Nonlinear theory of relativistic emitters based on straight free-electron beams

M. V. Kuzelev, V. A. Panin, A. P. Plotnikov, and A. A. Rukhadze

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The nonlinear dynamics of stimulated scattering of electromagnetic waves by dense relativistic electron beams is investigated. The mechanisms of nonlinear stabilization of the scattering processes are classified according to the electron beam's density and relativistic factor. It is shown that scattering by low-density relativistic beams evolves under novel conditions imposed by the energy phasing of the electrons. Analytic expressions are obtained in several cases for the amplitudes of the incident and scattered waves, and for the emission efficiency.

1. It is known that the processes of stimulated scattering of electromagnetic waves on relativistic free electron beams provide wide possibilities for the creation of powerful emission sources.[1,2] The radiation of these processes is undoubtedly important for understanding many phenomena occurring under normal conditions or in experiments in space. The fundamentals of the nonlinear theory of emitters on free electrons were expounded in Refs. 1–7. However, present theoretical results do not fully cover all possible regimes of stimulated scattering and mechanisms of their nonlinear stabilization.

This paper presents an analysis of possible mechanisms of beam instabilities in the field of two electromagnetic waves, and a development of a nonlinear theory of induced scattering processes of electromagnetic waves on dense relativistic electron beams. It is shown that in this case there is a new scattering mechanism due to energy grouping (phasing) of electrons. Moreover, we generalize the method[8] of expansion in perturbation of the electron trajectories to include the case of wave scattering on dense relativistic beams. Analytic expressions are obtained for the saturation amplitudes of interacting waves, the characteristic development times of processes, and the emission efficiencies.

2. In the most general formulation of the problem of the temporal evolution of stimulated scattering processes we assume only the presence of a strong longitudinal (in relation to the beam) magnetic field impeding the transverse motion of electrons. In this case the equations of motion of the beam electrons in the fields of the incident and scattered waves can be written in the form

$$\frac{dz}{dt}=v_z,\qquad \frac{dv_z}{dt}=\frac{e}{m}\left(1-\frac{v_z^2}{c^2}\right)^{3/2}(E_{z1}+E_{z2}).\tag{1}$$

Here z is the coordinate of the electron in the direction of beam propagation, v_z is its velocity, and E_{z1} and E_{z2} are the electric fields of the incident and scattered waves, respectively. Making the standard assumption[1] that the electron motion in fields

$$E_{z1}=A_1(t)\exp(-i\omega_1 t+ik_1 z)\text{ and }E_{z2}=A_2(t)\exp(-i\omega_2 t+ik_2 z)$$

is fast and in the field of the combination wave

$$E_{z1}E_{z2}^*=A_1A_2^*\exp(-i\omega_0 t+ik_0 z)\tag{2}$$

is slow, where $\omega_0=\omega_1-\omega_2$ and $k_0=k_1-k_2$, we represent the electron coordinate and velocity in the form

$$z=ut+z'+\tilde z,\qquad v_z=u+v'+\tilde v.\tag{3}$$

Here $\tilde z$ and $\tilde v$ are the fast oscillating electron coordinates and velocities in the fields E_{z1} and E_{z2}, z' and v' describe the slow motion in the field of the combination wave, and u is the velocity of the unperturbed beam. We note that representations (3) is correct if the following inequality is satisfied:

$$\max[t_0^{-2},\ \Omega_b^2]\ll(\omega_{1,2}-k_{1,2}u)^2,\ k_0^2u^2,\tag{4}$$

where t_0 is the characteristic time of wave amplitude change and Ω_b is the plasma oscillation frequency of beam electrons in the co-moving system of coordinates.[7]

When, in addition to Eq. (4), the following inequality is satisfied

$$|k_{1,2}\tilde z|\ll 1,\tag{5}$$

the fast oscillations are linear and the amplitudes $A_{1,2}$ are determined by the expression

$$\tilde z=\frac{e}{2m}\left(1-2\gamma_0^2\frac{u}{c^2}v'\right)^{3/2}\Omega_b^{-2}\times\sum_{j=1}^{2}\{A_j\exp[-i(\omega_j-k_ju)t+ik_jz']+\text{c.c.}\},\tag{6}$$

where

$$\Omega_0^2=(\omega_1-k_1u)^2\approx(\omega_2-k_2u)^2,\qquad \gamma_0=\left(1-\frac{u^2}{c^2}\right)^{-1/2}.$$

Substituting Eqs. (3) and (6) in the system (1) and carrying out the corresponding averaging over fast oscillations, we obtain equations of motion of beam electrons in the field of the combination wave for the slow components z' and v':

$$\frac{dz'}{dt}=v',$$

$$\frac{dv'}{dt}=-\frac{i}{4}\frac{e^2}{m^2}\left(1-2\gamma_0^2\frac{u}{c^2}v'\right)^3\gamma_0^{-6}\times\frac{k_0}{\Omega_0^2}[A_1A_2^*\exp(ik_0z'-i\tilde Dt)+\text{c.c.}],\tag{7}$$

where $\tilde D=\omega_0-k_0u$.

