Radiative corrections to the axial anomaly

A. A. Ansel'm and A. A. Iogansen

B. P. Konstantinov Leningrad Institute of Nuclear Physics, Academy of Sciences of the USSR
(Submitted 19 May 1989)
Zh. Eksp. Teor. Fiz. **96**, 1181–1203 (October 1989)

It is shown that, contrary to the generally held view, the axial anomaly is not saturated by the one-loop value. Contributions that were not previously taken into account arise in diagrams for scattering of light by light (or gluons by gluons). Radiative corrections are calculated in QED and in nonabelian gauge theories. It is shown that in supersymmetric QED these multi-loop contributions explain the apparent contradiction between the axial anomaly and the anomaly in the trace of the energy-momentum tensor. The situation is more complicated in supersymmetric nonabelian theories due to the gauge dependence of matrix elements off shell.

1. INTRODUCTION

It has been assumed for many years that the axial anomaly is saturated by the one-loop contribution (the Adler-Bardeen theorem[1]). In supersymmetric (SUSY) theories this anomaly is part of the same supermultiplet with the trace of the energy-momentum tensor. The latter is proportional to the β-function and therefore cannot be saturated by the one-loop contribution. A number of attempts have been made to resolve this apparent paradox.[2–15]

In this paper we will show that in any theory (whether SUSY or not is immaterial) the axial anomaly has multi-loop corrections. The source of these contributions are diagrams for scattering of light on light or gluons on gluons: in fact the fundamental role is played by evolution of the operators $F_{\mu\nu}\tilde{F}^{\mu\nu}$ (for QED) or $G_{\mu\nu,a}\tilde{G}_a^{\mu\nu}$ (for the Yang-Mills theory). Previously diagrams with scattering of light on light were thrown out on dimensional considerations, which are not valid in a theory with massless fermions.

In fact, there is a difference between the operator formulation of the equation for the anomaly and the equation for the matrix element of the divergence of the axial current. In actuality the one-loop character of the anomaly in the operator form is conditional: it is due to a particular choice of regularization of the axial current. For example, in the nonsupersymmetric case the one-loop form of the operator equation corresponds to the "natural" regularization, i.e, with the help of Pauli-Villars fermions. However, it can be shown that for a different gauge-invariant regularization of the current this equation can be of multi-loop form. Regularizations of this type may seem rather unnatural. Nonetheless, in a supersymmetric theory this is the only possibility for including the axial current j_μ^5 and the conserved energy-momentum tensor in the same supermultiplet, i.e., for preserving supersymmetry. For SUSY the ambiguity in the operator equation (its one-loop or multi-loop character) has been demonstrated previously in Ref. 14. In this work we will show that in terms of renormalized matrix elements the final result is of multi-loop character and does not depend on the regularization procedure.

Thus the widely held view that gauge invariance of the regularization is sufficient to unambiguously fix the result turns out to be only partly correct. It is actually correct for the renormalized matrix elements, while the form of the operator equations is conditional. Since the matrix elements have unambiguously multi-loop contributions, the Adler-Bardeen theorem must be recognized to be wrong.

The difference between the operator and the matrix-element forms of the anomaly equations was first recognized as the key point in the analysis of SUSY theory in Ref. 15. In actuality the multi-loop character of the anomaly has no relation to supersymmetry: the SUSY "paradox" is just a particular manifestation of a general situation.

The content of this paper is as follows. In Sec. 2 we discuss nonsupersymmetric QED and calculate radiative corrections to the divergence of the axial current. Nonsupersymmetric Yang-Mills theory is discussed in Sec. 3. In Sec. 4 we consider SUSY versions of QED and Yang-Mills theory. In the final Sec. 5 we summarize the main results of the paper and discuss some possible consequences. A brief description of the results of this work was presented in our letter, Ref. 16.

2. THE AXIAL ANOMALY IN QED

As is well known from the Adler-Bardeen analysis,[1] the two-loop corrections to the axial anomaly are absent (see the diagrams in Fig. 1). The authors of Ref. 13 interpret this fact as due to the so-called two-limit technique used in Ref. 1. In Appendix 1 we demonstrate with the help of explicit calculations by the background-field method that the latter assertion is incorrect: the two-loop corrections vanish already before integration over the coordinates of the vertices in the corresponding diagrams. Indeed, as we shall see below, these two-loop contributions are of conditional character, i.e., depend on the regularization scheme. In actuality the Adler-Bardeen assertion is connected with the Pauli-Villars procedure for regularization of the current. In Secs. 2 and 3 (i.e., for nonsupersymmetric theories) we shall adopt precisely this regularization.

Our main observation is that diagrams for scattering of light on light, which were thrown out in Ref. 1 on dimension-

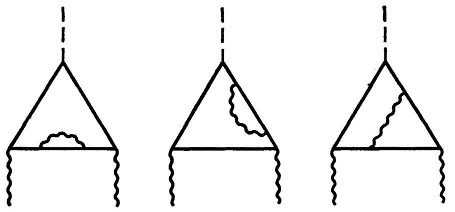

FIG. 1.

0038-5646/89/100670-13$04.00

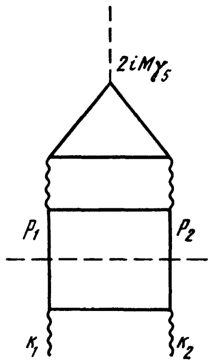

FIG. 2.

al grounds, should be taken into account in theories with massless fermions. Indeed, the diagrams for the axial anomaly (see Fig. 2) were estimated in Ref. 1 as $F_{\mu\nu}\tilde{F}^{\mu\nu}k_1k_2/m^2$ (m—fermion mass, $F_{\mu\nu}$—external field strength, k_1 and k_2—momenta of the external photons). Here the factor k_1k_2/m^2 is due to scattering of light on light. However, for $m=0$ (actually for $m^2 \ll |k_1^2|, |k_2^2|, |k_1k_2|$) this estimate is incorrect: the amplitude for scattering of light on light turns out to be of order unity.

We have explicitly calculated the diagram of Fig. 2 by the background-field method (see Appendix 2). In spite of the resultant unwieldy intermediate expressions, containing nontrivial dependence on the external momenta, the final result is very simple. We have for the matrix element $\langle \partial_\mu j_\mu^5 \rangle$ in the background electromagnetic field the following

$$\langle \partial_\mu j_\mu{}^5 \rangle = (F_{\mu\nu}\tilde{F}^{\mu\nu})_{ext}\frac{e_0{}^2}{8\pi^2}\left(1-\frac{3e_0{}^4}{64\pi^4}\ln\frac{\Lambda^2}{k^2}\right). \qquad (1)$$

Here Λ is the ultraviolet cut-off parameter, k is a typical external momentum, and the coefficient in front of $\ln \Lambda^2$ does not depend on the relations between k_1^2, k_2^2 and $k^2=(k_1+k_2)^2$. We have ignored in the brackets in Eq. (1) finite terms of order $\sim e_0^4$.

The expression, Eq. (1), may be understood as follows. Consider the diagram in Fig. 2. In calculating the divergence of the current j_μ^5, regularized with the help of the Pauli-Villars procedure, the upper triangle is formed by lines of the regulator fermion of mass M, with $2iM\gamma_5$ appearing at the vertex. As $M\to\infty$ the triangle may be contracted to a point, which corresponds to the operator $(e_0^2/8\pi^2)F\tilde{F}$. Suppose now that we cut the diagram, as is shown in Fig. 2. The remaining one-loop integration in the upper part of the diagram diverges logarithmically and gives

$$-\frac{3e_0{}^4}{64\pi^4}\ln\left(\frac{\Lambda^2}{k^2}\right)(p_1+p_2)_\mu(\bar{\psi}(p_1)\gamma_\mu\gamma_5\psi(p_2))$$

where $p_1+p_2=k$, and $\bar{\psi}(p_1)$ and $\psi(p_2)$ are the wavefunctions of the light virtual fermions. We now contract the entire upper part of the diagram to a point and run into the problem of evaluating the triangle diagram corresponding to the ordinary axial anomaly. The latter calculation gives $(e_0^2/8\pi^2)(F\tilde{F})_{ext}$. In this way we obtain the second term in Eq. (1).

The common factor e_0^2 in Eq. (1) is transformed into the square of the renormalized charge upon taking into account radiative corrections to the external photon lines. The expression in the brackets in Eq. (1) may be made cut-off independent with the help of multiplicative current renormalization $j_\mu^5 \to zj_\mu^5 = (j_\mu^5)_{ren}$. Taking into account that

$$e_0{}^2=e^2(k)[1+\beta_1e^2(k)\ln(\Lambda^2/k^2)], \quad \beta_1=1/12\pi^2,$$

we find

$$z=1+3e_0{}^2/64\pi^4\beta_1=1+9e_0{}^2/16\pi^2,$$

so that

$$\langle \partial_\mu j_\mu{}^5 \rangle_{ren}=z\langle \partial_\mu j_\mu{}^5 \rangle=(F\tilde{F})_{ext}\frac{e^2(k)}{8\pi^2}\left(1+\frac{9e^2(k)}{16\pi^2}\right). \qquad (2)$$

The need for renormalization of Eq. (1) is entirely natural since the current j_μ^5 is not conserved. Explicitly, renormalization of the current j_μ^5 (or $\partial_\mu j_\mu^5$) is determined by the diagram in Fig. 3, which, in fact, is the upper part of the diagram in Fig. 2. As was already mentioned, direct evalution of this diagram gives

$$\partial_\mu j_\mu{}^5 \to \partial_\mu j_\mu{}^5\left(1-\frac{3e_0{}^4}{64\pi^4}\ln\frac{\Lambda^2}{k^2}\right). \qquad (3)$$

It is clear that this amplitude is renormalized by the same factor $z=1+9e_0^2/16\pi^2$ as Eq. (1).

The multiplicative renormalization procedure for the operator j_μ^5 proposed here may at first glance seem dubious, since another current with the same quantum numbers and dimensions as j_μ^5 exists, namely the current K_μ where $\partial_\mu K_\mu = (e_0^2/8\pi^2)F\tilde{F}$. However the proposed renormalization recipe is correct because the gauge-noninvariant operator K_μ cannot be mixed with the gauge-invariant current j_μ^5. We shall discuss this question in more detail when investigating the operator equations.

Of course the need for renormalization of the operator j_μ^5 is well known (see, for example, the original work of Adler,[17] and also Ref. 18). However, to our knowledge, the contribution from scattering of light by light to the right-hand side of Eq. (1) has never been previously taken into account.

It is instructive to see what happens for nonzero mass m of the physical fermion. Evaluation of that same diagram (Fig. 2), but in the case $m\neq 0$ (see Appendix 2), gives

$$-\frac{3e_0{}^4}{64\pi^4}\ln\left(\frac{\Lambda^2}{k^2}\right)(1+2m^2I_{00})\frac{e_0{}^2}{8\pi^2}(F\tilde{F})_{ext},$$

$$I_{00}(k_1,k_2)=\int_0^1 dx\int_0^{1-x} dy[x(1-x)k_1{}^2+y(1-y)\times k_2{}^2+2xyk_1k_2-m^2]^{-1}. \qquad (4)$$

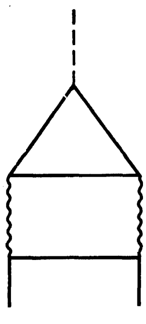

FIG. 3.

Here again the only contribution comes from the diagram with the regulator fermion in the triangle. The diagram with the physical fermion in the triangle may be ignored because the size of the triangle subdiagram drops off for large virtual photon momenta and, therefore, no contributions containing ln Λ appear.

