Diffusion in a medium with vortex flow

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The motion of a passive impurity in a medium with a specified vortical flow v is considered in the presence of a microscopic ("seed") diffusion D. Over sufficiently large scales the impurity transport is by diffusion and is described by an effective diffusion (tensor) coefficient $D_{\rm eff}$ that depends on the flow properties. If the Peclet number is large enough, $P = \lambda v/D \gg 1$ (λ is the characteristic dimension of the flow), power-law scaling $D_{\rm eff} \approx DP^{\alpha}$ obtains with an exponent α that depends on the velocity-field topology. For two-dimensional stationary "common-position" incompressible flow the result $\alpha = 10/13$ is expressed in terms of the critical exponents of the percolation problem. For a nonstationary velocity field specified in the form of a set of traveling vortices, an exact expression corresponding to an exponent $\alpha = 2$ is obtained in the two- and three-dimensional cases. The applicability of the results to plasma heat-conduction and electric-conduction problems is discussed.

1. INTRODUCTION

The recently renewed interest, after half a century of neglect, in vortex motion in continuous media has led to a proliferation of studies in this field. At the same time, notwithstanding the thorough investigation of a great variety of properties of the vortices themselves (see, e.g., the reviews in Refs. 1–3), the practical problem of their influence on transport processes in liquids, gases, and plasma has been little investigated. This question is of undisputed interest in view of the simple circumstance that vortex motion, via long-distance drift (convection) of the particles of the medium, should increase most effectively the kinetic flows compared with the traditionally considered wave motion.[4,5] We solve in the present paper several very simple problems dealing with diffusion of an impurity in a specified vortex flow of a continuous medium, without considering the origin of this flow. Such a formulation of the problem is the necessary step towards the solution of the more complicated self-consistent problem and reveals, in spite of its idealized character, a number of nontrivial effects.

The solution method employed reduces formally to spatial averaging of a diffusion equation with a convective term

$$\partial n/\partial t + \mathbf{v}\nabla n = D\Delta n \tag{1}$$

and of reducing it to the form

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial x_i} D_{ik}^{eff} \frac{\partial N}{\partial x_k}. \tag{2}$$

Here $N = \langle n \rangle$ is the local mean value of the impurity density (the averaging is over a scale $l \gg \lambda \approx |\nabla \ln v|^{-1}$). The flowing liquid is assumed to be incompressible (div $\mathbf{v} = 0$), and the coefficient D of the microscopic ("seed") diffusion is assumed constant. Principal attention is paid to two-dimensional flows in the (x,y) plane; these flows are characterized by different stream functions $\psi(x,y)$: $\mathbf{v} = \operatorname{curl} \psi \mathbf{e}_z$. Nonstationary vortices are considered in the simplest case of a constant topology of their streamlines. Naturaly, Eq. (1) does not always reduce to (2). It appears that an assuredly sufficient condition for this reducibility is one proposed below, that ψ be bounded. An interesting example of the insufficiency of the requirement $\langle \mathbf{v} \rangle = 0$ is given in Ref. 6.

It is easily seen that incompressible convective transport can only increase the average impurity flux. Indeed, by specifying on the closed surfaces S_1 and S_2 constant impurity densities $n_1 > n_2$ and integrating the quantity $(n\mathbf{v} - D\nabla n)\nabla n$ over the volume between them, we obtain for the stationary process

$$J(n_1 - n_2) = D\int (\nabla n)^2 d^3\mathbf{r}, \tag{3}$$

where J is the impurity flux between the surfaces. The right-hand side of (3) is a minimum for $\Delta n = 0$, i.e., in the absence convection [see Eq. (1) with $\partial/\partial t = 0$]. Nonstationary behavior of the process leads to the same answer but with addition of time averaging.

Therefore $D_{\rm eff} > D$ always. A substantial enhancement of the diffusion can be expected when the local convective transport exceeds substantially the diffusion transport, i.e., for a large Peclet number $P = \lambda v/D \gg 1$. In this limiting case it is natural to assume the presence of the power-law scaling

$$D_{eff} \approx DP^{\alpha} \tag{4}$$

with an exponent α that depends on the topology of the flow. It has been calculated so far for three flow classes: For

$$\psi = \psi(y) \tag{5}$$

we have $\alpha = 2$ (Ref. 6) and of importance to the answer is only that all the streamlines be infinite in the x direction. For

$$\psi = \psi_0 \operatorname{sign}[g(x,y)], \quad \langle \psi \rangle = 0 \tag{6}$$

we have $\alpha = 1$ (Refs. 6 and 7) and only the presence of narrow δ-function-like flows is important for the answer. For

$$\psi = \psi_0 \sin kx \sin ky \tag{7}$$

we have $\alpha = 1/2$ (Refs. 8 and 9) and only the formation of convective cells of finite size is essential for the answer. The numerical coefficient in (4) can also be calculated in all three cases.