The equations of motion (7) describe the dynamics of beam electrons only in the regime of single-particle or Thomson scattering.[7,9] This situation occurs in beams of small density. With the growth of beam electron density the instability shifts to the regime of collective or Raman scattering[7,9] and becomes a three-wave process because beam

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Langmuir waves are excited in addition to the elctromagnetic waves. Taking into account the beam plasma or Langmuir oscillations the second equation in system (7) can be rewritten in the following form[5,7]:

$$\frac{dv'}{dt}=-\frac{1}{2}i\left(1-2\gamma_0^2\frac{u}{c^2}v'\right)^{3/2}\left[\frac{\Omega_b^2}{k_0}\rho\exp(ik_0z')-\text{c.c.}\right]$$
$$+\frac{1}{4}\left(\frac{e}{m}\right)^2\left(1-2\gamma_0^2\frac{u}{c^2}v'\right)^3\gamma_0^{-6}$$
$$\times\frac{k_0}{\Omega_0^2}[A_1A_2^*\exp(ik_0z'-i\tilde{D}t)+\text{c.c.}],\tag{8}$$

where

$$\rho=\frac{k_0}{\pi}\int_0^{2\pi/k_0}\exp(-ik_0z')\,dz_0\tag{9}$$

is the amplitude of the beam Langmuir wave and $z_0\in[0,2\pi/k_0]$ are the initial positions of electrons in the beam. We further assume that the electromagnetic wave with index 1 is a scattered or signal wave and that with index 2 is an incident or pump wave. Because the amplitudes of the electromagnetic waves change in the process of scattering, one must add to the equations of motion equations for amplitudes A_1 and A_2 (Ref. 7):

$$\frac{dA_1}{dt}=-\frac{1}{2}i\frac{\omega_b^2}{\Omega_0^2}\gamma_0^{-3}\theta_1A_2\hat{\rho}\exp(i\tilde{D}t),$$
$$\frac{dA_2}{dt}=-\frac{1}{2}i\frac{\omega_b^2}{\Omega_0^2}\gamma_0^{-3}\theta_2A_1\hat{\rho}^*\exp(-i\tilde{D}t),\tag{10}$$
$$\hat{\rho}=\frac{k_0}{\pi}\int_0^{2\pi/k_0}\left(1-2\gamma_0^2\frac{u}{c^2}v'\right)^{3/2}\exp(-ik_0z_0)\,dz_0.$$

Here $\omega_b^2=4\pi e^2n_b/m$, n_b is the beam electron density, $\theta_{1,2}$ are coefficients determined by the actual geometry of the system (see Refs. 5, 7) and the factor $(1-2\gamma_0^2uv'/c^2)^{3/2}$ in the expression for $\hat{\rho}$ stems from the resolution, in the right-hand sides of the exact equations for A_1 and A_2, of the coordinate z into slow and fast components, and carrying out the average over $\tilde{z}$ (Ref. 1).

The system of equations in dimensionless variables, describing the nonlinear dynamics of stimulated scattering of two electromagnetic waves on free beam electrons, can be written in the form[5,7]

$$\frac{d\varepsilon_1}{d\tau}=-\nu\varepsilon_2\hat{\rho}\exp(i\eta_0\tau),$$
$$\frac{d\varepsilon_2}{d\tau}=\sigma\nu\varepsilon_1\hat{\rho}^*\exp(-i\eta_0\tau),\qquad\frac{dy}{d\tau}=\eta,\tag{11}$$
$$\frac{d\eta}{d\tau}=-\frac{1}{2}i(1-\mu\eta)^{3/2}\sum_{n=1}^{\infty}\frac{\alpha_n}{n}[\rho_n\exp(iny)-\text{c.c.}]$$
$$+\frac{1}{2}(1-\mu\eta)^3\nu[\varepsilon_1\varepsilon_2^*\exp(iy-i\eta_0\tau)+\text{c.c.}],$$
$$\rho_n=\frac{1}{\pi}\int_0^{2\pi}\exp(-iny)\,dy_0,\qquad\hat{\rho}=\frac{1}{\pi}\int_0^{2\pi}(1-\mu\eta)^{3/2}\exp(-iy)\,dy_0,$$

where $\tau=\Omega_0t$, $y=k_0z$, $\eta=k_0v'/\Omega_b$, ε_1 and ε_2 are dimensionless amplitudes of the electromagnetic waves, ν is a quantity inversely proportional to the beam electron density,[1)] and the frequency difference $\nu_0\equiv(\omega_0-k_0u)/\Omega_b$ is the resonance condition for three-wave interaction. The quantity η_0 takes the values ±1, where $\eta_0=+1$ signifies synchronism of the combination wave with the fast beam Langmuir wave and $\eta_0=-1$ signifies synchronism with the slow beam Langmuir wave. The parameter σ determines the type of three-wave process. When the slow beam wave ($\eta_0=-1$) is at resonance and the parameter $\sigma=1$, the system of equations (11) describes scattering with an increase of frequency. When the slow beam wave is excited and $\sigma=-1$, an explosive instability of the three interacting waves occurs.

The value

$$\mu=2\gamma_0^2\frac{\Omega_b}{k_0c}\frac{u}{c}\tag{12}$$

is the characteristic beam relativistic factor and is an important parameter both for the further development here and for all theories of radiating electron beams.[10] For $\mu=0$, Eqs. 11 coincide with those obtained in Ref. 7 for nonrelativistic beams. We note that in the fourth equation of system (11) the generation of high harmonics of the beam Langmuir wave is taken into account and the coefficients α_n which are determined by the geometry of the actual system are given in some specific cases in Ref. 8.