The expression, Eq. (4), should be added to the one-loop contribution to the anomaly. The latter has the same factor $1 + 2m^2 I_{00}$ as in Eq. (4). Thus we obtain

$$\langle \partial_\mu j_\mu^5 \rangle = \frac{e_0^2}{8\pi^2} (F\tilde{F})_{ext} \left(1 - \frac{3e_0^4}{64\pi^4} \ln \frac{\Lambda^2}{k^2}\right)(1+2m^2 I_{00}). \tag{5}$$

For $m^2 \gg k^2$, $1 + 2m^2 I_{00} \to 0$, i.e., the right-hand side of Eq. (5) vanishes. Here the vanishing of the one-loop contribution is actually due to the cancellation of the diagrams with the physical and regulator fermions (which is a manifestation of the Sutherland-Veltman theorem), while for the second term in Eq. (5) ($\sim \ln \Lambda$) this vanishing corresponds to the arguments of Adler and Bardeen on the suppression of scattering of light by light. Renormalization of the expression, Eq. (5), is accomplished by the same z-factor as for Eq. (4), the result being (for $\langle \partial_\mu j_\mu^5 \rangle$) a series in powers of $e^2(k)$. In this sense the Adler-Bardeen theorem is incorrect for the $m \neq 0$ case also.

Equation (1) permits a renormalization-group generalization. Making use of the standard technique (the Callan-Symanzik equation) we easily derive the following exact equations:

$$\langle \partial_\mu j_\mu^5 \rangle = \frac{e^2(k)}{8\pi^2} (F\tilde{F})_{ext} \varphi\left(e_0^2, \ln \frac{\Lambda^2}{k^2}\right)$$

$$= \frac{e^2(k)}{8\pi^2} (F\tilde{F})_{ext} \varphi(e^2(k), 0) \frac{z(e^2(k))}{z(e_0^2)} \tag{6}$$

where

$$z(e^2) = \exp \int_0^{e^2} \frac{\gamma(e_1^2)}{\beta(e_1^2)} de_1^2. \tag{7}$$

Here $\beta(e^2)$ is the Gell-Mann–Low function and $\gamma(e^2)$ is the anomalous dimension of the operator j_μ^5. The absence of two-loop corrections to the anomaly means that $\varphi(e_0^2, 0) = 1 + O(e_0^4)$. By using the perturbative values $\beta(e^2) = e^4/12\pi^2$ and $\gamma(e^2) = 3e^4/16\pi^4$ we are back to Eq. (2).

We have found radiative corrections to the amplitude for the transition of $\partial_\mu j_\mu^5$ into two photons. This contradicts the Adler-Bardeen theorem as formulated by the authors themselves[1] and as often cited in textbooks (see, for example, Ref. 19). However, some prefer to speak in terms of operators, and not their matrix elements. As we shall see below, in a certain sense the operator language permits the preservation of the one-loop character of the anomaly equation. To understand this we consider the evolution of the two operators $\partial_\mu j_\mu^5$ and $(e^2/8\pi^2)F\tilde{F}$. It follows from the diagrams of the type in Figs. 2 and 3 that the reduced matrix elements of these operators have the form

$$M = \begin{pmatrix} \langle\langle \psi\bar{\psi} | \partial_\mu j_\mu^5 | 0 \rangle\rangle & \langle\langle 2\gamma | \partial_\mu j_\mu^5 | 0 \rangle\rangle \\ \left\langle\left\langle \psi\bar{\psi} \left| \frac{e^2}{8\pi^2} F\tilde{F} \right| \right\rangle\right\rangle & \left\langle\left\langle 2\gamma \left| \frac{e^2}{8\pi^2} F\tilde{F} \right| 0 \right\rangle\right\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{3e_0^4}{64\pi^4} \ln \frac{\Lambda^2}{k^2} & 1 - \frac{3e_0^4}{64\pi^4} \ln \frac{\Lambda^2}{k^2} \\ -\frac{3e_0^4}{64\pi^4} \ln \frac{\Lambda^2}{k^2} & 1 - \frac{3e_0^4}{64\pi^4} \ln \frac{\Lambda^2}{k^2} \end{pmatrix}. \tag{8}$$

The order $\sim e_0^4$ contributions in the second column arise as a result of scattering of light by light. In Eq. (8), by definition,

$$\langle 2\gamma | \ldots | 0 \rangle = \frac{e^2}{8\pi^2} (F\tilde{F})_{ext} \langle\langle 2\gamma | \ldots | 0 \rangle\rangle, \quad \langle \psi\bar{\psi} | \ldots | 0 \rangle$$
$$= \partial_\mu (\bar{\psi}\gamma_\mu\gamma_5\psi)_{ext} \langle\langle \psi\bar{\psi} | \ldots | 0 \rangle\rangle,$$

where ψ_{ext} and F_{ext} are external fermion and photon fields.

We introduce the matrix z to renormalize the matrix elements:

$$z = \begin{pmatrix} 1+9e_0^2/16\pi^2 & 0 \\ 9e_0^2/16\pi^2 & 1 \end{pmatrix}. \tag{9}$$

By using the relation

$$e_0^2 = e^2(k)\left[1 + (e^2/12\pi^2) \ln (\Lambda^2/k^2)\right],$$

it can be seen that the product zM is finite.

It follows from Eq. (9) that $\partial_\mu j_\mu^5$ is renormalized multiplicatively. As was already mentioned, the reason is that the gauge-noninvariant current K_μ cannot be locally admixed to j_μ^5. The mixing which actually does take place is of the form $\delta j_\mu^5 = (k_\mu k_\nu / k^2) K_\nu$. This guarantees proportionality of $\partial_\mu j_\mu^5$ and $\partial_\mu K_\mu$, and the pole at $k^2 = 0$ in the matrix element $\langle 2\gamma | j_\mu^5 | 0 \rangle$ is well known.[20] The expression $(k_\mu k_\nu / k^2) K_\nu$ is nonlocal in coordinate space and requires no special counterterms in any order in e^2.

The matrix, Eq. (9), corresponds to the following matrix of anomalous dimensions[18]:

$$\gamma = \frac{d \ln z}{d \ln \Lambda^2} = \begin{pmatrix} 3e_0^4/64\pi^4 & 0 \\ 3e_0^4/64\pi^4 & 0 \end{pmatrix}. \tag{10}$$

From this we see immediately that the difference of the operators $\partial_\mu j_\mu^5 - (e_0^2/8\pi^2) F\tilde{F}$ is renormalization-group invariant, i.e., does not depend on the normalization point Λ. Consequently, for an arbitrary normalization point the operator equation for the anomaly,

$$\partial_\mu j_\mu^5 - (e^2/8\pi^2) F\tilde{F} = 0, \tag{11}$$

has no perturbative corrections. The charge e^2 and the operators in Eq. (11) are normalized, naturally, at the same point. In this fashion the operator form of the Adler-Bardeen theorem is reestablished.

Equation (11) is quite understandable in diagram language. Upon taking matrix elements of Eq. (11) we have

$$\langle 2\gamma | \partial_\mu j_\mu^5 | 0 \rangle = \langle 2\gamma | (e^2/8\pi^2) F\tilde{F} | 0 \rangle. \tag{12}$$

This means that diagrams of the form Fig. 2 with rescattering of photons are the same for both sides of Eq. (12). In other words, their size does not depend on whether the photons are emitted directly from the point two-photon vertex $(e^2/8\pi^2) F\tilde{F}$ or from the triangle formed by the regulator fermions.

Combining Eqs. (2) and (12) we also have

$$\langle 2\gamma|(e^2/8\pi^2)F\tilde{F}|0\rangle_{ren}=(e^2/8\pi^2)(F\tilde{F})_{ext}(1+9e^2/16\pi^2). \quad (13)$$

Consequently, the widely held view that the matrix element of the operator $e^2F\tilde{F}$ is not renormalized is erroneous.

3. THE AXIAL ANOMALY IN THE YANG-MILLS THEORY

In the Yang-Mills theory one must first investigate (two-loop) diagrams with Born rescattering of gluons (see Fig. 4). These diagrams give rise to corrections $\sim g_0^2$ in the equation for the anomaly. The contributions of diagrams of the type shown in Fig. 2 discussed above are $\sim g_0^4 \ln(\Lambda^2/k^2)$, which, in fact, are transformed to become of order $g(k^2)$ after renormalization of the current j_μ^5. Diagrams of the type shown in Figs. 2 and 3 differ in Yang-Mills theory, as compared to QED, only by the obvious group factor $T(R)C_2(R)$, where the fermions are in the representation R of the gauge group. As regards diagrams with one-loop gluon rescattering of the form, say, of Fig. 5, they give rise to renormalization of the contributions $\sim g_0^2$ from the diagrams in Fig. 4. It is clear that taking into account such diagrams (Fig. 5) does not result in the need for additional renormalization of the current j_μ^5 itself, similar to the one connected with the scattering of gluons through the fermion loop. This is clear from the structure of the diagrams for the evolution of the current $j_\mu^5 \to j_\mu^5$ (Fig. 3). For this reason in what follows we do not discuss diagrams of the type shown in Fig. 5.

In Appendix 3 we explicitly calculate contributions of order $\sim g_0^2$ from the diagrams in Fig. 4 (they are finite). The nontrivial part of this calculation has to do with the fact that the upper triangle, formed by the lines of the regulator fermion, must not be naively contracted to a point, although, at first glance, this could be done. The point is that following such contraction the remaining one-loop diagram is ill-defined and requires special regularization, with the result depending on the choice of regularization. In the diagram of Fig. 4 itself the triangle regularizes gluon-gluon scattering in a natural way. Gluon rescattering of the type shown in Fig. 4 was first considered in Ref. 15. The result obtained in Ref. 15 does not agree with ours (see below for a detailed comparison). The contribution of the diagram in Fig. 4 refers, in fact, to the quantity $\langle G_{\mu\nu,a}\tilde{G}_a^{\mu\nu}\rangle$ in the regularization of the operator $G\tilde{G}$ by means of insertion of the regulator triangle. It is shown in Appendix 3 that the expression for $\langle G\tilde{G}\rangle$ in the $SU(N)$ gauge group has the form

$$\langle G_{\mu\nu}{}^a\tilde{G}^{\mu\nu,a}\rangle=(G_{\mu\nu}{}^a\tilde{G}^{\mu\nu,a})_{ext}\Big\{1+\frac{Ng^2(k)}{4\pi^2}\Big(1-\frac{1}{2}k^2I_{00}\Big) -(\alpha-1)\frac{Ng^2(k)}{8\pi^2}\Big[1-\frac{1}{4}k^2\Big(k_1^2\frac{\partial}{\partial k_1^2}+k_2^2\frac{\partial}{\partial k_2^2}\Big)I_{00}\Big] -(\alpha-1)^2\frac{Ng^2(k)}{32\pi^2}\Big\}. \quad (14)$$

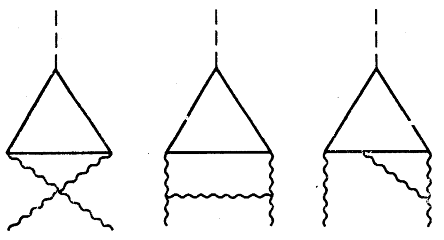

FIG. 4.

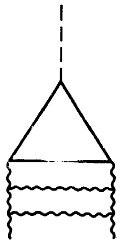

FIG. 5.

Here the function of external momenta $I_{00}=I_{00}(k_1^2,k_2^2,k_1k_2)$ is determined by Eq. (4) for $m=0$; $k=k_1+k_2$ is the total momentum of the external gluons. Derivatives with respect to k_1^2 and k_2^2 are taken at fixed value of k_1k_2. Equation (14) is obtained in the background-field gauge, which is fixed by the addition to the Lagrangian of the term $(1/2\alpha)(\nabla_\mu(B)A_\mu)^2$ (B_μ is the background field, A_μ is the quantum gluon field).