Since we shall refer hereafter to the velocity field (7), we estimate here the effective diffusion on quadratic vortex cells. The impurity flux in these cells is determined by the hydrodynamic component in a narrow boundary layer near the separatrix $\psi = 0$ (cell boundary). The width δ of this layer is equal to the diffusive displacement across the streamlines within a time λ/v on the order of the rotation

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around the cell: $\delta = (D\lambda/v)^{1/2} = \lambda P^{-1/2}$. In the presence of an average impurity-density gradient ∇N there exists in the medium a flux

$$\mathbf{j} \approx -\lambda \nabla N v \xi,$$

where $\xi = \delta/\lambda$ is the area fraction responsible for the transport; we obtain hence $D_{\text{eff}} \approx DP^{1/2}$.

The cases of convective cells of finite size and with infinite streamlines are respectively the most unfavorable and favorable for transport, so that we always have apparently $1/2 \leqslant \alpha \leqslant 2$. Moreover, in the "common position" case, when the fraction of infinite trajectories is zero, we have for stationary flow $\alpha < 1$, i.e., $D_{\text{eff}} \to 0$ as $D \to 0$ (Ref. 10).

The calculation of the values of α for different cases is in fact the subject of the present article.

In Sec. 2 we use percolation-theory results[11] to obtain for the "common-position" stationary flow $\psi(x,y)$ the result $\alpha = 10/13$, for which no full rigor is claimed. Control numerical calculations of the effective diffusion for this case are given in Sec. 3.

Sections 4 and 5 are devoted to nonstationary problems for the case of running vortices and rotating vortex pairs. Here α can reach its maximum value 2.

In the Conclusion we discuss the region in which the results are applicable to problems of plasma thermal and electric conduction, and assess certain results by other workers.

2. DIFFUSION ON A CHAOTIC TWO-DIMENSIONAL STATIONARY VELOCITY FIELD

Consider a two-dimensional zero-average-velocity stationary liquid flow specified by a bounded "common-position" stream function $\psi(x,y)$. Of course, the concept in the quotation marks cannot be defined rigorously, but we shall nevertheless use it, taking $\psi(x,y)$ to be on the average an isotropic oscillating function characterized only by characteristic "wavelength" and amplitude scales $|\psi/\nabla\psi| \approx \lambda$ and $|\psi| \approx \psi_0 = \lambda v$, and far from degeneracies such as periodicity or ordered height disposition of saddle points. These properties make the considered velocity field qualitatively different from the flows (5)–(7), and the field can be encountered in many physical problems (some are discussed in the Conclusion).

The presence of a universal exponent α that describes the effective diffusion in such a flow in the limit of large Peclet numbers [see (4)] is connected, in final analysis, with the universality of the percolation-theory[11] critical exponents in terms of which α is expressed.

The streamlines, which are level lines $\psi(x,y) = h$, can be illustratively interpreted as shore lines produced by successively flooding a hilly terrain $z = \psi(x,y)$.[11] An abrupt transition takes place here from individual lakes on infinite dry land to individual islands in an infinite ocean. At the transition point ($h = \langle\psi\rangle = 0$) there exists at least one shore line of infinite length. The maximum transverse dimension a_h of the level lines $\psi(x,y) = h$ is finite if $h \neq 0$ and its increase as $h \to 0$ is given by

$$a_h \approx \lambda (\psi_0/h)^{\nu}. \tag{8}$$

Since the streamlines are highly sinuous (their curvature radius is of the order of $\lambda \ll a_h$, see Fig. 1), the length $\mathcal{L}_h$ of a closed streamline exceeds greatly its diameter:

$$\mathcal{L}_h \approx \lambda (a_h/\lambda)^d \approx \lambda (\psi_0/h)^{\nu d}. \tag{9}$$

The correlation exponent ν and the fractal dimensionality d of a randomly meandering non-self-intersecting curve were investigated in detail by numerical methods, and were also recently obtained analytically: $\nu = 4/3$, $d = 1 + 1/\nu = 7/4$.[12]

Knowing the exponents ν and d we can estimate D_{eff} by the procedure used for the stream function (7) in the Introduction. The only difference is that now the expression for the average impurity flux must be summed over different convection cells:

$$\mathbf{j} \approx -\int_0^\infty \frac{dh}{h} n_h a_h \nabla N \langle v_\parallel \rangle \delta_h \mathcal{L}_h. \tag{10}$$

Here $\delta_h = (D\mathcal{L}_h/v)^{1/2}$ is the width of the diffusion boundary layer responsible for the transport, $\langle v_\parallel \rangle \approx v a_h/\mathcal{L}_h$ is the average velocity component in the ∇N direction, and n_h

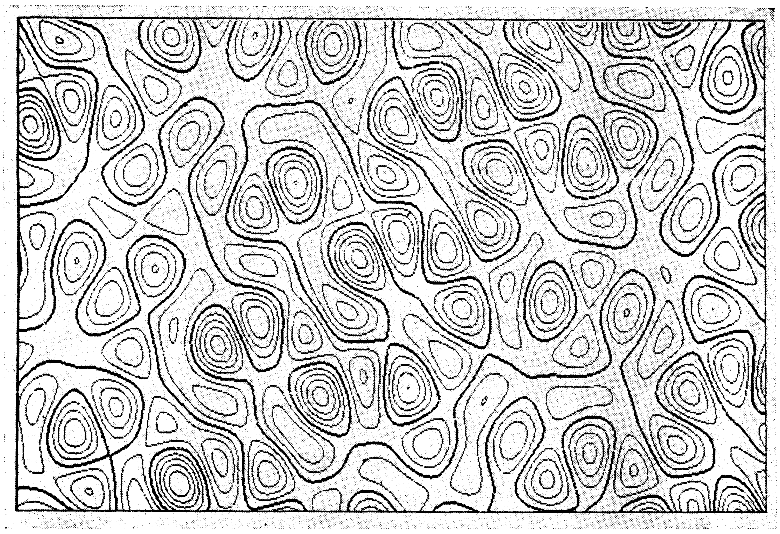

FIG. 1. Streamlines of "chaotic" flow ψ_5 in a 35×64 rectangle. The thick lines are the zero-level lines.