Finally, the efficiency of scattering on free electrons is expressed in terms of the amplitudes ε_1 and ε_2, and is determined by one of the following equations:

$$W=1-\frac{1}{2\pi}\int_0^{2\pi}(1-\mu\eta)^{-1/2}\,dy_0$$
$$=\frac{\mu}{8}(|\varepsilon_1|^2-|\varepsilon_{10}|^2)=-\sigma\frac{\mu}{8}(|\varepsilon_2|^2-|\varepsilon_{20}|^2),\tag{13}$$

where ε_{10} and ε_{20} are the initial amplitudes of the signal and pump waves, respectively. One can be easily convinced of the validity of Eq. (13) by writing first integrals of system (11).

3. The system of nonlinear equations (11) is universal in the sense that its form does not depend on the actual geometry of the beam system.[8] To simplify the presentation we consider in what follows the case of undulator radiation for which the amplitude of the pump wave ε_2 can be considered constant, i.e., $\varepsilon_2\equiv\varepsilon_{20}$ (Ref. 5).

Before considering mechanisms for the nonlinear stabilization of stimulated wave scattering, we carry out an analysis of linear regimes. In the case of undulator radiation, the dispersion relation for the growth rate δ normalized to the frequency Ω_b can be written in the form

$$(\delta-\eta_0)(\delta^2-1)=\nu^2|\varepsilon_{20}|^2(1-{}^3/_2\mu\delta).\tag{14}$$

We consider initially the nonrelativistic limit when the parameter $\mu\ll1$. In the case of high-density beams with

$$\nu'\equiv\nu|\varepsilon_{20}|\ll1\tag{15}$$

collective or Raman scattering on density oscillations occurs and the instability growth rate is determined by the expression[9]

$$\delta=-1+i\frac{\nu'}{2^{1/2}}\left(1+\frac{3}{2}\mu\right)^{1/2},\tag{16}$$

for a frequency difference $\eta_0=-1$, since conversion of kinetic electron energy into radiation energy occurs only for excitation of the slow beam Langmuir wave.

For low density beams when $\nu' \gg 1$ and correspondingly $\operatorname{Im}\delta \gg 1$, the scattering process is of the single particle type.[9] In this case, when the inequality

$$1 \ll \nu' \ll \mu^{-3/2}, \tag{17}$$

is satisfied, usual Thomson scattering of electromagnetic waves occurs with a growth rate[9]:

$$\delta = \frac{-1+3^{1/2}i}{2}(\nu')^{2/3}. \tag{18}$$

Finally in the limit

$$\nu' \gg \mu^{-3/2}, \tag{19}$$

one can see from dispersion equation (14) that the induced scattering occurs in a new regime with a growth rate

$$\delta = i(^3/_2\mu)^{1/2}\nu'. \tag{20}$$

In spite of the fact that expression (20) does not contain a real addition to the frequency, the instability developed under conditions (19) is of the radiation type, but its mechanism, in contrast to the one considered previously, is completely different and is connected with the effect of energy phasing (see below). Thus, in the weakly relativistic limit, besides the two well known scattering mechanisms with growth rates (16) and (18), a new instability mechanism occurs.

We proceed to a consideration of strongly relativistic beams when $\mu \gg 1$. If the condition

$$\nu' \ll \mu^{-1/2}, \tag{21}$$

is satisfied, the previously considered regime of collective scattering is preserved, but the expression for the growth rate takes the following form:

$$\delta = -1 + i\nu'(^3/_4\mu)^{1/2}. \tag{22}$$

We emphasize that the instability with growth rate (22) develops in the Raman regime in large-density relativistic beams, but the beam relativistic factor is limited by inequality (21). In the strongly relativistic regime when inequality (21) is violated, namely for

$$\nu' \gg \mu^{-1/2}, \tag{23}$$

scattering of electromagnetic waves is due to energy phasing with growth rate (20).

Because both in the limit (19) and in the limit (23) the growth rate of the scattering process $\delta \gg 1$, the energy phasing effect is a single-particle one. Moreover, in relativistic beams when $\mu \gg 1$, normal Thomson scattering of electromagnetic waves does not occur.

4. We consider mechanisms of nonlinear stabilization of induced scattering processes on electron beams in correspondence with the enumerated linear regimes. For nonrelativistic electrons when $\mu = 0$, Eqs. (11) (recall that in system (11) it is necessary to exclude the second equation for ε_2 because we are considering undulator radiation) depend on only one density parameter ν because the amplitude $|\varepsilon_{20}|$ can be taken as 1 without limiting the generality.

The time dependences of $|\varepsilon_1|$ and $|\rho_1|$ are shown in Fig. 1a for $\nu = 3$ and $\mu = 0$ (low density beam) and in Fig. 1b the electron phase space at specific times is presented. One can see well that the saturation mechanism of single-particle scattering processes is the trapping of beam electrons by the combined electromagnetic wave[1,5,7] in this case. We note that the chosen values of parameters ν and μ automatically satisfy inequality (17). Taking into account the finite but small values of the relativisitic parameter μ does not significantly change the nonlinear dynamics of induced undulator emission, as shown by the results of numerical calculations. The radiation efficiency in this case is proportional to $\mu\nu^{2/3}$ (Ref. 10).

