^2} = \lim_{T\to\infty}(1/T)\int_{-T/2}^{T/2}dt\,\mu^2(t)$ is the conditional mean modulation level of the generation field (or of the external field for μ_1).

We now determine the sensitivity threshold of the detector from the equality of the two terms in (9), i.e., we require that the level of information-containing modulation be equal to the level of stochastic modulation because of self noise of the laser:

$$qn_0(1-\Delta)C\overline{\mu^2}=\Delta\omega(1+2q\xi C/\Gamma).\tag{10}$$

This is a cumbersome power-law relative to the power of the external field and is not solvable analytically in general form. However, it is not necessary in fact to solve it in general form, since we are dealing with the threshold of sensitivity and therefore can require $I_1 \ll 1$, which leads to the requirement $x \ll 1$. Moreover, we also require that the power generation under the influence of the external field also change insignificantly, i.e., $\Delta \ll 1$, and hence we obtain a lower bound for the power generation, $I_0 \gg x$. Taking all the smallnesses into account, we obtain a quadratic equation which is already easily solved and which allows us to write down the explicit equation for the sensitivity threshold $\eta = SS_1(\mu^2/\overline{\mu^2}/\Delta\omega)^{1/2}$, which is customarily expressed in terms of the power of the light referred to the square root of the bandwidth $\Delta\omega$:

$$\eta=\hbar\omega_{ac}\gamma_{ac}\Delta_0\frac{I_0}{1+\mathcal{I}_0}\frac{\gamma_a}{\gamma_{a\to c}}\frac{S}{\lambda_{ac}^2}\frac{|1-R_1|^2}{|T_1|^2}\tag{11}$$

where Δ_0 is the solution of the indicated quadratic equation but written down not for I_1, but for $\Delta \propto I_1$:

$$\Delta_0=\left[1+2q\frac{1+I_0}{I_0}\left(\frac{1}{I_0}-\frac{1}{2}\frac{\gamma_b}{\gamma_a+\gamma_b}\right)\right](qCn_0)^{-1}.$$

The negative term here occurs only for the case of regular excitation of the operating level.[3]

We now estimate the threshold η for a helium-neon laser, assuming that the generation takes place at the wavelength $\lambda_{ab} = 0.63\ \mu$m, and the external field is at resonance with the transition $\lambda_{ac} = 3.39\ \mu$m. We obtain the following values for the linear gain A: $A_{0.63} = 1\times10^6\ \text{s}^{-1}$, $A_{3.39} = 6\times10^8\ \text{s}^{-1}$, for the saturation parameter $\beta_+ = 10^{-10}$, and for a stationary number of photons $n_0 = 10^9$, which corresponds to a dimensionless generation power $I_0 = 10^{-1}$. Knowledge of the following constants is also required: $\gamma_{ab}/kU = 10^{-1}\ \text{s}^{-1}$ (kU is the inhomogeneous width of the luminescence line at the working transition), $\gamma_a/2\pi = 1.8\times10^7\ \text{s}^{-1}$, $\gamma_b/2\pi = 4\times10^7\ \text{s}^{-1}$, and $\gamma_c/2\pi = 1\times10^6\ \text{s}^{-1}$. These data enable us to obtain the estimate $\eta = 10^{-9}\ \text{W/Hz}^{1/2}$. This quantity depends very weakly on the generation power: the threshold falls off with increase in the power: at an increase in power by three orders of magnitude (which is hardly realistic for a single-mode helium-neon laser), η decreases by a single order.

It follows from Eq. (11) that the higher the power the lower the threshold. The decrease of the threshold with increase in I_0 is determined by the fact that the self noise of the laser decreases here more rapidly than the characteristic modulation of generation.

There is still another important circumstance that follows from (9): the ratio of the signal to the noise (of the second term to the first) in the two limiting cases ($I_0 \ll 1$ and $I_0 \gg 1$) is proportional to the generation power, i.e., even in this sense, it is better to have larger generation power.

If the detector operates on the basis of a laser generating a quantum field with sub-Poisson photon statistics, then, in comparison with the ordinary case, an additional factor $(\gamma_a/\gamma_b)^{1/2}$ arises in η. This factor, as expected, lowers the threshold (for a laser radiating quantum light, the inequalities $I_0^{-1} \ll \gamma_a/\gamma_b \ll 1$ (Ref. 3) should be satisfied. However, this decrease is not too effective. In order to obtain a gain by an order of magnitude, we must have $\gamma_a/\gamma_b \sim 10^{-2}$.

[1]W. E. Lamb and M. O. Scully, Phys. Rev. **159**, 208 (1967).
[2]R. Glauber, Optical Coherence and Photon Statistics, in: *Quantum Optics and Electronics*, Gordon and Breach, New York, 1965.
[3]Yu. M. Golubev and P. V. Sokolov, Zh. Eksp. Teor. Fiz. **87**, 408 (1984) [Sov. Phys. JETP **60**, 234 (1984)].

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