The dependence of the right-hand side of Eq. (14) on the gauge parameter α should not be surprising since the quantity $\langle G\tilde{G}\rangle$ is being calculated for off-shell external gluons: $k_1^2\neq 0$, $k_2^2\neq 0$. Passage to the mass shell is impossible due to the singular infrared behavior for $k_1^2,k_2^2\to 0$. One may, however, consider the limit $k_1^2,k_2^2\ll k^2=(k_1+k_2)^2$. The leading (doubly logarithmic) asymptotic behavior is contained in the term $\sim I_{00}$ and does not depend on α:

$$\langle G_{\mu\nu}{}^a\tilde{G}^{\mu\nu,a}\rangle\approx(G_{\mu\nu}{}^a\tilde{G}^{\mu\nu,a})_{ext}\Big[1-\frac{Ng^2(k)}{8\pi^2}\ln\frac{k^2}{k_1^2}\ln\frac{k^2}{k_2^2}\Big]. \quad (15)$$

One may also attempt to get around the difficulty with the passage to the mass shell by introducing an infrared cut off for the momenta of the virtual gluons, for example by introducing into the gluon propagator a mass μ. Let $k_1^2,k_2^2\ll\mu^2\ll k^2$. It can be shown that in that case the terms $\sim(\alpha-1)$ and $\sim(\alpha-1)^2$ go to zero algebraically as k_1^2/μ^2, k_2^2/μ^2. There remain in Eq. (14) only the first two terms, corresponding to the Feynman gauge $\alpha=1$ (it was this expression that was presented in our work, Ref. 16). We also note that for $k_1^2,k_2^2,(k_1+k_2)^2\ll\mu^2$ all radiative corrections in Eq. (14) disappear.

Let us compare Eq. (14) with the answer given in Ref. 15. The case considered there was $k_1^2,k_2^2=0,\mu^2\ll(k_1+k_2)^2$. As already mentioned only the first two terms in Eq. (14) are relevant in that case, the main contribution coming from the doubly logarithmic asymptotic form $k^2I_{00}\approx[\ln(k^2/\mu^2)]^2$. In Ref. 15 the term with I_{00} was omitted. Thus our answer disagrees with the result of Ref. 15.

We give now the full expression for the renormalized matrix element of the divergence of the axial current with the contribution of gluon rescattering taken into account through the fermion loop:

$$\langle\partial_\mu j_\mu{}^5\rangle=(G_{\mu\nu}{}^a\tilde{G}^{\mu\nu,a})_{ext}\frac{g^2(k)T(R)}{8\pi^2}\Big[1-\frac{3g^2(k)}{4\pi^2}\frac{T(R)C_2(R)}{{}^{11}/_3N-{}^4/_3T(R)}+\frac{Ng^2(k)}{4\pi^2}\Big(1-\frac{1}{2}k^2I_{00}\Big)\Big]. \quad (16)$$

Equation (16) is given for the case of the Feynman gauge $\alpha=1$; terms $\sim(\alpha-1)$ and $\sim(\alpha-1)^2$ are given in Eq. (14).

Up to now we have been discussing the size of the matrix element of the operator $\partial_\mu j_\mu^5$. As far as the operator equation for the anomaly is concerned it is easy to see that the matrix of anomalous dimensions in the Yang-Mills theory remains the same [with the obvious substitution $e^4 \to g^4 T(R) C_2(R)$] as in Eq. (10). This means that in the framework of regularization here adopted (Pauli-Villars) the operator equation has the one-loop form and coincides with Eq. (11) [after the replacement $e^2 \to g^2 T(R)$].

4. ANOMALY IN SUPERSYMMETRIC THEORIES

We turn now to SUSY theories. The following set of superfields is present in SUSY QED: the vector superfield V (the super field strength W_α contains $F_{\mu\nu}$ and the photino field λ_α) and two chiral matter superfields T and U. In this model there are two anomalous axial currents, which are the components of two supermultiplets $J = U^+ e^{-V} U + T^+ e^V T$ and $J_{\alpha\dot\alpha}$ (see, for example, Ref. 15),

$$J_{\alpha\dot\alpha} = (1/g^2) W_\alpha \overline{W}_{\dot\alpha} - {}^1/_6 D_\alpha (e^V T) e^{-V} D_{\dot\alpha}(e^V \overline{T}) + {}^1/_6 T e^V D_\alpha (e^{-V} \overline{D}_{\dot\alpha}(e^V \overline{T})) + {}^1/_6 T \overline{D}_{\dot\alpha}(e^V D_\alpha \overline{T}) + (T \to U,\ V \to -V).$$

Here, in contrast to the previous sections, we have included the coupling constant in the definition of the field V. The composite superfield $J_{\alpha\dot\alpha}$ contains in addition to j_μ^5 the energy-momentum tensor $\theta_{\mu\nu}$ and the supercurrent $S_{\mu\alpha}$. At the quantum level the current divergences $\overline{D}^2 J$ and $\overline{D}^\alpha J_{\alpha\dot\alpha}$ are proportional to $W^2 \equiv W^\alpha W_\alpha$: for the supermultiplet $\overline{D}^2 J$ this corresponds to the so-called Konishi anomaly; the supersymmetric anomaly in $\overline{D}^\alpha J_{\alpha\dot\alpha}$ contains in addition to the axial anomaly the conformal and superconformal anomalies. The quantity $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is contained in the imaginary part of the F-term of the chiral superfield W^2. Therefore to take into account scattering of light on light it is necessary to find radiative corrections to $\langle W^2 \rangle$ in the external superfield $V_{\rm ext}$.

We consider the generating functional Z in the presence of the external vector superfield $V_{\rm ext}$:

$$Z = \int DVDUDT \exp[iS(V + V_{ext}, U, T)],$$
$$S = \int d^4x \left\{ \frac{1}{4g_0^2}([W^2]_F + \text{h.c.}) + [U^+ e^{-V} U + T^+ e^V T]_D \right\}. \tag{17}$$

We have omitted the gauge-fixing terms because in electrodynamics they are irrelevant to what follows. The logarithmic derivative of Z with respect to g_0^2 gives the exact expression for the vacuum expectation value $\langle W^2 \rangle$ in the external superfield $V_{\rm ext}$, integrated over $d^4x d^2\theta$:

$$\int d^4x\, d^2\theta \langle W^2 \rangle + \text{h.c.} = -4i[\partial/\partial(1/g_0^2)] \ln Z. \tag{18}$$

On the other hand, ln Z is nothing but the effective action:

$$Z = \exp(iS_{eff}), \quad S_{eff} = \int d^4x\, d^2\theta \frac{1}{4g^2(k)} W_{ext}^2 + \text{h.c.} \tag{19}$$

Here k is a typical momentum of the external field. In Eq. (19) we have kept only the term quadratic in $W_{\rm ext}$. This equation is valid for $k^2 \gg W_{\rm ext}^2$. If we now assume that the integration over x and θ in Eq. (19) may be omitted, then upon comparison of Eqs. (18) and (19) we obtain separate equations for the chiral superfields W^2 and $\overline{W}^2$:

$$\langle W^2 \rangle = \frac{\partial}{\partial(1/g_0^2)} \frac{1}{g^2(k)} W_{ext}^2 = \frac{g_0^4 \beta(g^2(k))}{g^4(k) \beta(g_0^2)} W_{ext}^2$$
$$= \frac{\beta(g^2)}{\beta_1(g^2)} \frac{\beta_1(g_0^2)}{\beta(g_0^2)} W_{ext}^2, \tag{20}$$

where β and β_1 are, respectively, the exact and the one-loop Gell-Mann–Low functions. We have verified Eq. (20) in the two-loop approximation $\sim g_0^4 \ln(\Lambda^2/k^2)$ with the help of an explicit calculation by the method of supergraphs developed in Ref. 21. The details of this calculation are given in Appendix 4.

It was shown in Ref. 14 that there exist two regularized expressions for the superfield operator $J_{\alpha\dot\alpha}$. One of them corresponds to the absence of two-loop corrections in the operator equation for $\overline{D}^\alpha J_{\alpha\dot\alpha}$, and therefore also for $\partial_\mu j_\mu^5$, so that accurate to $\sim g^2$ the operator $\overline{D}^\alpha J_{\alpha\dot\alpha}$ seems to obey the generalized Adler-Bardeen theorem. However, due to scattering of light on light the three-loop contribution $\sim g_0^2 \ln(\Lambda^2/k^2)$ to the corresponding matrix element $\langle \overline{D}^\alpha J_{\alpha\dot\alpha} \rangle$ does not vanish [as can be seen from Eq. (20)]. After the appropriate multiplicative renormalization with the help of the factor $z(g_0) = \beta(g_0^2)/\beta_1(g_0^2)$ the matrix element of the anomalous divergence is proportional to $\beta(g^2)/\beta_1(g^2)$. The situation turns out to be fully analogous to the nonsupersymmetric case, and the validity of the expression given above for $z(g_0^2)$ can be verified by explicit calculation of the anomalous dimension of the current. In this way, in the regularization being considered the operator equation for $\partial_\mu j_\mu^5$ has no multi-loop corrections, but the matrix element is given by

$$\langle \partial_\mu j_\mu^5 \rangle = (1/8\pi^2)[\beta(g^2)/\beta_1(g^2)](F\tilde{F})_{ext}.$$

However, as was shown in Ref. 14, this regularization is unsatisfactory in view of the circumstance that the component $\theta_{\mu\nu}$ of the superfield $J_{\alpha\dot\alpha}$ is not conserved, in spite of the fact that at the classical level $\theta_{\mu\nu}$ corresponds to the energy-momentum tensor.

With another definition of the regularized current operator $J_{\alpha\dot\alpha}$ it contains the conserved component $\theta_{\mu\nu}$, corresponding to the energy-momentum tensor not only at the classical but also at the quantum level. In that case the authors of Ref. 14 found two-loop corrections to the anomaly equation, which agree with the appearance in the anomaly equation of the exact β-function. This is not surprising since now the current j_μ^5 needs no renormalization (its anomalous dimension, just like the anomalous dimension of the conserved tensor $\theta_{\mu\nu}$, vanishes). This means that the corrections, not connected with scattering of light on light, ensure the appearance of the factor $\beta(g_0^2)/\beta_1(g_0^2)$, which cancels the factor $\beta_1(g_0^2)/\beta(g_0^2)$ in Eq. (18) for the evolution of W^2. As a result the anomaly again turns out to be proportional to $\beta(g_0^2)/\beta_1(g_0^2)$. However now this factor should be ascribed rather to the function $\varphi(g^2, 0)$ in Eq. (6), while $z = 1$.

In this way we should make a precise distinction between the SUSY axial current and the Adler-Bardeen current. For the definition of the SUSY current we may utilize the condition that its matrix elements should be finite without any renormalization. As far as the Adler-Bardeen current is concerned, it may be defined by the requirement that all the radiative corrections to its matrix elements should vanish at the normalization point $k^2 = \Lambda^2$, where Λ is the ultraviolet cut-off parameter. This means that perturbatively such a current does not contain so-called finite terms of the type $1 + c_1 g_0^2 + c_2 g_0^4 + \dots$. Obviously, the matrix elements of such a current are proportional to a certain factor $z(g^2)/z(g_0^2)$ [see Eq. (6)].

We have discussed the appearance of the full β-function in the anomaly equation for $J_{\alpha\dot\alpha}$, but not for J. However now it is obvious that the superdivergence of the Konishi current J is also proportional to β. The real reason for this has to do with the fact that the multi-loop corrections are always determined by scattering of light on light, i.e., by the evolution of the quantity W^2, and do not depend on the type of current, $J_{\alpha\dot\alpha}$ or J. This argument is correct not only for SUSY QED but also for SUSY Yang-Mills theory, which will be considered below. In addition, in SUSY QED the Konishi current J includes the same charged fermions as $J_{\alpha\dot\alpha}$. Therefore, no matter what the operator form of the anomaly equation, the renormalized matrix elements of the operators $\bar D^2 J$ and $\bar D^{\dot\alpha} J_{\alpha\dot\alpha}$ should be the same, i.e., proportional to the β-function. In other words, the final result for the renormalized matrix elements is of the form

$$\langle \bar D^{\dot\alpha} J_{\alpha\dot\alpha}\rangle_{ren} = -[\beta(g^2)/6g^4] D_\alpha W^2_{ext},$$
$$\langle \bar D^2 J\rangle_{ren} = [\beta(g^2)/g^4] W_{ext}. \qquad (21)$$

To our knowledge, the multi-loop form of the Konishi equation [the second of Eqs. (21)] has not been discussed previously.