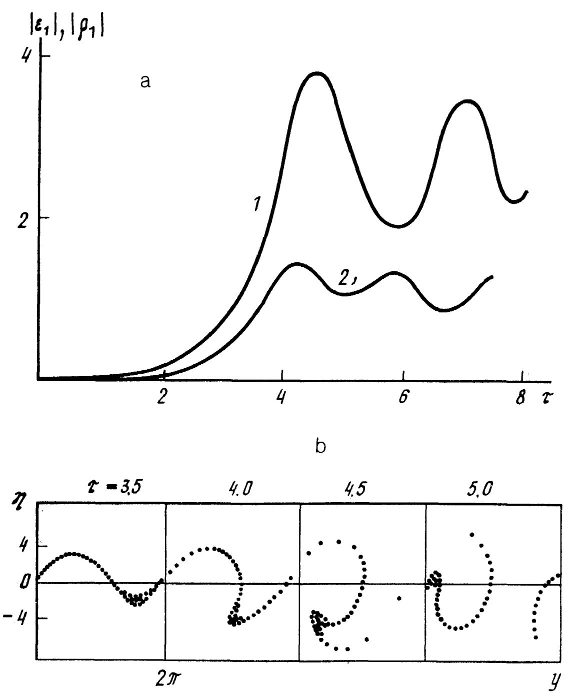

FIG. 1. a) The amplitudes $|\varepsilon_1|$ (1) and $|\rho_1|$ (2) as functions of time for $\nu = 3$ and $\mu = 0$; b) beam electron phase for various times τ for $\nu = 3$ and $\mu = 0$.

The same quantities are shown in Fig. 2a and b for a beam of intermediate density with $\nu = 0.3$ and $\mu = 0$. In this case the instability passes to the regime of collective or Raman scattering and is stabilized by trapping of the beam electrons by a Langmuir wave with formation of multistream flow and subsequent beam turbulence.[5] The latter can be well seen on the phase plane. In agreement with the results of Ref. 5, the radiation efficiency for this case is of order $\mu\nu$.

Finally, for the case of high-density beams,[2)] when the parameter $\nu \ll 1$ and $\mu = 0$, the saturation is due to a nonlinear frequency shift.[7,11] The effect of the nonlinear frequency shift for nonrelativistic emitters on free electrons is determined by processes of beam deceleration and generation of high harmonics of the beam Langmuir wave,[8,12] and corresponds mathematically to taking into account only nonlinear terms of third order in the expansion of the particle equations in powers of the wave amplitude.[11] Because the resonance width for three-wave interaction is of the order of the growth rate, i.e., $\sim\nu$, for $\nu \ll 1$ the decrease in average velocity of beam electrons or the generation of high harmonics of the beam Langmuir wave leads to rapid violation of the resonance condition ($\omega_0 - k_0 u = -\Omega_b$) until the amplitude of the first resonance harmonic of the beam Langmuir

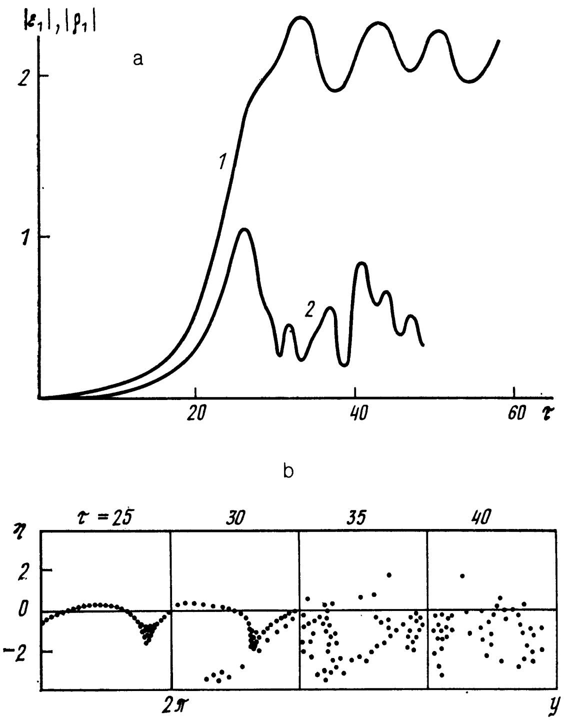

FIG. 2. a) Amplitudes $|\varepsilon_1|$ (1) and $|\rho_1|$ (2) as functions of time for $\nu = 0.3$ and $\mu = 0$; b) beam electron phase spaces for various times τ for $\nu = 0.3$ and $\mu = 0$.

wave becomes large and strongly nonlinear processes set in. As already stated, cubic nonlinearity plays the decisive role here.

It follows that for $\nu \ll 1$ Eq. (11) can be expanded in powers of the wave amplitude with an accuracy to cubic terms. This approach was used in Ref. 7 for an analytic consideration of the nonlinear dynamics of scattering on nonrelativistic beams. However, only one of two possible mechanisms, deceleration of the electron beam, was taken into account in this work. A more general approach, the method of expansion in trajectory perturbations, in which generation of high harmonics of the beam Langmuir wave is taken into account in addition to deceleration, was proposed in Refs. 8, 12. The methods and results of Refs. 7, 8, 12 for $\mu = 0$ can be used without change for the undulator radiation considered here. Thus, we will not consider them separately here because they are particular cases of a somewhat more general result which will be examined below. Namely, for finite (even large) values of μ, when only the strong inequality (21) is satisfied, the instability is stabilized as before by a nonlinear frequency shift in which, together with deceleration and the generation of high harmonics, relativistic effects provide a substantial contribution. The corresponding analytic treatment will be given somewhat later.