is the number of convection cells with diameter of order a_h (i.e., with diameter from $a_h/2$ to a_h) per unit area. By convection cell we mean a bundle of imbedded trajectories having a diameter not less than half that of the largest of them and having a common portion of width

$$\Delta_h \approx \lambda h/\psi_0. \quad (11)$$

Convection cells of size a are numbered by h in accord with the rule $a_h = a$. Since the last relation defines $h(a)$ in principle accurate to a value 2, the integration in (10) is in a logarithmic scale of h.

To calculate the density n_h of the convection cells we construct a square $a_h \times a_h$ near each of them and repeat in it a mental experiment with "flooding." We arrive then at the conclusion that the cells are closely packed and their diameters are of the same order, so that $n_h \simeq a_h^{-2}$.

We get thus from (10) the following expression for the effective-diffusion coefficient:

$$D_{eff} \approx DP^{1/2} \int_0^\infty \frac{dh}{h} \left(\frac{\mathscr{L}_h}{\lambda}\right)^{1/2}, \quad (12)$$

which according to (8) diverges on the lower integration limit $h \to 0$, i.e., for infinitely large convection cells. To regularize the integral (12), we take it into account that in the derivation of (10) we assumed implicitly that the diffusion boundary layer δ_h is small compared with the hydrodynamic width (11) of the cell; otherwise the cell cannot contain the boundary layer and makes no significant contribution to the flux (10).

The cutoff parameter

$$h_{min} = \psi_0 P^{-1/(2+\nu d)}$$

estimated from the relation $\delta_h = \Delta_h$ leads to the following result for the effective-diffusion coefficient:

$$D_{eff} \approx DP^\alpha, \quad \alpha = (\nu d + 1)/(\nu d + 2) = {}^{10}/_{13}. \quad (13)$$

The solution shows that the impurity motion averaged over a scale $l \gg a_{h_{min}}$ is diffusive. The characteristic mixing dimension in a two-dimensional chaotic velocity field is equal to

$$a = \lambda P^{4/13}. \quad (14)$$

The foregoing result uses essentially the equivalence of the critical exponents ν and d for different types of percolation two-dimensional problems—continual and lattice percolation (the analytic values were obtained in Ref. 12 for ν and d just for the lattice problem). The causes of this equivalence are well manifested in the solution of the problem of effective diffusion on weakly perturbed flow (7):

$$\psi = \psi_0 \sin kx \sin ky + \varepsilon \psi_1(x, y), \quad (15)$$
$$|\psi_1| \simeq \psi_0, \quad |\nabla \psi_1| \approx k\psi_0, \quad \varepsilon \ll 1$$

with perturbation $\psi_1(x,y)$ of the common position.

The streamlines $\psi(x,y) = h$ for $h \lesssim \psi_0$ are broken lines that are "hulls" of clusters of bonds on a dual quadratic lattice with sites at the points where

$$\sin kx \sin ky = \operatorname{sign} h,$$

and the topology of the streamlines turns out to be connected with the topology of the lattice clusters. As a result we obtain for the flow (15)

$$D_{eff} \approx DP^{10/13} \varepsilon^{7/13}, \quad P^{-1/2} < \varepsilon < 1. \quad (16)$$

Even for small $\varepsilon \gg P^{-1/2}$ (when the splitting of the sites in the separatrix lattice exceeds the width of the boundary layer) the effective diffusion is substantially higher than in unperturbed flow: $D_{eff} \gg DP^{1/2}$, while at the upper limit of the applicability $\varepsilon = 1$, corresponding to strong perturbation, we obtain, as expected, the result (13) for a flow of general form.

3. NUMERICAL SIMULATION OF TWO-DIMENSIONAL EFFECTIVE DIFFUSION

To check on the analytic estimate (13) we simulated two-dimensional effective diffusion numerically by the method of particles. For each of 256 independent particles we solved the equation of motion

$$dx/dt = \partial\psi/\partial y,$$
$$dy/dt = -\partial\psi/\partial x,$$

and in each step Δt of integration by the Runge–Kutta method we displaced the particles by a distance $(4D\Delta t)^{1/2}$ in a direction determined randomly each time; this simulates seed diffusion with a coefficient D.

The stream function $\psi(x,y)$ was chosen to satisfy the equation of stationary motion of an ideal liquid $(\psi,\Delta\psi) = 0$ [where (...,...) is a Jacobian]:

$$\psi_N(r) = N^{-1/2} \sum_{i=1}^{N} \sin(\mathbf{k}_i \mathbf{r}), \quad (17)$$

where $|\mathbf{k}_i| \equiv 1$, and the direction of $\mathbf{k}_i$ on the plane was chosen at random so as to simulate a common-position function. The factor $N^{-1/2}$ in (17) was introduced to preserve the average amplitude $\langle \psi_N^2 \rangle = 1/2$.