We consider now the operator formulation of the anomaly equations in SUSY QED. In that theory there are three superfield operators which may mix with each other, namely $\bar D^{\dot\alpha} J_{\alpha\dot\alpha}$, $D_\alpha W^2$ and $D_\alpha \bar D^2 J$. By making use of Eq. (20) and analyzing the ladder structure of the diagrams with rescattering of light (see below) we are able to obtain matrix elements of these operators in external superfields V_{ext}, T_{ext}, and U_{ext}. We have

$$\begin{pmatrix} \langle \bar D^{\dot\alpha} J_{\alpha\dot\alpha}\rangle \\ \langle D_\alpha W^2\rangle \\ \langle D_\alpha \bar D^2 J\rangle \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{\beta(g^2)}{6g^4} & -\dfrac{1}{6}\left[\dfrac{\beta(g^2)}{\beta_1(g^2)} - \dfrac{\beta(g_0^2)}{\beta_1(g_0^2)}\right] \\ 0 & \dfrac{\beta(g^2)}{\beta_1(g^2)}\dfrac{\beta_1(g_0^2)}{\beta(g_0^2)} & \left[\dfrac{\beta(g^2)}{\beta_1(g^2)} - \dfrac{\beta(g_0^2)}{\beta_1(g_0^2)}\right]\dfrac{g_0^4}{\beta(g_0^2)} \\ 0 & \dfrac{\beta(g^2)}{g^4}\dfrac{\beta_1(g_0^2)}{\beta(g_0^2)} & \dfrac{\beta(g^2)}{\beta_1(g^2)}\dfrac{\beta_1(g_0^2)}{\beta(g_0^2)} \end{pmatrix} \begin{pmatrix} \bar D^{\dot\alpha} J_{\alpha\dot\alpha} \\ D_\alpha W \\ D_\alpha \bar D^2 J \end{pmatrix}. \qquad (22)$$

Generally speaking, the specific form of the matrix in Eq. (22) is fixed by three conditions: 1) the Konishi supermultiplet J is regularized in the usual manner, say by means of Pauli-Villars superfields; 2) the current $J_{\alpha\dot\alpha}$ is regularized in such a way as to contain the conserved operator $\theta_{\mu\nu}$; and 3) the composite operator W^2 is determined in agreement with Eq. (20). We now make clear in greater detail the form of the matrix elements in Eq. (22). We take the value of the amplitude $\langle J_{\alpha\dot\alpha} \to J_{\alpha\dot\alpha}\rangle$ to be equal to unity, since $J_{\alpha\dot\alpha}$ contains the conserved energy-momentum tensor $\theta_{\mu\nu}$ and, consequently, is not renormalized. This circumstance also explains the two zero matrix elements in Eq. (22): if that were not so the further evolution of the operators W^2 and J in $J_{\alpha\dot\alpha}$ would induce corrections to the amplitude $\langle J_{\alpha\dot\alpha} \to J_{\alpha\dot\alpha}\rangle$. The amplitude $\langle J_{\alpha\dot\alpha} \to W^2\rangle$ corresponds to Eq. (21), while the expression for the amplitude $\langle J_{\alpha\dot\alpha} \to J\rangle$ is not so obvious. To find that last one we write out the equation that follows directly from the factorization property of the amplitude. It is analogous to the one that was explained for QED [see Eq. (1) and the discussion that follows] and has the form

$$\langle J_{\alpha\dot\alpha} \to W^2\rangle - \langle J_{\alpha\dot\alpha} \to W^2\rangle_0 = \langle J_{\alpha\dot\alpha} \to J\rangle \langle J \to W^2\rangle_{1\text{-loop}}. \qquad (23)$$

In that equation the transition amplitude $\langle J_{\alpha\dot\alpha} \to W^2\rangle_0$ includes by definition only corrections from diagrams not containing scattering of light on light, i.e., the one-loop contribution and corrections of the type shown in Fig. 1. Upon regularization, in which the conserved tensor $\theta_{\mu\nu}$ is a component of $J_{\alpha\dot\alpha}$, these diagrams give

$$\langle J_{\alpha\dot\alpha} \to W^2\rangle_0 = -\beta(g_0^2)/6g_0^4,$$

while

$$\langle J_{\alpha\dot\alpha} \to W^2\rangle = -\beta(g^2)/6g^4$$

in accordance with Eq. (21). The one-loop amplitude is

$$\langle J \to W^2\rangle_{1\text{-loop}} = \beta_1(g^2)/g^4 = 1/8\pi^2$$

(one-loop Konishi anomaly). In this way we obtain from Eq. (23) the expression for $\langle J_{\alpha\dot\alpha} \to J\rangle$ shown in Eq. (22). To obtain the amplitude $\langle W^2 \to J\rangle$ we make use of the factorization relation analogous to Eq. (23):

$$\langle J_{\alpha\dot\alpha} \to J\rangle = \langle J_{\alpha\dot\alpha} \to W^2\rangle_0 \langle W^2 \to J\rangle, \qquad (24)$$

and substitute here for the amplitude $\langle J_{\alpha\dot\alpha} \to J\rangle$ the expression found above. The amplitude $\langle W^2 \to W^2\rangle$ is given by Eq. (20). The matrix element $\langle J \to W^2\rangle$ is nothing other than the multi-loop Konishi anomaly. Lastly, analogous arguments making use of the factorization property permit one to also obtain the amplitude $\langle J \to J\rangle$.

To make the matrix in Eq. (22) finite (i.e., dependent on $g^2(k)$ and not on g_0^2), it is necessary to multiply Eq. (22) from the left by the matrix $z_{ij}(g_0)$:

$$z = \begin{pmatrix} 1 & \dfrac{\beta(g_0^2) - \beta_1(g_0^2)}{6g_0^4} & -\dfrac{1}{6}\left[\dfrac{\beta(g_0^2)}{\beta_1(g_0^2)} - 1\right] \\ 0 & 1 & \left[\dfrac{\beta(g_0^2)}{\beta_1(g_0^2)} - 1\right]\dfrac{g_0^4}{\beta_1(g_0^2)} \\ 0 & 0 & \beta(g_0^2)/\beta_1(g_0^2) \end{pmatrix}. \qquad (25)$$

From Eqs. (22) and (25) we obtain the renormalization-invariant combinations

$$O_i^{ren} = \sum_j z_{ij} O_j,$$

where the O_i refer to the operators in Eq. (22). It is more convenient to make use of certain linear combinations of the operators O_i^{ren}. They may be chosen as follows:

$$\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} + [\beta(g_0^2)/6g_0^4]\, D_\alpha W^2 = c_1,$$
$$D_\alpha \bar{D}^2 J - [\beta_1(g_0^2)/g_0^4]\, D_\alpha W^2 = c_2,$$
$$[\beta(g_0^2)/\beta_1(g_0^2)]\, D_\alpha \bar{D}^2 J = c_3.$$

The constants c_1 and c_2 may be fixed by considering the limit $g_0 \to 0$, when, obviously, $c_1 = c_2 = 0$ as a consequence of the one-loop approximation in the anomaly equations. This gives rise to the following operator equations:

$$\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} = -\frac{\beta(g^2)}{6g^4} D_\alpha W^2, \quad \bar{D}^2 J = \frac{1}{8\pi^2} W^2. \qquad (26)$$

In these equations we have moved from the normalization point Λ to an arbitrary point. Consequently, the operator equation for $J_{\alpha\dot\alpha}$ contains the β-function, and J is subject to the one-loop equation. It is clear from what has been said above that this distinction is connected with the choice of the Pauli-Villars procedure (typical of the Adler-Bardeen current) in the regularization of the operator J, while the regularized operator $J_{\alpha\dot\alpha}$ contains the conserved energy-momentum tensor $\theta_{\mu\nu}$.

We discuss briefly the situation in supersymmetric Yang-Mills theory. At first sight it may seem that the matrix element of the operator W^2 is determined as before by Eq. (20), since formally the action has a form analogous to Eq. (17). We know, however, from evaluation of the matrix element $\langle G\tilde{G}\rangle$ in nonsupersymmetric Yang-Mills theory that this is not so: the matrix element $\langle W^2\rangle$, contained in the imaginary part of the F-component of $G\tilde{G}$, should include not only a series in powers of $g^2(k)$ but also the functions k_1^2, k_2^2, k^2 and should moreover depend on the gauge parameter α. Technically, the derivation of Eq. (20) is incorrect in this case because we have failed to follow explicitly the gauge-fixing terms, which give in a nonabelian theory a nonvanishing contribution on differentiation with respect to g_0^2. In SUSY QED these gauge-noninvariant terms do not contribute as a consequence of neutrality of the superfield W. Indeed one may verify[22] that, for example, in nonsupersymmetric Yang-Mills theory the anomaly in the trace of the energy-momentum tensor $\langle\theta_{\mu\mu}\rangle \sim \beta(g^2)\, G_{\text{ext}}^2$ is obtained for the operator $\theta_{\mu\mu}$, which includes explicitly gauge-dependent terms. The specific form of these terms (the full expression for $\theta_{\mu\mu}$) is established, of course, with the help of the Noether theorem. In SUSY Yang-Mills theory the expression for $J_{\alpha\dot\alpha}$, containing $\theta_{\mu\nu}$ and j_μ^5, also contains explicitly gauge-dependent terms.[4] Now in the framework of the supersymmetric gauge the relation $\langle\theta_{\mu\mu}\rangle \sim \beta G_{\text{ext}}^2$ is generalized to $\langle\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha}\rangle \sim \beta D_\alpha W_{\text{ext}}^2$. This, apparently, corresponds to a certain redefinition of the operators and $G\tilde{G}$ off-shell in a supersymmetric manner including explicitly gauge-dependent terms. We do not know how to perform the corresponding redefinition in the Wess-Zumino gauge, which would permit a comparison between the SUSY case and the nonsupersymmetric result.

5. CONCLUSION

The following must be kept in mind in judging the consequences of the assertions given above. The multi-loop corrections to the anomalies are determined by diagrams with rescattering of vector fields (photons, gluons). Therefore to cancel the anomalies in any order of perturbation theory it is sufficient to cancel the one-loop contributions. Precisely for this reason the multi-loop corrections to the anomalies present no difficulties for the standard model. It is easy to see that also in the other interesting case, namely in the discussion of the t' Hooft conditions for fusing anomalies at the preon and quark-lepton levels, no new results are obtained: the one-loop fusion rules lead to an exact coincidence of the anomalies.

It may be that the simplest consequence of the multi-loop character of the axial anomaly is a correction to the amplitude for the decay $\pi^0 \to 2\gamma$. Taking into account photon rescattering results in the appearance of the additional factor $1 + 9e^2(k)/16\pi^2$ in the amplitude for $\pi^0 \to 2\gamma$, where k^2 is the invariant mass of the (virtual) pion for $k^2 > m_u^2, m_d^2$ (m_u and m_d are the masses of, respectively, the u- and d-quark).

We discuss now the question of renormalization of the θ-term. At first sight it may seem that it should be renormalized due to the presence of radiative corrections to $\langle G\tilde{G}\rangle$. However, the situation is not that simple. We shall demonstrate that for the example of QED, because the really interesting case of the Yang-Mills theory is even more complicated.

We assume that the external electromagnetic field satisfies the condition

$$Q_{ext} = \int d^4x \,(F_{\mu\nu}\tilde{F}^{\mu\nu})_{ext} \neq 0, \qquad (27)$$

where it is understood that we have passed to integration over a Euclidean space and the fields have been appropriately redefined. Although it is known that instanton-type fields are absent in electrodynamics, the external field may be chosen in such a fashion that the above integral is different from zero if one gives up the requirement that the action be finite. We consider the case $Q_{\text{ext}} \neq 0$ since, obviously, for $Q_{\text{ext}} = 0$ the problem of renormalization of the operator Q does not arise.