If, however, inequality (21) ceases to be strong (intermediate-density beam), the instability develops as before in the collective regime, but it is no longer stabilized by a nonlinear frequency shift. The stabilization mechanism is trapping of electrons in the field of the beam Langmuir wave. The results of numerical calculations in this case do not differ qualitatively from those given above (see Figs. 2a, b) and thus will not be discussed in detail here (see also Ref. 5).

In conditions when inequality (19) or (23) is fulfilled, the results of numerical calculations differ qualitatively from the preceding ones. As an example, the results of the calculation with $\mu = 0.8$ and $\nu = 1$ are given in Figs. 3 a, b. It can be seen that an exceptionally sharp growth of $|\hat{\rho}|$ occurs against the background of a rather smooth growth of amplitude of the signal wave $|\varepsilon_1|$ and of the amplitude of the first harmonic of the beam Langmuir wave $|\rho_1|$. The quantity $|\hat{\rho}|$ characterizes the electron-beam energy modulation while $|\rho_1|$ describes spatial modulation. This behavior of $|\rho_1|$ and $|\hat{\rho}|$ is connected with the above-noted effect of energy phasing which we proceed to examine now. We note that for $\mu = 0.8$ and $\nu = 1$ inequalities (19) and (23) only begin to be satisfied. Energy phasing will become even more dominant when these inequalities become stronger.

5. We introduce for convenience a definition of electron "momentum"

$$p = (1 - \mu\eta)^{-1/2} \tag{24}$$

and transform Eq. (11) for the considered case of undulator radiation to the form

$$\frac{d\varepsilon_1}{d\tau} = -\nu\varepsilon_{20}\hat{\rho}\exp(i\eta_0\tau),$$

$$\frac{dp}{d\tau} = -\frac{\mu}{4} i \sum_{n=1}^{\infty} \frac{\alpha_n}{n} [\rho_n \exp(iny) - \text{c.c.}]$$

$$+ \frac{\mu}{4}\nu\frac{1}{p^3}[\varepsilon_1\varepsilon_{20}^* \exp(iy - i\eta_0\tau) + \text{c.c.}],$$

$$\frac{dy}{d\tau} = \frac{1}{\mu}\frac{p^2-1}{p^2}, \quad \hat{\rho} = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{p^3}\exp(-iy)\,dy_0. \tag{25}$$

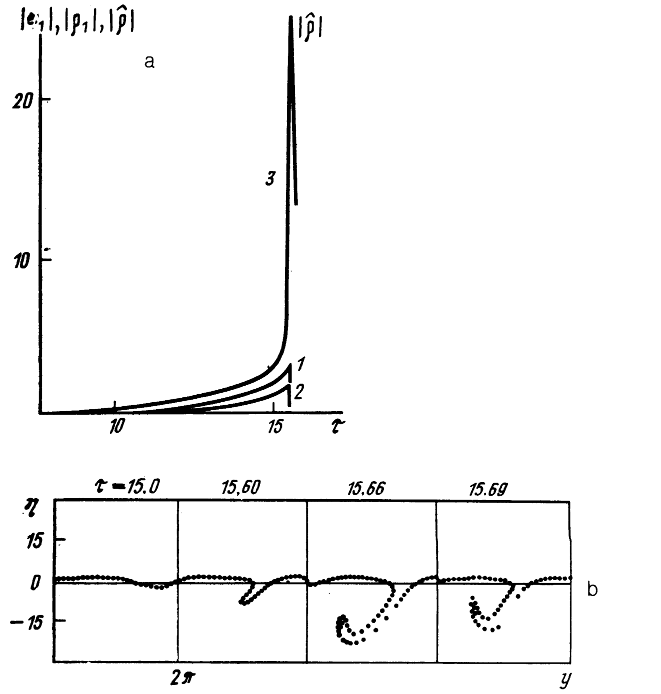

FIG. 3. a) Amplitudes $|\varepsilon_1|$ (1), $|\rho_1|$ (2), and $|\hat{\rho}|$ (3) as functions of time for $\nu = 1$ and $\mu = 0.8$; b) beam electron phase spaces at various times τ for $\nu = 1$ and $\mu = 0.8$.

It can be seen from the third equation of system (25) that for $\mu \gg 1$ we have $y \sim y_0$, i.e., the beam spatial modulation is small. At the same time the electron momentum p changes very rapidly within wide limits. In fact, in strongly relativistic beams a small velocity change produces a small change of electron position and a significant change of momentum, as confirmed by the behavior of $|\rho_1|$ and $|\hat{\rho}|$ in Fig. 3a. It is just this sharp change of momentum which causes the energy phasing. We note that under conditions (23), owing to the significant decrease of momentum p in the first and second equations of system (25), definite difficulties arise in the numerical integration of this system.