For $N = 2$, the sum of the sine functions in (17) reduces to a product of trigonometric functions, so that the stream forms a periodic system of convection cells of finite size, which differ from that considered in Refs. 8 and 9 only in that the rectangles are replaced by parallelegrams. As already noted, such a stream function is structurally unstable (also in the framework of the equation $(\psi,\Delta\psi) = 0$). Starting with $N = 3$, $\psi(x,y)$ is no longer periodic,[1)] all the saddle points have different heights, and the streamlines can be arbitrarily long (see Fig. 1).

The calculations were performed for the stream functions (17) for $N = 2$, 6, 12, 18, 25, 30. The seed-diffusion range was $0.001 \leqslant D \leqslant 0.2$, corresponding to Peclet numbers $5 \leqslant P \leqslant 1000$. At $t = 0$ the particles filled uniformly an 8×8 square near the origin. The effective-diffusion coefficient was calculated to be the steady-state value of the quantity

$$D_{eff}(t) = (1/4t) \langle r^2 \rangle_{256}, \quad (18)$$

where $\langle \ldots \rangle_{256}$ denotes averaging over the particles. A typical behavior of (18) is shown in Fig. 2. In all the calculations D_{eff} assumes a constant value within a characteristic time of order λ^2/D, after which it varies during the entire calculation time $t \simeq 10^3$–10^4 only within the limits of the rms error, thus confirming the diffusive motion of the particles.

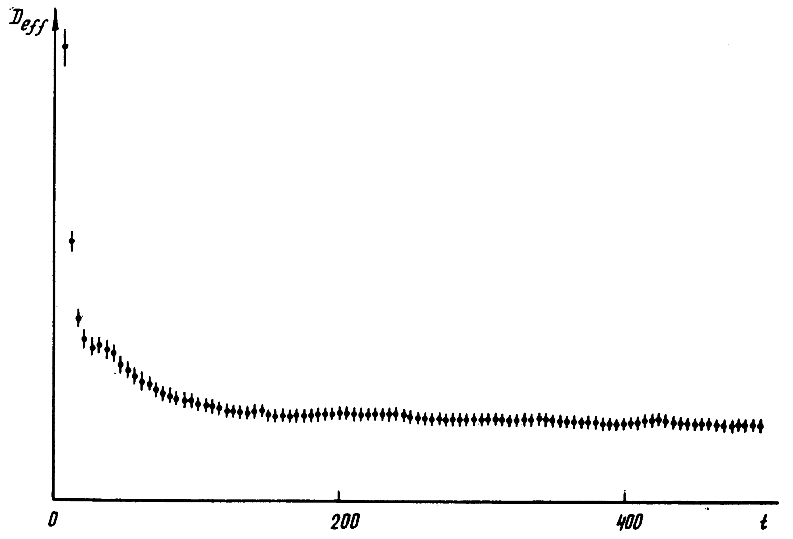

FIG. 2. Plot of $D_{\rm eff}(t)$ for a velocity field ψ_{25} at a seed diffusion $D = 0.1$.

Figure 3 shows the dependence of $D_{\rm eff}$ on D for various N. Attention is called to the substantial difference of the effective diffusion on the flow ψ_2, for which $D_{\rm eff}$ is substantially lower than for the other considered flows. The exponent

$$\alpha = 1 - \partial \ln D_{eff} / \partial \ln D,$$

obtained by least squares is listed in Table I. The result for $N = 2$ agrees with Refs. 8 and 9 ($\alpha = 1/2$), while for $N > 2$ we have $1/2 < \alpha < 1$, with the results close to the analytically obtained $\alpha = 10/13 \approx 0.77$. The absence of better agreement can be attributed to shortcomings of the assumed common-position function (17) for finite N.

4. DIFFUSION IN A GAS OF TRAVELING VORTICES

We consider impurity diffusion in a nonstationary vortical field comprising a set of traveling two-dimensional vortices, in which the gas parameter $\xi = (\lambda / l)^2$ (λ is the characteristic vortex dimension and l is the distance between vortices) is small enough to neglect collisions between individual vortices, so that each vortex remains stationary in its own reference frame. By "individual vortex" we mean a traveling localized solution of the set of equations describing the motion of a continuous medium containing a separatrix that envelops the liquid carried by the vortex. A typical picture of streamlines in a reference frame traveling at the vortex velocity has a characteristic Φ-shaped form (see, e.g., Ref. 1).