The radiative corrections to the quantity $\langle F\tilde{F}\rangle$ are described by the diagram of Fig. 6 and were calculated in the text in the case of a weak field. Implicitly this presumes that the Dirac operator in the external field has no discrete modes. But it is known that for $Q_{\text{ext}} \neq 0$ the Dirac operator has normalizable zero modes (the Atiyah-Singer theorem) and in that sense the field now considered is strong. The calculation of the matrix element $\langle F\tilde{F}\rangle$ should be carried out with this circumstance taken into account. An infinite renormalization, $\sim \ln \Lambda$, of the matrix element $\langle F\tilde{F}\rangle$ arises from integration over the loop containing the photon lines. Even a strong external field has no effect on this integration because the momentum flowing through the loop is $\sim \Lambda \gg |F_{\mu\nu}^{\text{ext}}|$. As a result of the integration over this loop we have

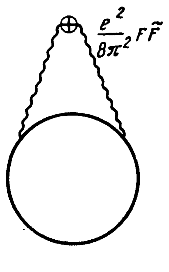

FIG. 6.

$$\langle F\tilde{F}\rangle_{2\text{-}loop}=-\frac{3e_0^2}{8\pi^2}\ln(\Lambda^2)\langle\partial_\mu j_\mu{}^5\rangle_{1\text{-}loop}, \quad (28)$$

where the resultant divergence of the axial current was regularized in a gauge-invariant way by means of the upper part of the diagram in Fig. 6. Equation (28) was explained in Sec. 2. To evaluate the correction to the quantity $\langle Q\rangle$ we integrate Eq. (28) over d^4x:

$$\langle Q\rangle_{2\text{-}loop}=\int d^4x\langle F\tilde{F}\rangle_{2\text{-}loop}=-\frac{3e_0^2}{8\pi^2}\ln(\Lambda^2)\int d^4x\langle\partial_\mu j_\mu{}^5\rangle_{1\text{-}loop}. \quad (29)$$

In the case under discussion it is not possible to consider from the very beginning the strictly massless theory due to the infrared divergences connected with the presence of zero modes. If we introduce a small fermion mass $m\neq 0$ we obtain (in the one-loop approximation)

$$\langle\partial_\mu j_\mu{}^5\rangle_{1\text{-}loop}=-2m\sum_n\frac{\overline{\psi}_n(x)\gamma_5\psi_n(x)}{m+\lambda_n}+\frac{e^2}{8\pi^2}(F\tilde{F})_{ext}, \quad (30)$$

where $\psi_n(x)$ and λ_n are the eigenfunctions and eigenvalues of the Dirac operator in the external field. Even for $m\to 0$ the first term does not vanish: the contribution of the zero modes $\psi_0(\lambda_0=0)$ survives. Integration of Eq. (30) over x gives

$$\int\langle\partial_\mu j_\mu{}^5\rangle d^4x=-2\int\overline{\psi}_0\gamma_5\psi_0\,d^4x+\frac{e^2}{8\pi^2}\int(F\tilde{F})_{ext}\,d^4x=0, \quad (31)$$

i.e., the contribution of the zero modes balances the quantity $Q_{\rm ext}\neq 0$ (the Atiyah-Singer theorem). Returning to Eq. (29) we see that

$$\langle Q\rangle_{2\text{-}loop}=0. \quad (32)$$

The analysis of the situation in Yang-Mills theory is made more difficult by the fact that, due to gluon self-interaction, there are corrections to $\langle G\tilde{G}\rangle$ that are not proportional to $\langle\partial_\mu j_\mu^5\rangle_{1-\rm loop}$, as is the case in Eq. (29).

We summarize the main results of this work as follows.

1. We have shown that the Adler-Bardeen theorem for the matrix element of the divergence of the axial current in an external gauge field is false: the matrix element $\langle\partial_\mu j_\mu^5\rangle$ has multi-loop corrections both in quantum electrodynamics and in Yang-Mills theory. These corrections are connected to rescattering of vector bosons.

2. In QED we have shown that, although at first glance these corrections are absent to lowest order in the coupling constant e_0^2, they start with terms $\sim e_0^4\ln\Lambda$ (photon scattering through fermion loop) and after renormalization of the current j_μ^5 contribute $\sim e^2(k)$, where $e^2(k)$ is the running coupling constant. The same is also true for Yang-Mills theory; however, in that case there also occurs direct, Born, rescattering of gluons.

We have explicitly calculated the corrections $\sim e^2(k)$, $g^2(k)$ for QED and Yang-Mills theory. In contrast to QED the corrections $\sim g^2(k)$ in Yang-Mills theory turn out to be gauge-dependent since they refer to an amplitude with charged external lines (gluons) off-shell. Passage to the mass shell is impossible due to infrared singularities, but the leading contribution for $k_1^2, k_2^2\to 0$ is gauge-independent.

3. We have derived the equation for the operator $\partial_\mu j_\mu^5$, with operator mixing taken into account. We have demonstrated that for standard operator regularization (Pauli-Villars) the operator equation for $\partial_\mu j_\mu^5$ in QED and in Yang-Mills theory has one-loop character. However this assertion makes little sense physically: whether the character of the operator equations is one-loop or multi-loop depends on the choice of regularization scheme for the composite operators j_μ^5 and $G_{\mu\nu}\tilde{G}^{\mu\nu}$.

4. We have investigated SUSY QED and derived exact equations for the anomalous divergences of the supercurrent $J_{\alpha\dot\alpha}$ and the Konishi current J, in both the matrix-element and the operator forms. As in the nonsupersymmetric case physical meaning is to be ascribed to the equations for the matrix elements, with the anomalous divergences of both currents $J_{\alpha\dot\alpha}$ and J being proportional to the β-function. The precise analysis of the anomalous axial divergence in supersymmetric Yang-Mills theory is made difficult by the gauge dependence of the matrix elements off-shell.

We are grateful for numerous discussions to our colleagues and friends: Ya. Balitskii, A. I. Vainshtein, D. I. D'yakonov, I. T. Dyatlov, L. N. Lipatov, V. A. Novikov, L. B. Okun', V. Petrov, A. M. Polyakov, L. L. Frankfurt, M. A. Shifman, M. I. Eides, and A. Yung. None of them, however, is in any way responsible for the assertions made in this work.

APPENDICES

1. Evaluation of the diagrams in Fig. 1.

In the case of a zero-mass physical fermion the contribution to the two-loop amplitude for the transition of the operator $\partial_\mu j_\mu^5$ into two photons is described by the diagrams of Fig. 1, containing loops of the regulator field with mass M much larger than any of the external momenta. In the background-field method these diagrams are summarized by the diagram of Fig. 7, where the wavy line corresponds to the free photon propagator including the regularizing factor $\Lambda^2/(\Lambda^2+\partial^2)$, where Λ is the cut-off parameter; the matter field propagators are considered in the external field. The contribution of the diagram in Fig. 7 may be represented as follows

$$\int d^4x_1\,d^4x_2\left(\frac{-\Lambda^2}{\partial^2(\partial^2+\Lambda^2)}\right)_{12}$$
$$\times 2iM\,\mathrm{Sp}\,\gamma_5\left(\frac{M+i\hat\nabla}{M^2+\hat\nabla^2}\right)_{31}\gamma_\mu\left(\frac{M+i\hat\nabla}{M^2+\hat\nabla^2}\right)_{12}$$
$$\times\gamma_\mu\left(\frac{M+i\hat\nabla}{M^2+\hat\nabla^2}\right)_{23}, \quad (\mathrm{A}1)$$

where $\nabla_\mu=\partial_\mu-iB_\mu$, B_μ is the external photon field and we have introduced the following notation for matrix elements of operators: $(O)_{12}=O\delta^4(x_1-x_2)$ with the operator

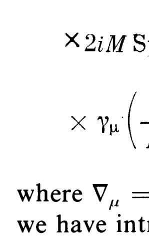

FIG. 7.

O acting on the coordinate x_1. Actually, the integrand in Eq. (A1) is suppressed by a factor k/M, where k is the external momentum, and vanishes for $M\to\infty$ prior to integration over x_1 and x_2. In order to be convinced of that we transform Eq. (A1) identically into the following form (we omit the photon propagator and transfer the derivative from the last factor to the first):

$$-i\partial_\rho^{(3)}\Big[2iM\,\mathrm{Sp}\,\gamma_5\Big(\frac{M+i\hat\nabla}{M^2+\hat\nabla^2}\Big)_{31}$$
$$\times\gamma_\mu\Big(\frac{M+i\hat\nabla}{M^2+\hat\nabla^2}\Big)_{12}\gamma_\mu\Big(\frac{1}{M^2+\hat\nabla^2}\Big)_{23}\gamma_\rho\Big]$$
$$+2iM\,\mathrm{Sp}\,\gamma_5\Big(\frac{(M-i\hat\nabla)(M+i\hat\nabla)}{M^2+\hat\nabla^2}\Big)_{31}$$
$$\times\gamma_\mu\Big(\frac{M+i\hat\nabla}{M^2+\hat\nabla^2}\Big)_{12}\gamma_\mu\Big(\frac{1}{M^2+\hat\nabla^2}\Big)_{23}. \qquad (A2)$$

The first term in Eq. (A2) is a total derivative with respect to x_3 of a gauge-invariant expression and is of order $\sim(k/M)F\tilde F$. In the second term, contributions linear and bilinear in the external field are absent. That is easily shown by using the expansion ($\sigma F=\sigma_{\mu\nu}F^{\mu\nu}$)

$$\frac{1}{M^2+\hat\nabla^2}=\frac{1}{M^2+\nabla^2}-\frac{1}{M^2+\nabla^2}\frac{1}{2}\sigma F\frac{1}{M^2+\nabla^2}$$
$$+\frac{1}{M^2+\nabla^2}\frac{1}{2}\sigma F\frac{1}{M^2+\nabla^2}\frac{1}{2}\sigma F\frac{1}{M^2+\nabla^2}+\cdots \qquad (A3)$$

and calculating the trace in Eq. (A2). Thus the diagram of Fig. 7 does not contribute to the amplitude $\partial_\mu j^5_\mu\to F\tilde F$ regardless of the relation between the parameters M and Λ.

2. Evaluation of the diagrams of the type in Fig. 2 with scattering of light on light in QED

In the background-field method the sum of diagrams of the type in Fig. 2 with scattering of light on light is determined by the diagram in Fig. 8, where the fermion propagator is taken in the external field. Below we are interested only in the ultraviolet-divergent contributions to the amplitude $\partial_\mu j^5_\mu\to F\tilde F$. Clearly, the diagram of the form shown in Fig. 2, corresponding to the propagation of a physical fermion field with mass m in the upper triangle, is finite. We therefore analyze only diagrams in Fig. 2 with the regulator field of mass M propagating around the triangle. Typical momenta in the triangle are in that case $\sim M$ and (for $\Lambda\lesssim M$) much bigger than typical momenta of the virtual photon lines. Therefore the triangle may be contracted to a point. The result is the diagram shown in Fig. 6, with the upper vertex corresponding to the operator $(F\tilde F)e_0^2/8\pi^2$. The amplitude for scattering of light on light is finite (the contribution of the regulator loop "dies out" when gauge invariance is explicitly adhered to) and the logarithm $\ln\Lambda$ arises in integration over the loop containing the virtual photon lines.

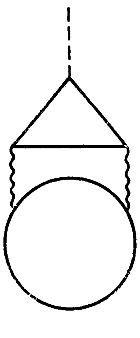

FIG. 8.

In the coordinate representation the expression for the diagram in Fig. 6 has the form

$$\Gamma(x_3)=\frac{e_0^6}{8\pi^2}2\varepsilon_{\mu\nu\alpha\beta}\int d^4x_1\,d^4x_2\Big(\frac{\partial_\alpha}{\partial^2}\Big)_{31}\Big(\frac{\partial_\beta}{\partial^2}\Big)_{32}$$
$$\times\mathrm{Sp}\,\gamma_\mu\Big(\frac{1}{m-i\hat\nabla}\Big)_{12}\gamma_\nu\Big(\frac{1}{m-i\hat\nabla}\Big)_{21}. \qquad (A4)$$

To evaluate Γ it is convenient to make use of Eq. (A3) and the following expansion:

$$\frac{\hat\nabla}{m^2+\hat\nabla^2}=\frac{1}{2}\Big\{\hat\nabla,\frac{1}{m^2+\nabla^2}-\frac{1}{m^2+\nabla^2}\frac{1}{2}\sigma F\frac{1}{m^2+\nabla^2}$$
$$+\frac{1}{m^2+\nabla^2}\frac{1}{2}\sigma F\frac{1}{m^2+\nabla^2}\frac{1}{2}\sigma F\frac{1}{m^2+\nabla^2}+\ldots\Big\}, \qquad (A5)$$

where $\{\ldots\}$ denotes an anticommutator. The use of an anticommutator in Eq. (A5) may seem artificial, but it allows simplification of the calculations that follow. The terms in Eq. (A5), explicitly proportional to the mass m, will not be further considered since, as is easily verified, they are ultraviolet-finite.