We introduce the new variables

$$\tau'=\tau(\nu')^{2/3}, \quad \varepsilon=\varepsilon_1(\nu')^{-1/3}, \quad \mu'=\mu(\nu')^{2/3}, \quad \nu'=\nu\varepsilon_{20}. \tag{26}$$

Since the spatial modulation in strongly relativistic beams is small, we have $\rho_n \cong 0$, $\eta_0 \cong 0$, and Eqs. (25) can be written in the form

$$\begin{aligned} \frac{d\varepsilon}{d\tau'} &= -\frac{1}{\pi}\int_0^{2\pi}\frac{1}{p^3}\exp(-iy_0)\,dy_0, \\ \frac{dp}{d\tau'} &= -\frac{1}{4}\mu'\frac{1}{p^3}[\varepsilon\exp(iy_0)+\text{ c.c. }]. \end{aligned} \tag{27}$$

Further, assuming $\varepsilon = dS/d\tau'$ (the function S can be taken as real without limiting the generality), we integrate the equation for the momentum p:

$$p=(1+2\mu' S\cos y_0)^{1/4}. \tag{28}$$

In this case the first equation of system (27) can be rewritten as

$$\frac{d^2S}{d\tau'^2} = -\frac{1}{\pi}\int_0^{2\pi}\frac{\cos y_0\,dy_0}{(1+2\mu' S\cos y_0)^{3/4}} \tag{29}$$

The growth rate (20) follows from Eq. (29) in the linear approximation. Using the new variables $q = 2\mu' S$ and $\xi = (3/2\mu')^{1/2}\tau'$ and the definition (13) of radiation efficiency, we obtain

$$\begin{aligned} \frac{d^2q}{d\xi^2} &= -\frac{4}{3\pi}\int_0^{2\pi}\frac{\cos y_0\,dy_0}{(1+q\cos y_0)^{3/4}}, \\ W &= \frac{3}{64}\left(\frac{dq}{d\xi}\right)^2. \end{aligned} \tag{30}$$

To estimate the maximum radiation efficiency we integrate the first equation of (30) with an adiabatic initial condition

$$\left(\frac{dq}{d\xi}\right)^2 = \frac{64}{3}\left\{1-\frac{1}{2\pi}\int_0^{2\pi}(1+q\cos y_0)^{1/4}dy_0\right\}. \tag{31}$$

Since it can be seen from the right-hand side of Eq. (30) that $q_{\max} = 1$, we have

$$W=1-\frac{1}{2\pi}\int_0^{2\pi}(1+\cos y_0)^{1/4}\,dy_0 \approx 0.16. \tag{32}$$

The result (32) is rigorous for $\nu\mu^{1/2} \gg 1$ and $\mu \gg 1$. However, numerical calculations show that it is actually reached already at $\mu \lesssim 1$. Thus the value of W obtained from the calculated results (Fig. 3) is 0.22 which is sufficiently close to Eq. (32). It follows from an analysis of the linear dispersion equation that for $\nu'\mu^{3/2} \gg 1$ and $\mu \ll 1$ the stimulated scattering is also due to energy grouping. However, for weakly relativistic beams ($\mu \ll 1$) one can see from the second and third equations of the system (25) that the effective energy grouping competes with spatial grouping.

We note that the presence of a pump wave is very important for the effect of energy phasing. In simpler systems, when beam emission occurs without a pump (e.g., a beam in a plasma or in a retarding system) multipliers of type p^{-3} are absent in the right-hand side of Eq. (25) and the effect considered does not arise.[13] Hence, the nonlinear dynamics of stimulated wave scattering on strongly relativistic beams of free electrons differs qualitatively from the dynamics of usual beam instability.

6. We consider the nonlinear dynamics of stimulated wave scattering on relativistic beams for conditions when inequality (21) is satisfied.

We represent the electron coordinate and momentum in the form

$$\begin{aligned} y &= y_0+W(\tau)+\frac{1}{2}\sum_{s=1}^{\infty}[b_s(\tau)\exp(isy_0)+\text{ c.c. }], \\ p &= \langle p\rangle+\frac{1}{2}\sum_{s=1}^{\infty}[a_s(\tau)\exp(isy_0)+\text{ c.c. }]. \end{aligned} \tag{33}$$

Here $W(\tau)$ is the average displacement of the electron beam, $\langle p\rangle$ is the average electron momentum, and $b_s(\tau)$ and $a_s(\tau)$ are the oscillation amplitudes of the corresponding coordinate and momentum in the field of the combination wave. As noted above, under conditions (21) the instability is eliminated by nonlinearities of cubic type. This assertion was corroborated in detail in Ref. 8 for nonrelativistic beams, when only the first expression of (33) is used. Substituting Eq. (33) into system (25) and discarding nonlinearities of higher than third order, we obtain the following system of equations:

$$\begin{aligned} \frac{d\varepsilon_1}{d\tau} &= -\nu\varepsilon_{20}\hat{\rho}\exp(i\eta_0\tau), \\ \frac{da_1}{d\tau} &= \frac{\mu}{2}\left[-b_1+(1-\alpha_2)\left(ib_1^*b_2+\frac{1}{2}|b_1|^2b_1\right)\right] \\ &\quad +\frac{1}{2}\nu\mu\varepsilon_1\varepsilon_{20}^*\exp(iW-i\eta_0\tau), \\ \frac{da_2}{d\tau} &= \frac{\mu}{2}\left[-\alpha_2b_2+\frac{i}{2}(\alpha_2-1)b_1^2\right], \\ \frac{db_1}{d\tau} &= \frac{1}{\mu}\left[2a_1+\frac{3}{4}\mu(|\varepsilon_1|^2-|\varepsilon_{10}|^2)a_1-3a_1^*a_2+3a_1|a_1|^2\right], \\ \frac{db_2}{d\tau} &= \frac{1}{\mu}\left(2a_2-\frac{3}{2}a_1^2\right), \\ \frac{dW}{d\tau} &= -\frac{1}{4}(|\varepsilon_1|^2-|\varepsilon_{10}|^2)-\frac{3\mu}{2}|a_1|^2, \\ \hat{\rho} &= (3a_1+ib_1)\exp(-iW). \end{aligned} \tag{34}$$