Let us estimate first the order of magnitude of the effective diffusion. The impurity density inside the vortex settles within a time of order $\tau = \lambda^2/D$. During this time the trapped liquid "remembers" the impurity density outside the separatrix, so that the mixing length is $a = u\tau$, whence

$$D_{eff} \approx \xi a^2/\tau = \xi \lambda^2 v^2 / D. \tag{19}$$

Thus, owing to the capture of the liquid and hence to the

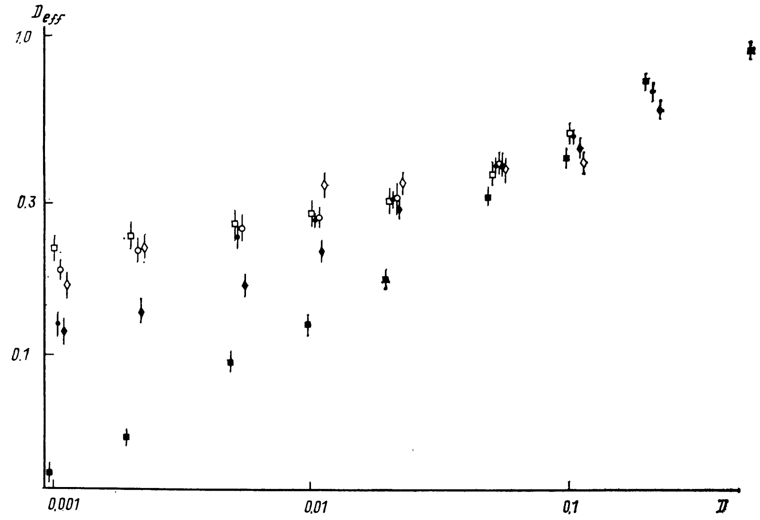

FIG. 3. Effective diffusion on ψ_N flow vs the seed diffusion for $N = 2$ (■), 6 (□), 12 (●), 18 (○), 25 (◆), 30 (◇).

TABLE I. Effective diffusion exponent α in a flow ψ_n as a function of N.

N	2	6	12	18	25	30
α	0.48±0.02	0.85±0.04	0.72±0.03	0.81±0.03	0.69±0.02	0.81±0.03

large mixing length $a \gg \lambda$ we obtain for traveling vortices a maximum exponent $\alpha = 2$.

An exact calculation of the effective-diffusion tensor is based on the possibility of reducing the two-dimensional diffusion-convection inside the vortex at $P \gg 1$ to a one-dimensional diffusion across the streamlines (see Appendix 1):

$$\frac{\partial n(I,t)}{\partial t} = \frac{D}{\partial I} \tilde{D}(I) \frac{\partial n(I,t)}{\partial I}. \tag{20}$$

Here $I(\psi)$ is the area, divided by 2π, of the region inside the line $\psi = \text{const}$, and $\tilde{D}(I)$ is defined in terms of the known vortex parameters by Eq. (A2). If the characteristic dimension $N/|\nabla N|$ of the average inhomogeneity exceeds substantialy the mixing length a, the average impurity density N can be regarded as a slowly varying boundary condition for (20) at $I = I_0 = S_{\text{in}}/(2\pi)$ (S_{in} is the area of the region inside the separatrix), so that (20) can be solved by successive approximations:

$$n^{(0)}(I,t) = n(I_0,t) = N(t),$$

$$\frac{\partial}{\partial I} \tilde{D}(I) \frac{\partial n^{(1)}}{\partial I} = \frac{\partial n^{(0)}}{\partial t},$$

from which we get the average impurity density in the vortex

$$\langle n_{\text{in}} \rangle = N(t) - \tau \dot{N}(t) \approx N(t-\tau). \tag{21}$$

The relaxation time τ estimated above is given by

$$\tau = \frac{1}{I_0} \int_0^{I_0} dI \int_I^{I_0} \frac{I' dI'}{\tilde{D}(I')}. \tag{22}$$

From (21) we obtain the average impurity flux

$$\mathbf{j} = \mathscr{L}^{-2} \int n\mathbf{v}\, d^2\mathbf{r} = \mathscr{L}^{-2} \sum_{\text{B}} \int n(\mathbf{u}_{\text{in}} + [\nabla\psi, \mathbf{e}_z])\, d^2\mathbf{r} = \mathscr{L}^{-2}$$

$$\times \sum_{\text{B}} (\langle n_{\text{in}} \rangle - N) S_{\text{in}} \mathbf{u}_{\text{in}}. \tag{23}$$

We can change in (23) from integration over the entire region $\mathscr{L} \times \mathscr{L}$ to summation over the vortices because outside the vortices, apart from a thin boundary layer $\delta = \lambda P^{-1/2}$ on the outside of the separatrix, the density for a sufficiently small gas parameter $\xi \ll P^{-1/2}$ has time to level off and is practicaliy locally homogeneous. The integral in (23) reduces then to integrals over the vortex separatrices. Expanding $N(t) = N(\mathbf{r}_0 + \mathbf{u}t)$ in a series in the small parameter $a/\mathscr{L}$ we obtain, with allowance for (21 and (23),

$$j_i = -D_{ik}^{\text{eff}}\, \partial N / \partial x_k,$$

$$D_{ik}^{\text{eff}} = \xi \langle \tau u_i u_k \rangle, \quad \xi = \mathscr{L}^{-2} \sum_i S_{\text{in}}, \tag{24}$$

where ξ is the gas parameter, and the averaging over the vortices in (24) is carried out with a weight S_{in}. In the case of two separatrices in a dipole vortex, the averaging in (24) is carried out independently over each separatrix. The limits of applicability of the result (24) are determined by the inequalities $P^{-2} \ll \xi \ll P^{-1/2}$, the left side of which corresponds to the condition $D_{\text{eff}} \gg D$.