We substitute now the expansion Eq. (A5) into Eq. (A4) and calculate the trace with the help of the identities

$$\Big\{\hat\nabla,\frac{1}{\nabla^2+m^2}\sigma F\frac{1}{\nabla^2+m^2}\Big\}=-2i\gamma_\rho\Big[\nabla_\lambda,\frac{1}{\nabla^2+m^2}F_{\lambda\rho}\frac{1}{\nabla^2+m^2}\Big]$$
$$-2\gamma_\rho\gamma_5\Big\{\nabla_\lambda,\frac{1}{\nabla^2+m^2}\tilde F_{\lambda\rho}\frac{1}{\nabla^2+m^2}\Big\}, \qquad (A6)$$

$$\Big\{\hat\nabla,\frac{1}{m^2+\nabla^2}\sigma F\frac{1}{m^2+\nabla^2}\sigma F\frac{1}{m^2+\nabla^2}\Big\}$$
$$=2i\gamma_5\Big[\hat\nabla,\frac{1}{m^2+\nabla^2}F_{\mu\nu}\frac{1}{m^2+\nabla^2}\tilde F^{\mu\nu}\frac{1}{m^2+\nabla^2}\Big]$$
$$+4\gamma_\rho\Big[\nabla_\lambda,\frac{1}{m^2+\nabla^2}F_{\lambda\beta}\frac{1}{m^2+\nabla^2}F_{\rho\beta}\frac{1}{m^2+\nabla^2}-(\lambda\rightleftarrows\rho)\Big]$$
$$-4i\gamma_\rho\gamma_5\varepsilon_{\lambda\rho\tau\sigma}\Big\{\nabla_\lambda,\frac{1}{m^2+\nabla^2}F_{\tau\beta}\frac{1}{m^2+\nabla^2}F_{\sigma\beta}\frac{1}{m^2+\nabla^2}\Big\}$$
$$+2\Big\{\hat\nabla,\frac{1}{m^2+\nabla^2}F_{\mu\nu}\frac{1}{m^2+\nabla^2}F^{\mu\nu}\frac{1}{m^2+\nabla^2}\Big\}. \qquad (A7)$$

Keeping in Γ only the divergent terms we obtain for the part quadratic in the field (regulator factors in photon lines are omitted)

$$\Gamma^{\mathrm{qu}}=-8e_0^6\int d^4x_1\,d^4x_2\Big[\Big(\frac{\partial_\alpha}{\partial^2}\Big)_{31}\Big(\frac{\partial_\beta}{\partial^2}\Big)_{32}-(\alpha\rightleftarrows\beta)\Big]$$
$$\times\Big[\Big(\frac{\partial_\alpha}{\partial^2}\Big)_{12}\Big(\Big[\partial_\beta,\frac{1}{\partial^2+m^2}F_{\mu\nu}\frac{1}{\partial^2+m^2}\tilde F_{\mu\nu}\frac{1}{\partial^2+m^2}\Big]$$
$$+2\varepsilon_{\lambda\beta\tau\sigma}\Big\{\partial_\lambda,\frac{1}{\partial^2+m^2}F_{\tau\eta}\frac{1}{\partial^2+m^2}F_{\sigma\eta}\frac{1}{\partial^2+m^2}\Big\}\Big)_{21}$$
$$+i\Big\{\nabla_\alpha,\frac{1}{m^2+\nabla^2}\Big\}_{12}\Big\{\nabla_\lambda,\frac{1}{m^2+\nabla^2}\tilde F_{\lambda\beta}\frac{1}{m^2+\nabla^2}\Big\}_{21}\Big], \qquad (A8)$$

where the last term should be expanded in powers of the external field and the quadratic term extracted.

To obtain an estimate of the resultant expression we go over into the momentum representation. For the first two terms in Eq. (A8) we obtain

$$-\frac{3}{128\pi^4}\ln(\Lambda^2)(F\tilde{F})_{ext}$$

$$\times\int_0^1 dx\int_0^{1-x} dy\,\frac{(k_1+k_2)^2-2k_1k_2(1-x-y)}{k_1^2x(1-x)+k_2^2y(1-y)+2k_1k_2xy-m^2},\quad (A9)$$

where k_1 and k_2 are the momenta of the external photon lines, $k^2=(k_1+k_2)^2$, and $(F\tilde{F})_{ext}=F(k_1)\tilde{F}(k_2)+(k_1\rightleftarrows k_2)$. The analysis of the contribution of the last term in Eq. (A8) is somewhat more complicated and leads to the following result, independent of the gauge of the external field,

$$-\frac{3}{128\pi^4}\ln(\Lambda^2)(F\tilde{F})_{ext}+\frac{3}{128\pi^4}\ln(\Lambda^2)(F\tilde{F})_{ext}$$

$$\times\int_0^1 dx\int_0^{1-x} dy\,\frac{[k_1x+k_2(1-y)]^2+[k_2y+k_1(1-x)]^2-2m^2}{k_1^2x(1-x)+k_2^2y(1-y)+2xyk_1k_2-m^2}.\quad (A10)$$

Adding Eqs. (A9) and (A10) we obtain

$$\Gamma^{qu}=-\frac{3e_0^6}{64\pi^4}\ln(\Lambda^2)(F\tilde{F})_{ext}\Big(1+2m^2\int_0^1 dx\int_0^{1-x} dy\,(k_1^2x(1-x)+k_2^2y(1-y)+2xyk_1k_2-m^2]^{-1}\Big).\quad (A11)$$

We note that in Eq. (A11) for $m=0$ the dependence on k_1 and k_2 [present in each of the Eqs. (A9) and (A10)] in the coefficient of $F\tilde{F}\ln\Lambda$ has disappeared and, therefore, that coefficient does not depend on the relation between k_1^2, k_2^2, and k_1k_2.

3. Corrections to $\langle G\tilde{G}\rangle$ and $\langle\partial_\mu j_\mu^5\rangle$ connected with born scattering of gluons on gluons

In the background-field method diagrams of the type in Fig. 4 for the divergence of the (singlet) axial current are summarized by the one diagram in Fig. 9a. In that diagram the wavy line denotes the gluon propagator in the external gluon field, and the straight lines correspond to the regulator fermion field (in the representation R of the gauge group) of mass M. Clearly, the same diagram describes the one-loop correction to the quantity $(g_0^2T(R)/8\pi^2)\langle G\tilde{G}\rangle$, arising in the contraction of the triangle (Fig. 9b), with the triangle playing the role of the regularizing insertion.

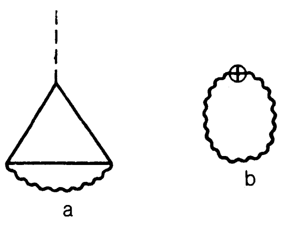

FIG. 9.

For arbitrary α-gauge in the external field B_μ one adds to the Lagrangian the gauge-fixing term $(1/2\alpha)(\nabla_\mu(B)A_\mu)^2$. The gluon propagator in the external gluon field has the form

$$\left(\frac{1}{-\nabla^2+2iG+\nabla\times\nabla-\alpha^{-1}\nabla\times\nabla}\right)^{ab}_{\mu\nu},\quad (A12)$$

where

$$\nabla_\mu=\delta^{ab}\partial_\mu+f_{abc}B_{\mu,c},\qquad G_{\mu\nu}{}^{ab}=if_{abc}G_{\mu\nu}{}^c,$$
$$(\nabla\times\nabla)_{\mu\nu}=\nabla_\mu\nabla_\nu.$$

To find the two-loop diagram Γ (Fig. 9a) it is convenient to add and subtract the same diagram with a massive virtual gluon of mass m, $\Gamma_{m\neq0}$:

$$\Gamma=(\Gamma-\Gamma_{m\neq0})+\Gamma_{m\neq0},$$

where $k\ll m\ll M$ (k—external momentum). In order to evaluate the diagram Γ in an arbitrary α-gauge it is convenient to obtain it first in the Feynman gauge ($\alpha=1$), and then calculate the contributions depending on $\alpha-1$. Consider for $\alpha=1$ the diagram $\Gamma_{m\neq0}$ (Fig. 9a), in which the propagator corresponding to the virtual gluon is $[-\nabla^2+2iG-m^2]^{-1}$. Transforming the fermion triangle in the same manner as was done when estimating two-loop corrections to $\langle\partial_\mu j_\mu^5\rangle$ in electrodynamics (see Appendix 1), one can show that the diagram $\Gamma_{m\neq0}$ is equal to the product of a gauge-invariant expression and the sum of momenta of the incident gluons $k=k_1+k_2$. Therefore it is of order $(k/m)(G\tilde{G})_{ext}$. As far as the difference $\Gamma-\Gamma_{m\neq0}$ is concerned, the regulator fermion triangle in it may be contracted to a point since the typical momentum of the virtual gluon is $p\lesssim m\ll M$.

In this way the diagram $\Gamma-\Gamma_{m\neq0}$, and with it also Γ, reduces to the difference of the one-loop diagrams of Fig. 9b corresponding to the massless and massive virtual gluons, with the vertex corresponding to the operator $(g_0^2T(R)/8\pi^2)\mathrm{Tr}\,G\tilde{G}$. To evaluate the one-loop contribution to $\langle G\tilde{G}\rangle$, viz., $\langle G\tilde{G}\rangle_{1-\mathrm{loop}}$, one should introduce the external field $(B_\mu^a)_{ext}: A_\mu^a\to A_\mu^a+(B_\mu^a)_{ext}$ and take the quantity $G\tilde{G}$ to second order in the quantum field A_μ^a:

$$\mathrm{Tr}\,G\tilde{G}\to2\,\mathrm{Tr}(\partial_\mu\varepsilon_{\mu\nu\alpha\beta}A_\nu\nabla_\alpha A_\beta),$$

where $\nabla_\alpha=\partial_\alpha-i(B_\alpha)_{ext}$. Using the expression for the gluon propagator in an external field we obtain (we omit the factor $g_0^2T(R)/8\pi^2$ that arises from contraction of the triangle)

$$\langle G\tilde{G}\rangle_{1\text{-}loop}=-2i\partial_\mu\varepsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left(\nabla_\alpha\left(\frac{1}{-\nabla^2+2iG}\right)_{\beta\nu}-\nabla_\alpha\left(\frac{1}{-\nabla^2+2iG-m^2}\right)_{\beta\nu}\right)_{11}.\quad (A13)$$

Here we used for matrix elements of operators the notation $(\ldots)_{11}$ introduced in Appendix 1. We expand Eq. (A13) in powers of the external field strength G:

$$\langle G\tilde{G}\rangle_{1\text{-}loop}=-2i\varepsilon_{\mu\nu\alpha\beta}\partial_\mu\mathrm{Tr}\Big(-2i\nabla_\alpha\frac{1}{\nabla^2}G_{\beta\nu}\frac{1}{\nabla^2}+4\nabla_\alpha\frac{1}{\nabla^2}G_{\beta\rho}\frac{1}{\nabla^2}G_{\rho\nu}\frac{1}{\nabla^2}+2i\nabla_\alpha\frac{1}{\nabla^2+m^2}G_{\beta\nu}\frac{1}{\nabla^2+m^2}$$
$$-4\nabla_\alpha\frac{1}{\nabla^2+m^2}G_{\beta\rho}\frac{1}{\nabla^2+m^2}G_{\rho\nu}\frac{1}{\nabla^2+m^2}\Big)_{11}.\quad (A14)$$

The common differentiation ∂_μ in Eq. (A14) may be rewritten as the commutator of ∇_μ with the expression in brackets. As a result we obtain (in the quadratic approximation in the external field)

$$\langle G\tilde G\rangle_{1\text{-}loop} = -8i\,\mathrm{Tr}\Big(G_{\mu\alpha}\frac{1}{\partial^2+m^2}\tilde G_{\mu\alpha}\frac{1}{\partial^2+m^2} - \frac{1}{\partial^2+m^2}\{\partial_\alpha\{\partial_\tau G_{\mu\tau}\}\}\frac{1}{\partial^2+m^2}\tilde G_{\mu\alpha}\frac{1}{\partial^2+m^2} - \Big\{\partial_\alpha, \frac{1}{\partial^2+m^2}(\partial_\rho \tilde G_{\alpha\beta})\frac{1}{\partial^2+m^2}G_{\rho\beta}\frac{1}{\partial^2+m^2}\Big\}\Big)_{11} - (m=0). \quad \text{(A15)}$$

Going over to the momentum representation and evaluating the corresponding integrals we obtain

$$\langle G\tilde G\rangle_{1\text{-}loop} = \frac{T(G)g^2}{4\pi^2}\Big(1-\frac12 k^2 I_{00}\Big), \quad \text{(A16)}$$

where $k = k_1 + k_2$ is the total momentum of the external gluons and I_{00} is defined by Eq. (4) for $m = 0$. The first term in Eq. (A16) arises from the diagram with the regulator gluon ($-\Gamma_{m\neq 0}$), while the nonlocal correction $k^2 I_{00}$ is determined by the contribution of the diagram with the massless virtual gluon (Γ).