We note that since $\tau = |\Omega_b|t$, the coefficient $\alpha_1 \equiv 1$. Further considering that

$$a_1=\tilde{a}_1(\tau)\exp[i(\tau+W)],\quad a_2=\tilde{a}_2(\tau)\exp[2i(\tau+W)],$$

$$b_1=\tilde{b}_1(\tau)\exp[i(\tau+W)],\quad b_2=\tilde{b}_2(\tau)\exp[2i(\tau+W)],\tag{35}$$

where $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$ and $\tilde{b}_2$ are slowly varying functions of time, and the frequency difference $\eta_0 = -1$, we rewrite the system (34) in the simpler form:

$$\frac{d\varepsilon_1}{d\tau}=i\nu\varepsilon_{20}(1+{}^3/_2\mu)\tilde{b}_1,\quad \frac{d\tilde{b}_1}{d\tau}+i\Delta\tilde{b}_1=-i\frac{\nu}{2}\varepsilon_1\varepsilon_{20}^*,\tag{36}$$

$$\Delta=-\frac{1}{4}\left(1+\frac{3}{4}\mu\right)(|\varepsilon_1|^2-|\varepsilon_{10}|^2)+\frac{1}{4(\alpha_2-4)}\left\{3(\alpha_2-1)+\frac{3}{2}\mu\left[(4-\alpha_2)\left(1+\frac{\mu}{2}\right)+2(\alpha_2-1)+\frac{3}{8}\mu\alpha_2\right]\right\}|\tilde{b}_1|^2.$$

The case of resonance excitation of the second harmonic of the beam Langmuir wave for $\alpha_2 = 4$ is excluded from consideration in Eqs. (36) and requires a separate examination.[14] Using the two first integrals it is easy to reduce the system of equations (36) to one equation for $X = |\tilde{b}_1|^2$, whose solution can be expressed in the standard way with elliptic functions:

$$X=\frac{4\nu\varepsilon_{20}|\varepsilon_{10}|^2(2q)^{1/2}\,\mathrm{sn}^2(y,r)}{\lambda|\varepsilon_{10}|^2+8\nu\varepsilon_{20}(2q)^{3/2}\,\mathrm{cn}^2(y,r)},$$

$$r=1-\frac{\lambda|\varepsilon_{10}|^2}{8\nu\varepsilon_{20}(2q^3)^{1/2}},\quad y=\nu\varepsilon_{20}\left(\frac{q}{2}\right)^{1/2}\tau,\quad q=1+{}^3/_2\mu,\tag{37}$$

$$\lambda=1+\frac{9}{4}\mu\left(1+\frac{\mu}{2}\right)-\frac{3}{2}\frac{(\alpha_2-1)}{(\alpha_2-4)}-\frac{3}{4}\frac{\mu}{(\alpha_2-4)}\left[2(\alpha_2-1)+3\left(1+\frac{\mu}{2}\right)+\frac{3}{8}\mu\alpha_2\right].$$

The characteristic instability development time is determined by the expression

$$\tau_{\text{H}}=\frac{1}{\nu\varepsilon_{20}}\left(\frac{2}{q}\right)^{1/2}\ln\frac{8}{|\varepsilon_{10}|}\left[\frac{\nu\varepsilon_{20}(2q^3)^{1/2}}{\lambda}\right]^{1/2},\tag{38}$$

and the maximum efficiency of electromagnetic radiation in the considered system has the form:

$$W=2^{1/2}\nu\varepsilon_{20}\mu\frac{q^{1/2}}{\lambda}.\tag{39}$$

In the case of an adiabatic application of the field, when $\varepsilon_{10} = 0$, the solution of Eq. (36) can be written in the form:

$$X=\frac{4\nu\varepsilon_{20}(2q)^{1/2}}{\lambda}\left\{1-\left[\frac{1-\exp(2\nu\varepsilon_{20}(2q)^{1/2}\tau)}{1+\exp(2\nu\varepsilon_{20}(2q)^{1/2}\tau)}\right]^2\right\}^{1/2}.\tag{40}$$

The radiation efficiency is determined as before by expression (39). We note that Eqs. (37) and (40) correspond as $\mu\to 0$ to the results obtained in Ref. 15.