We emphasize that the result is valid for the case when the vortex collision effects can be neglected. In the opposite "essentially nonstationary" case, when the flow topology varies with time, e.g., if the vortices exchange trapped liquid, effective diffusion can occur also in the absence of a seed diffusion.[15]

We conclude this section by considering the exactly solvable case of diffusion on three-dimensional traveling vortices in an ideal liquid. Progress can be made here because of Arnol'd's topological classification of stationary common-position vortices.[16] The streamlines and the vortex lines of such flows are located on a system of imbedded tori secluded from the external flow by a separatrix. It is possible to carry out in this case the averaging of (1) over the invariant tori and reduce the three-dimensional diffusion-convection to one-dimensional diffusion along the coordinate I that numbers the tori (see Appendix 1). The result takes the form (22) and (24), with $\tilde{D}(I)$ defined in accordance with (A3).

5. DIFFUSION ON ASYMMETRIC VORTEX PAIRS

Two-dimensional vortices in a homogeneous medium are described by the vorticity transport equation $d\Omega/dt = 0$, which conserves respectively the momentum and angular-momentum integrals:

$$\mathbf{p} = \int \Omega \mathbf{r}\, d^2\mathbf{r}, \quad M = \int \Omega r^2\, d^2 r.$$

It follows from the momentum conservation that traveling vortices should have a zero integrated vorticity

$$Q = \int \Omega\, d^2\mathbf{r},$$

i.e., they should be dipoles. The simplest representation of a dipole is a pair of unlike point vortices. As a vortex turbulence evolves, a situation can arise in which the combining pairs are opposite monopoles that do not cancel each other. It follows from momentum and angular-momentum conservation that asymmetric pairs move in a homogeneous medium along circles. For a weakly symmetric vortex pair $q = |Q_1 + Q_2| \ll |Q_1|$, the rotation vector is $r_0 = \lambda |Q_1|/q$ (λ is the distance between the monopoles).

In the case $\omega\tau \ll 1$ ($\omega = u/r_0$ is the angular velocity of the rotating asymmetric pair) the diffusion on the rotating vortices can be calculated as in Sec. 4. The impurity flux is equal to

$$\mathbf{j} = \xi \langle \mathbf{u}(t) [N(\mathbf{r}(t-\tau) - N(\mathbf{r}(t))] \rangle, \tag{25}$$

where $\mathbf{r}(t) = (r_0 \cos \omega t, r_0 \sin \omega t)$ is the coordinate of the vortex, and $\mathbf{u}(t) = \dot{\mathbf{r}}(t)$ is its velocity. Expanding $N(\mathbf{r})$ in a series we obtain from (25), after averaging over time,

$$j_i = -D_{ik}^{eff}\, \partial N / \partial x_k,$$
$$D_{ik}^{eff} = {}^1/_2 \xi \langle \tau u^2 \rangle \delta_{ik}. \qquad (26)$$

Equation (26) is similar to (24). The vortex rotation results in isotopy of the effective-diffusion tensor.

In the opposite limiting case $\omega\tau \gg 1$ the effective diffusion is suppressed by the decrease of the mixing length $r_0 \ll u\tau$ and by the fact that only a small region inside the separatrix participates in the transport; a rapidly oscillating perturbation of the impurity density has time to penetrate into this region (analog of the skin effect). If the skin layer exceeds substantially the boundary layer $(\lambda(\omega\tau)^{-1/2} \gg \delta$, i.e., $1 \ll \omega\tau \ll P)$ the result takes the form

$$D_{ik}^{eff} = \frac{\xi}{2} \left\langle \frac{u^2}{\omega (2\omega\tau_0)^{1/2}} \right\rangle \delta_{ik},$$
$$\tau_0 = I_0{}^2 / \bar{D}(I_0) \approx \tau. \qquad (27)$$

In the considered limiting case, a system of rotating vortices is qualitatively analogous, with respect to diffusion, to a square lattice of vortices. The function $D_{\text{eff}}(D)$ corresponds therefore to an exponent $\alpha = 1/2$.

6. CONCLUSION

Effective diffusion on vortices depends thus strongly on the type of the specified vortex flow even in the non-self-consistent approximation. This attests to the complexity of a self-consistent theory in which the vortex flow is itself induced, e.g., by temperature and density gradients in a plasma. Some problems of transport in a plasma can nevertheless be reduced to calculation of the effective diffusion on a specified system of vortices. The first example of this type is effective electronic specific heat in fast processes described by electronic magnetic hydrodynamics (EMH).[17] If the effects due to plasma inhomogeneity and to the finite electron Larmor radius are small, the electrons are described by the freezing-in equation for the curl of their generalized momentum, an equation having known vortical solutions,[17,18] while the evolution of the electron temperature is described by the convection-diffusion equation (1), in which $D = \chi_e$ is the electronic thermal diffusivity. If the plasma contains traveling electronic vortices, this makes it possible to use the estimate for the thermal diffusivity[19] (thus estimate is valid also for three-dimensional vortices):

$$\chi_e^{eff} \approx \xi \tilde{\omega}_{He}{}^2 c^4 / \omega_{pe}{}^4 \chi_e,$$

where $\tilde{\omega}_{He} = e\tilde{H}/m_e c$, $\tilde{H}$ is the characteristic magnetic field of the vortices ($\text{curl}\, \tilde{H} = -4\pi e n_0 v/c$), ω_{pe} is the plasma frequency, and ξ is the volume fraction occupied by the vortices.