We now calculate the contributions that depend on $\alpha - 1$. It is convenient to introduce the notation $T = \nabla^2 - 2iG$, $\xi = 1 - 1/\alpha$. The propagator, Eq. (A12), can be represented as follows:

$$\frac{1}{-\nabla^2+2iG+\xi\nabla\times\nabla} = -\frac1T - \frac1T\nabla\times\frac{\xi}{1-\xi\nabla(1/T)\nabla}\times\nabla\frac1T. \quad \text{(A17)}$$

Here the quantity $\nabla(1/T)\nabla$ denotes the operator $\nabla_\mu(1/T)_{\mu\nu}\nabla_\nu$. The first term in Eq. (A17) contributes to the diagram Γ, corresponding to the choice $\alpha = 1$, and was evaluated above. The dependence of Γ on $\xi = 1 - 1/\alpha$ is determined by the expression corresponding to the contribution of the second term in Eq. (A17):

$$\langle G\tilde G\rangle^{\xi}_{1\text{-}loop} = 2i\varepsilon_{\mu\nu\alpha\beta}\partial_\mu\,\mathrm{Tr}\Big(\nabla_\alpha\Big(\frac1T\nabla\times\frac{\xi}{1-\xi\nabla(1/T)\nabla}\times\nabla\frac1T\Big)_{\beta\nu}\Big)_{11}. \quad \text{(A18)}$$

To evaluate this contribution we expand it in powers of the external field. To this end it is convenient to make use of the following expansions:

$$\nabla\frac1T\nabla = 1 - \nabla_\mu\frac{1}{\nabla^2}i[\nabla_\rho G_{\mu\rho}]\frac{1}{\nabla^2} + 2\nabla_\mu\frac{1}{\nabla^2}G_{\mu\rho}\frac{1}{\nabla^2}[\nabla_\nu G_{\rho\nu}]\frac{1}{\nabla^2}, \quad \text{(A19)}$$

$$\frac{\xi}{1-\xi\nabla(1/T)\nabla} = \frac{\xi}{1-\xi} + \Big(\frac{\xi}{1-\xi}\Big)^2\Big(-\nabla_\mu\frac{1}{\nabla^2}i[\nabla_\rho G_{\mu\rho}]\frac{1}{\nabla^2} + 2\nabla_\mu\frac{1}{\nabla^2}G_{\mu\rho}\frac{1}{\nabla^2}[\nabla_\nu G_{\rho\nu}]\frac{1}{\nabla^2}\Big) - \Big(\frac{\xi}{1-\xi}\Big)^3\nabla_\mu\frac{1}{\nabla^2}[\nabla_\rho G_{\mu\rho}] \times\frac{1}{\nabla^2}\nabla_\nu\frac{1}{\nabla^2}[\nabla_\tau G_{\nu\tau}]\frac{1}{\nabla^2}. \quad \text{(A20)}$$

Before substituting Eq. (A20) into Eq. (A18) it is helpful to rewrite the latter as

$$\langle G\tilde G\rangle^{\xi}_{1\text{-}loop} = 2i\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\Big[\nabla_\mu, \nabla_\alpha\Big(\frac1T\nabla\times\frac{\xi}{1-\xi\nabla(1/T)\nabla}\times\nabla\frac1T\Big)_{\beta\nu}\Big]_{11}$$
$$= 2i\varepsilon_{\mu\nu\alpha\beta}\,\mathrm{Tr}\Big[\Big(\nabla_\mu\nabla_\alpha\Big(\frac1T\nabla\times\frac{\xi}{1-\xi\nabla(1/T)\nabla}\times\nabla\frac1T\Big)_{\beta\nu} - \nabla_\alpha\Big(\frac1T\nabla\times\frac{\xi}{1-\xi\nabla(1/T)\nabla}\times\nabla\frac1T\Big)_{\beta\nu}\nabla_\mu\Big)_{11}\Big]. \quad \text{(A21)}$$

We substitute the expansion, Eq. (A20), into Eq. (A21). We evaluate first the contribution of the term in Eq. (A20) that is linear in $\xi/(1-\xi) = \alpha - 1$. The first term in the square brackets in Eq. (A21) gives (one must expand the operator $1/T$ in powers of the field strength $G_{\mu\nu}$)

$$\frac{\xi}{1-\xi}\Big(-\frac14 G_{\mu\nu}\frac{1}{\partial^2}G_{\alpha\beta}\frac{1}{\partial^2} + 2G_{\mu\nu}\frac{1}{\partial^2}G_{\alpha\rho}\frac{\partial_\rho\partial_\beta}{\partial^4}\Big)_{11}. \quad \text{(A22)}$$

The second term in the square brackets in Eq. (A21) leads to the expression

$$\frac{\xi}{1-\xi}\Big(-\frac14 G_{\alpha\beta}\frac{1}{\partial^2}G_{\mu\nu}\frac{1}{\partial^2} - \frac12 G_{\alpha\beta}\frac{\partial_\rho}{\partial^4}(\partial_\rho G_{\mu\nu})\frac{1}{\partial^2} + \frac12 G_{\alpha\beta}\frac{\partial_\nu}{\partial^4}(\partial_\rho G_{\mu\rho})\frac{1}{\partial^2} + \frac12\frac{\partial_\alpha}{\partial^2}(\partial_\rho G_{\beta\rho})\frac{1}{\partial^2}G_{\mu\nu}\frac{1}{\partial^2} - \frac{\partial_\alpha}{\partial^2}(\partial_\rho G_{\beta\rho})\frac{\partial_\tau}{\partial^4}(\partial_\tau G_{\mu\nu})\frac{1}{\partial^2} + \frac{\partial_\alpha}{\partial^2}(\partial_\rho G_{\beta\rho})\frac{\partial_\nu}{\partial^4}(\partial_\tau G_{\mu\tau})\frac{1}{\partial^2}\Big)_{11}. \quad \text{(A23)}$$

In deriving this result, use was made of the following relations that result from taking into account the antisymmetry in the indices μ,ν,α,β of the quantity in the square brackets in Eq. (A21):

$$\nabla_\alpha\Big(\frac1T\Big)_{\beta\rho}\nabla_\rho = -\frac{i}{2}G_{\alpha\beta}\frac{1}{\partial^2} - \frac{\partial_\alpha}{\partial^2}i(\partial_\rho G_{\beta\rho})\frac{1}{\partial^2}, \quad \text{(A24)}$$
$$\nabla_\tau\Big(\frac1T\Big)_{\tau\nu}\nabla_\mu = \frac{i}{2}G_{\mu\nu}\frac{1}{\partial^2} - i\frac{\partial_\rho}{\partial^2}(\partial_\rho G_{\mu\nu})\frac{1}{\partial^2} + i\frac{\partial_\nu}{\partial^2}(\partial_\rho G_{\mu\rho})\frac{1}{\partial^2}.$$

The terms in the square brackets in Eq. (A21) that are proportional to $[\xi/(1-\xi)]^2$ are calculated analogously and have the form

$$-\frac12\Big(\frac{\xi}{1-\xi}\Big)^2\Big[\partial_\mu, G_{\alpha\beta}\frac{\partial_\rho}{\partial^4}(\partial_\tau G_{\rho\tau})\frac{\partial_\nu}{\partial^4}\Big]. \quad \text{(A25)}$$

Contributions of higher order in $\xi/(1-\xi) = \alpha - 1$ vanish. Collecting the contributions, Eqs. (A22), (A23), and (A25), taking into account the contributions of the regulator fields (as was explained above for the case $\alpha = 1$), and going over to the momentum representation one readily obtains Eq. (14) in the main text.

4. Evaluation of two-loop corrections to the matrix element $\langle W^2\rangle$ in an external vector field in $N=1$ supersymmetric QED

Here we exploit the technique developed in Ref. 21. One-loop corrections to $\langle W^2\rangle$ are absent in SUSY QED. In two-loop approximation the corrections are described by the diagrams of Fig. 10. Solid lines correspond to propagators of matter superfields in an external vector superfield, and wavy lines correspond to free propagators of the vector superfield.

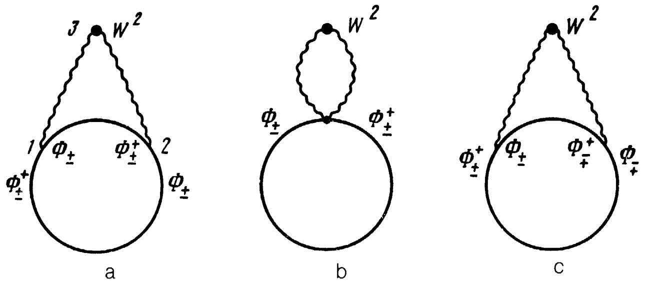

FIG. 10.

Integrals over the full superspace measure, $\int d^4x_1 d^4\theta_1$ and $\int d^4x_2 d^4\theta_2$, appear at the vertices 1 and 2, respectively. Vertex 3 corresponds to the operator $W^2(x_3,\theta_3)$. Below we make use of the following notation for the matrix element of the operator O in superspace:

$$(O)_{12}=O\delta^{(4)}(x_1-x_2)\delta^{(4)}(\theta_1-\theta_2)$$

(the operator O acts on the coordinates x_1 and θ_1). As before we shall be interested only in contributions $\sim \ln \Lambda$.