7. We noted above that when inequalities (19) or (23) are satisfied, numerical integration of Eqs. (11) or (25) encounters definite difficulties. This circumstance is due to the approximations in which these systems of equations were obtained. It is necessary in the more general case to start from electron equations of motion written in terms of the momentum p_z:

$$\frac{dz}{dt}=\frac{p_z/m}{(1+p_z^2/m^2c^2)^{1/2}},\quad \frac{dp_z}{dt}=e(E_{z1}+E_{z2}).\tag{41}$$

Using for the coordinate and momentum an expansion analogous to Eq. (3) ($p_z = p' + \tilde{p}$ and $z = z' + \tilde{z}$) and averaging the equations of motion and the wave amplitude equations over the variables z and p_z, we finally obtain the following system of relativistic equations:

$$\frac{d\varepsilon_1}{d\tau}=-\nu\varepsilon_2\hat{\rho}\exp(i\eta_0\tau),\quad \frac{d\varepsilon_2}{d\tau}=\sigma\nu\varepsilon_1\hat{\rho}^*\exp(-i\eta_0\tau),$$

$$\frac{dy}{d\tau}=\frac{p}{(1+Q^2p^2)^{1/2}}-\frac{u}{c}Q^{-1},$$

$$\frac{dp}{d\tau}=-\frac{i}{2}\gamma_0^3\sum_{n=1}^{\infty}\left[\frac{\alpha_n}{n}\rho_n\exp(iny)-\text{c.c.}\right]+\frac{1}{2}\nu\gamma_0^{-6}\frac{1}{(1+Q^2p^2)^{3/2}}[\varepsilon_1\varepsilon_2^*\exp(iy-i\eta_0\tau)+\text{c.c.}],\tag{42}$$

$$\rho_n=\frac{1}{\pi}\int_0^{2\pi}\exp(-iny)\,dy_0,$$

$$\hat{\rho}=\frac{1}{\pi}\int_0^{2\pi}\gamma_0^{-3}(1+Q^2p^2)^{-3/2}\exp(-iy)\,dy_0.$$

Here $p = k_0p'/m\Omega_b$ and $Q = \Omega_b/k_0c$.

Integration of Eq. (41) on a computer does not entail difficulties because the denominator $(1 + Q^2p^2)^{3/2}$ on the right-hand sides of Eq. (42) cannot become less than unity. Numerical solutions of Eq. (42) for large Q and γ_0 (it is easy to see that $\mu = 2\gamma_0^2Q$) confirm the presence of the energy phasing effect in a beam and are not given here because their form is analogous to Fig. 3a.

It is easy to see that the dimensionless electron momentum p is connected with the slow oscillation of the velocity v' in the following way:

$$p=\frac{k_0}{\Omega_b}\hat{v}\left(1-\frac{\hat{v}^2}{c^2}\right)^{-1/2};\quad \hat{v}=u+v'.\tag{43}$$

When the inequality

$$v'\ll u,\tag{44}$$

is satisfied, after substitution of Eq. (44) in Eq. (43) and carrying out corresponding expansions, Eqs. (42) reduce completely to Eqs. (11) and (25).

We note that when high-frequency radiation instabilities of electron beams evolve (including the various processes of wave scattering), the inequality (44) is satisfied with a wide margin. An exception is emission due to the energy phasing effect in relativistic beams, when separate electrons are retarded so strongly that condition (44) begins to be violated.

8. The basic results of the work on nonlinear dynamics of stimulated undulator radiation are given in Table I. The results of the numerical integration of the general nonlinear equations (25) and the asymptotic forms of analytic equations (39) for large and small values of μ were used in constructing the table. We note that for weakly relativistic electron beams, together with classical mechanisms of Thomson and Raman scattering, there is a new radiation mechanism due to electron energy phasing. However, there are only two instability mechanisms for scattering of electromagnetic

TABLE I.

Conditions on μ and ν'	Instability increment	Type of process	Stabilization mechanism	W
$\mu \ll 1, \nu' < 1$	$-1 + i\frac{\nu'}{2^{1/2}}\left(1+\frac{3}{2}\mu\right)^{1/2}$	Collective or Raman scattering	Trapping by the beam Langmuir wave for for $\nu_1 \lesssim 1$; nonlinear frequency shift for $\nu' \ll 1$	$\mu\nu$
$\mu \ll 1$, $1 < \nu < \mu^{-3/2}$	$\frac{-1 + 3^{1/2}i}{2}(\nu')^{2/3}$	Single-particle or Thomson scattering	Trapping by the combination wave	$\mu\nu^{2/3}$
$\mu \ll 1$, $\nu' > \mu^{-3/2}$	$i\left(\frac{3}{2}\mu\right)^{1/2}\nu'$	Energy phasing	Total momentum modulation of the beam	0.16
$\mu \gg 1$, $\nu' < \mu^{-1/2}$	$-1 + i\nu'\left(\frac{3}{4}\mu\right)^{1/2}$	Collective or Raman scattering	Trapping by the beam Langmuir wave for $\nu'\mu^{1/2} \lesssim 1$; nonlinear frequency shift for $\nu'\mu^{1/2} \ll 1$	$\nu\mu^{1/2}$
$\mu \gg 1$, $\nu' > \mu^{-1/2}$	$i\left(\frac{3}{2}\mu\right)^{1/2}\nu'$	Energy phasing	Total momentum modulation of the beam	0.16

waves on strongly relativistic beams: collective Raman scattering and the energy phasing effect. Single-particle Thomson scattering on the beams does not occur in this case.

[1)]The choice of dimensionless variables is the same here as in Ref. 7.

[2)]In Ref. 8 the actual inequality determining the distinction between intermediate and high-density beam cases was determined on the basis of numerical calculations and turned out to be $\nu \lesssim 0.1$.

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