Another example is the EMH resistance produced in a non-uniform plasma by convective Hall-drift of the magnetic energy $H^2/8\pi$, and its increased dissipation by various obstacles. The obstacles considered in Ref. 17, such as the plasma–vacuum interface or metallic electrodes, lead to a universal resistance estimate that does not depend on the seed conductivity σ and corresponds to total dissipation of the entire magnetic energy moving at the current velocity. In the two-dimensional case the obstacles to Hall-current transfer are also the plasma inhomogeneities far from its boundary.

Let the magnetic field of the current $\mathbf{j}(x,y) \perp \mathbf{e}_z$ be weak compared with the external uniform magnetic field $\mathbf{H}_0 \| \mathbf{e}_z$. The stationary current flow is then described by the equation

$$\mathbf{j} + \beta(x, y)[\mathbf{j}, \mathbf{e}_z] = \sigma(x, y)\mathbf{E},$$
$$\text{curl}\, \mathbf{E} = 0, \ \text{div}\, \mathbf{j} = 0. \qquad (28)$$

Here $\beta = \omega_{He}\tau_e \gg 1$ is the Hall parameter and $\sigma = n_0 e^2 \tau_e / m_e$ is the plasma conductivity. Putting $\mathbf{j} = [\nabla h(x,y), \mathbf{e}_z]$, we obtain from (28) the equation

$$\mathbf{v}\nabla h = \text{div} \frac{1}{\sigma} \nabla h, \qquad \mathbf{v} = \left[\nabla \frac{\beta}{\sigma}, \mathbf{e}_z\right], \qquad (29)$$

which is the stationary analog of the convection-diffusion equation (1) with a stream function

$$\psi = \beta/\sigma = cH_0 / (4\pi e n_0(x, y)).$$

For a randomly inhomogeneous plasma without an average density gradient, ψ is the general-form stream function considered in Sec. 2, the analog of the Peclet function being the Hall parameter β. Assuming the fluctuations of the plasma density $n_0(x,y)$ to be on the average isotropic and of the order of the mean plasma density, we obtain from (29) an expression for the average flux of the quantity h:

$$\left\langle -\frac{1}{\sigma} \nabla h + \mathbf{v}h \right\rangle = -D_{eff} \nabla \langle h \rangle,$$
$$D_{eff} \approx \sigma^{-1} \beta^{10/13}. \qquad (30)$$

Integrating in (30) by parts, we obtain a relation similar to (28) between the mean values of the current and of the electric field:

$$\langle \mathbf{j} \rangle + \beta_{eff}[\langle \mathbf{j} \rangle, \mathbf{e}_z] = \sigma_{eff} \langle \mathbf{E} \rangle,$$

where

$$\sigma_{eff} = D_{eff}^{-1} \approx \sigma\beta^{-10/13}, \qquad \beta_{eff} = D_{eff} \langle \beta/\sigma \rangle \approx \beta^{3/13}. \qquad (31)$$

Note that in the two-dimensional case the analogy between the problem of effective conductivity of an inhomogeneous medium and the problem of diffusion on vortices is not merely formal,[6] but is connected with the freezing of the magnetic field in the electrons and the diffusion of the field on account of the finite seed conductivity.

The effective conductivity and the effective Hall parameter were obtained by a different method in Ref. 7 for an inhomogeneity model in the form of a set of several phases, i.e., regions with constant microscopic values of σ and β. The result of Ref. 7 corresponds in our notation to an exponent $\alpha = 1$, since an artificial small parameter was introduced in the model of Ref. 7, viz., the ratio of the width of the transition layer to the dimension of the diffusion boundary layer in the corresponding convective-diffusion problem. For a flow of type (5) it is easy to estimate that $D_{\text{eff}} \approx |\psi|$, meaning $\alpha = 1$. Equations (31) are valid for smooth inhomogeneities.

Note that the estimate (31) is valid if the averaging is over a dimension larger than the mixing length (14). For a smaller scale, the transport is not diffusive but convective in channels $\psi = \text{const}$ of width $\Delta(l) \approx \lambda(\lambda/l)^{1/\nu}$ and length $\mathscr{L}(l) \approx \lambda(l/\lambda)^d$. The result is a size effect, i.e., a dependence of the effective conductivity on the sample size $l \times l$ (see also Ref. 17):

$$\sigma_{eff}(l) \approx \begin{cases} \sigma(\lambda/l)^{5/2}, & \lambda < l < \lambda\beta^{4/13} \\ \sigma\beta^{-10/13}, & l > \lambda\beta^{4/13} \end{cases}.$$

Normal-resistance effects when a Hall current flows in an inhomogeneously doped semiconductor[19] have the same character as the EMH resistance in a plasma, and can be described by analogous methods.

Some comments must be made concerning a recent paper[20] in which, in connection with quantum diffusion in crystals, the average convective-diffusive particle drift in a random velocity field was calculated in the framework of Eqs. (1). Averaging over a statistical ensemble of incompressible flows yielded in Ref. 20 a strange result, a superdiffusive drift $\langle r^2\rangle/t \sim \ln t \to \infty$, corresponding to infinite effective diffusion. In our opinion the reason is that in an ensemble of flows having stream functions of same order of magnitude contains some flows in which the mixing lengths exceed any prescribed value. Ensemble-averaging of the contribution of such realizations leads to a nondiffusive wandering over some finite dimension. Actually, a random velocity field must be taken not in the sense of averaging over a statistical ensemble of flows, but in the sense of structural stability of an individual flow, a situation that can be defined, in particular, as conservation of the characteristic exponent α for a small change of the velocity field.

Our conclusions, in summary, are the following:

1. Any vortical convection in a diffusion problem increases the transport. Over scales exceeding a certain mixing dimension, this transport is also diffusive with an effective diffusion coefficient (tensor). The power-law scaling (4) obtains then in the limit of large Peclet numbers.

2. Our equations (13), (24), (26), and (27) for the effective diffusion describe a large class of two-dimensional flows and certain three dimensional flows of liquids.

3. A number of problems of electric conductivity and thermal conductivity of a plasma in the presence of inhomogeneities or vortices reduce to the considered convective-diffusion problem.

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APPENDIX I

REDUCTION OF TWO-DIMENSIONAL DIFFUSION TO ONE-DIMENSIONAL

We solve (1) by successive approximations in terms of the small parameter P^{-1}. In the zeroth approximation, corresponding to discarding the right-hand side, we obtain a function of the integral of motion $n = n(I, \varphi - \omega(I)t)$ of the Hamiltonian system $\dot{x} = \partial\psi/\partial y$, $\dot{y} = -\partial\psi/\partial x$. The integrals are expressed here as functions of action and angle (I,φ)[14]:

$$\omega(I) = -d\psi/dI,$$

$$\partial(I,\varphi)/\partial(x,y) = 1, \quad \mathbf{v}\nabla = \omega(I)\partial/\partial\varphi,$$

$I(\psi)$ is the area, divided by 2π, inside the closed phase trajectory $\psi = \text{const}$, and φ is the periodic (mod 2π) coordinate on the trajectory. In the next approximation with respect to the small right-hand side there appears also a dependence of n on the "slow" time εt. Transforming to the curvilinear coordinates $(x^1, x^2) = I(\varphi)$, we get

$$\frac{\partial n(I, \varphi - \omega(I)t, \varepsilon t)}{\partial \varepsilon t} = D\frac{\partial}{\partial x^i} g^{ik}\frac{\partial n}{\partial x^k} \tag{A1}$$

Here $g^{ik}(i,\varphi)$ are the components of a metric tensor ($g^{ik} = \nabla x^i \nabla x^k$). Averaging (A1) over the fast time t and over the angle φ we obtain for the average concentration (20) an equation in which

$$\tilde{D}(I) = D\langle(\nabla I)^2\rangle_\varphi. \tag{A2}$$

APPENDIX 2

REDUCTION OF THREE-DIMENSIONAL DIFFUSION TO ONE-DIMENSIONAL

It was shown in Ref. 16 for stationary three-dimensional flows of an ideal common-position liquid there exist closed cells (vortices) inside of which one can introduce curvilinear coordinates $(I, \varphi_1, \varphi_2)$ such that I is the volume, divided by $(2\pi)^2$, made up of the stream lines $\mathbf{v}$ and of the vortex lines curl $\mathbf{v}$, i.e., I numbers the system of imbedded invariant currents, while φ_1, and φ_2(mod 2π) are the angle coordinates on them. The Jacobian of the transformation is $\partial(I,\varphi_1,\varphi_2)/\partial(x,y,z) = 1$ and the operator of differentiation along the stream line is

$$\mathbf{v}\nabla = \omega_1(I)\partial/\partial\varphi_1 + \omega_2(I)\partial/\partial\varphi_2.$$

The direct analogy of these coordinates with the action–angle variables on a two-dimensional phase plane makes it possible, by analogy with Appendix 1, to reduce three-dimensional diffusion–convection at $P \gg 1$ to a one-dimensional diffusion along the coordinate I. The result takes the form (20), in which

$$\tilde{D}(I) = D\langle(\nabla I)^2\rangle_{\varphi_1,\varphi_2}. \tag{A3}$$

[1)] If $N \geqslant 3$ the $\psi(x,y)$ level lines can be represented as an intersection of a family of $(N-1)$-dimensional periodic hypersurfaces $\sin x_1 + \dots + \sin x_N = \text{const}$, in an N-dimensional space with a two-dimensional hyperplane $x_n = \mathbf{k}_n \cdot \mathbf{r}, n = 1,\dots,N$. The flows considered have thus a definite "periodicity trace" typical of quasicrystals,[13] which should vanish asymptotically as $N \to \infty$. We note also without proof a special case that takes place at $N = 3$, when the infinite $\psi(x,y)$ level lines are non-fractal (with $d = 1$) since they are contained entirely in parallel strips of finite width. An exponent $\alpha = 5/6$ can be obtained for this special flow.

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