The diagrams of Fig. 10a and b correspond to the expressions

$$\Gamma^{(a)}=\int d^4x_1\,d^4\theta_1\int d^4x_2\,d^4\theta_2\,\frac{1}{2}\left(\bar{D}^2D^\beta\frac{1}{\partial^2}\right)_{31}\left(\bar{D}^2D_\beta\frac{1}{\partial^2}\right)_{32}$$
$$\times\frac{1}{16}\left(\nabla^2\frac{1}{m^2+\Box_+}\bar{\nabla}^2\right)_{12}\frac{1}{16}\left(\nabla^2\frac{1}{m^2+\Box_+}\bar{\nabla}^2\right)_{21},\qquad\text{(A26)}$$

$$\Gamma^{(b)}=\int d^4x_1\,d^4\theta_1\cdot\frac{1}{2}\left(\bar{D}^2D^\beta\frac{1}{\partial^2}\right)_{31}$$
$$\times\left(\bar{D}^2D_\beta\frac{1}{\partial^2}\right)_{31}\frac{1}{16}\left(\nabla^2\frac{1}{m^2+\Box_+}\bar{\nabla}^2\right)_{11},\qquad\text{(A27)}$$

where

$$\Box_+=\Box-(W^\alpha\nabla_\alpha+\nabla^\alpha W_\alpha)/4,$$

$$\nabla_\alpha=D_\alpha-i\Gamma_\alpha,\qquad \Box=\nabla_\mu{}^2,\qquad \nabla_\mu=\partial_\mu-i\Gamma_\mu,$$

Γ_α and Γ_μ are respectively the spinor and vector connections. In deriving Eqs. (A26) and (A27) use was made of the following expression[21] for the propagator of the chiral superfields Φ in an external vector superfield:

$$i\langle T\Phi_+{}^+(1)\Phi_+(2)\rangle={}^1/_{16}\left(\bar{\nabla}^2\frac{1}{m^2+\Box_+}\nabla^2\right)_{12},$$

and for the propagator of the vector superfield V in the corresponding supergauge:

$$i\langle TV(1)V(2)\rangle=2(1/\partial^2)_{12}.$$

The contributions of the diagrams proportional to m^2, Fig. 10c, are written analogously. In this section of the Appendix we omit for calculational convenience the coupling constant. It is easily restored in the final expression. We recall some definitions and relations:

$$\{\nabla_\alpha\bar{\nabla}_{\dot\alpha}\}=2i\nabla_{\alpha\dot\alpha},\qquad \nabla_{\alpha\dot\alpha}\equiv\sigma_{\alpha\dot\alpha}{}^\mu\nabla_\mu,$$
$$W^\alpha={}^1/_2 i[\bar{\nabla}^{\dot\alpha},\nabla^{\dot\alpha\alpha}],\qquad \bar{W}_{\dot\alpha}={}^1/_2 i[\nabla^\alpha,\nabla_{\dot\alpha\alpha}],$$
$$\bar{\nabla}^2=\bar{\nabla}_{\dot\alpha}\bar{\nabla}^{\dot\alpha},\qquad \nabla^2=\nabla^\alpha\nabla_\alpha,\qquad \Box={}^1/_2\nabla_{\alpha\dot\alpha}\nabla^{\dot\alpha\alpha},$$
$$\bar{\nabla}^2\nabla^2\bar{\nabla}^2=-16\Box_+\bar{\nabla}^2,\qquad \nabla^2\bar{\nabla}^2\nabla^2=-16\Box_-\nabla^2,$$
$$[\nabla_\alpha,\bar{\nabla}^2]=2i\{\nabla_{\alpha\dot\alpha},\bar{\nabla}^{\dot\alpha}\},\qquad [\bar{\nabla}^{\dot\alpha},\nabla^2]=2i\{\nabla^{\dot\alpha\alpha},\nabla_\alpha\},$$
$$[\nabla_\alpha,\nabla_{\beta\dot\beta}]=-i\varepsilon_{\alpha\beta}\bar{W}_{\dot\beta},\qquad [\bar{\nabla}^{\dot\alpha},\nabla^{\dot\beta\beta}]=-i\varepsilon^{\dot\alpha\dot\beta}W^\beta,$$
$$[\nabla^\alpha,\Box]=-{}^1/_2 i\{\bar{W}_{\dot\alpha},\nabla^{\dot\alpha\alpha}\},\qquad [\bar{\nabla}^{\dot\alpha},\Box]={}^1/_2 i\{W_\beta,\nabla^{\dot\alpha\beta}\}.$$

In the calculations that follow it is convenient to use the identity

$$\nabla^2\frac{1}{m^2+\Box_+}\bar{\nabla}^2=\nabla^2A\bar{\nabla}^2,\qquad\text{(A28)}$$

where A is an operator not containing spinor derivatives ∇_α and $\bar{\nabla}^{\dot\alpha}$ (we exploit here their Grassmann character, $\nabla^3=0$). The explicit form of the operator A may be obtained from the expansion of $(m^2+\Box_+)^{-1}$ in powers of $(W^\alpha\nabla_\alpha+\nabla^\alpha W_\alpha)$. We have

$$A=\frac{1}{m^2+\Box}-\frac{1}{4}\frac{1}{m^2+\Box}(DW)\frac{1}{m^2+\Box}$$
$$-\frac{i}{2}\frac{1}{m^2+\Box}\bar{W}_{\dot\beta}\partial^{\dot\beta\alpha}\frac{1}{m^2+\Box}W_\alpha\frac{1}{m^2+\Box}$$
$$+\frac{1}{16}\frac{1}{m^2+\Box}(DW)\frac{1}{m^2+\Box}(DW)\frac{1}{m^2+\Box}+O(V^3),$$
$$\text{(A29)}$$

where $(DW)=(D^\alpha W_\alpha)$.

By making use of Eqs. (A28) and (A29) the expressions for the diagrams of Fig. 10 may be transformed by integrating by parts and transferring the derivatives ∇^2 and $\bar{\nabla}^2$, acting inside the operator $(\nabla^2(m^2+\Box_+)^{-1}\bar{\nabla}^2)_{12}$, to act on other factors. Further, taking into account the identity

$$\delta^{(4)}(\theta_1-\theta_2)\nabla^2\bar{\nabla}^2\delta^{(4)}(\theta_1-\theta_2)=16\delta^{(4)}(\theta_1-\theta_2),\qquad\text{(A30)}$$

we go over to the following expression for the sum of the diagrams of Fig. 10:

$$\Gamma=\frac{1}{4}\int d^4x_1\,d^4x_2\,d^4\theta\left(\bar{D}^2\partial_{\alpha\dot\alpha}\frac{1}{\partial^2}\delta^{(4)}(\theta_3-\theta)\right)_{32}$$
$$\times\left(\bar{D}^2D^2\frac{1}{\partial^2}\delta^{(4)}(\theta_3-\theta)\right)_{31}(A)_{12}$$
$$\times\left(\bar{\nabla}^{\dot\alpha\alpha}A+\frac{i}{2}\{\nabla^\alpha[\bar{\nabla}^{\dot\alpha}A]\}\right)_{21},\qquad\text{(A31)}$$

where the operators inside the brackets act only on the spatial δ-function [for example, $(A)_{12}=A(x_1)\delta^{(4)}\times(x_1-x_2)$]; the last two factors depend on the Grassmann coordinate θ. We have omitted in Eq. (A31) certain obviously finite contributions $\sim m^2$.

Let us take outside the integral the operator $\bar{D}^2$ from the first factor in Eq. (A31) (we make use of the identity $\bar{D}^3=0$). Then, with Eq. (A30) taken into account, we obtain

$$\Gamma=4\bar{D}^2\int d^4x_1\,d^4x_2\left(\frac{\partial_{\alpha\dot\alpha}}{\partial^2}\right)_{32}\left(\frac{1}{\partial^2}\right)_{31}(A)_{12}$$
$$\left(\nabla^{\dot\alpha\alpha}A+\frac{i}{2}\{\nabla^\alpha[\bar{\nabla}^{\dot\alpha}A]\}\right)_{21}\qquad\text{(A32)}$$

Further calculations can be simplified by the observation that Eq. (A32) can be transformed to the following form:

$$\Gamma=4\int d^4x_1\,d^4x_2\left(\frac{\partial_{\alpha\dot\alpha}}{\partial^2}\right)_{32}\left(\frac{1}{\partial^2}\right)_{31}$$
$$\left(\{\bar{\nabla}_{\dot\beta}[\bar{\nabla}^{\dot\beta},A]\}_{12}\left(\nabla^{\dot\alpha\alpha}A+\frac{i}{2}\{\nabla^\alpha[\bar{\nabla}^{\dot\alpha},A]\}\right)_{21}\right.$$
$$+2[\bar{\nabla}_{\dot\beta},A]_{12}\left[\bar{\nabla}^{\dot\beta},\nabla^{\dot\alpha\alpha}A+\frac{i}{2}\{\nabla^\alpha[\bar{\nabla}^{\dot\alpha}A]\}\right]_{21}$$
$$\left.+(A)_{12}\left\{\bar{\nabla}_{\dot\beta},\left[\bar{\nabla}^{\dot\beta},\nabla^{\dot\alpha\alpha}A+\frac{i}{2}\{\nabla^\alpha,[\bar{\nabla}^{\dot\alpha}A]\}\right\}_{21}\right).\qquad\text{(A33)}$$

Formally it is convenient to prove Eq. (A33) by choosing the gauge for the external superfield in which $\overline{\nabla}^{\dot\alpha} = \overline{D}^{\dot\alpha}$, and by bringing the derivatives $\overline{D}$ under the integral sign (the quantity Γ is supergauge-invariant). Evaluating the (anti) commutators in Eq. (A33) and keeping only the terms quadratic in the field (the linear contributions vanish) we obtain for Γ the following expression already in terms of the superfield strengths:

$$\Gamma=8\left(\frac{\partial_{\dot\alpha\alpha}}{\partial^2}\right)_{32}\left(\frac{1}{\partial^2}\right)_{31}\Bigg\{\left(\frac{1}{m^2+\partial^2}\right)_{12}$$

$$\times\left(\partial^{\dot\alpha\beta}\frac{1}{m^2+\partial^2}W_\beta\frac{1}{m^2+\partial^2}W^\alpha\right)_{21}$$

$$+\left(\partial_{\gamma\dot\beta}\frac{1}{m^2+\partial^2}W^\gamma\frac{1}{m^2+\partial^2}\right)_{12}$$

$$\times\left(\frac{\partial^{\dot\alpha\alpha}\partial^{\dot\beta\beta}}{m^2+\partial^2}W_\beta\frac{1}{m^2+\partial^2}\right)_{21}$$

$$+\left(\partial_{\gamma\dot\beta}\frac{1}{m^2+\partial^2}W^\gamma\frac{1}{m^2+\partial^2}\right)_{12}$$

$$\times\left(\varepsilon^{\dot\alpha\dot\beta}W^\alpha\frac{1}{m^2+\partial^2}+\frac{1}{m^2+\partial^2}\partial^{\dot\alpha\beta}(\partial^{\dot\beta\alpha}W_\beta)\frac{1}{m^2+\partial^2}\right)_{21}$$

$$-m^2\left(\frac{\partial_{\alpha\dot\alpha}}{m^2+\partial^2}\right)_{21}$$

$$\times\left(\frac{1}{m^2+\partial^2}W^\beta\frac{1}{m^2+\partial^2}W_\beta\frac{1}{m^2+\partial^2}\right)_{12}$$

$$-m^2\left(\frac{1}{m^2+\partial^2}\right)_{12}$$

$$\times\left(\frac{1}{m^2+\partial^2}W^\beta\frac{1}{m^2+\partial^2}W_\beta\frac{1}{m^2+\partial^2}\partial^{\dot\alpha\alpha}\right)_{21}\Bigg\}. \qquad \text{(A34)}$$

We go over to the momentum representation and evaluate the corresponding integrals. The ultraviolet-divergent contribution to Γ arises only from the third and fourth terms in Eq. (A34):

$$\Gamma\approx-\frac{1}{32\pi^4}W^2\ln\left(\frac{\Lambda^2}{k^2}\right)(1+2m^2I_{00}). \qquad \text{(A35)}$$

For $m^2\gg k_1^2, k_2^2$ and (k_1k_2) ($k_{1,2}$ are momenta of the external fields) the right-hand side vanishes in accordance with the general theorem on the absence of multi-loop corrections to the effective action in the presence of an infrared cut off.[15] For $m=0$ we have for the matrix element $\langle W^2\rangle$ [restoring the factor e_0^4 in Eq. (A35)]

$$\langle W^2\rangle=\left(1-\frac{e_0^4}{32\pi^4}\ln\frac{\Lambda^2}{k^2}\right)W^2_{ext}. \qquad \text{(A36)}$$

The expression in brackets in Eq. (A36) equals

$$1-\frac{e_0^4}{32\pi^4}\ln\frac{\Lambda^2}{k^2}\approx\frac{\beta(e^2)}{e^4}\frac{e_0^4}{\beta(e_0^2)}$$

$$\approx 1+\frac{\beta_2}{\beta_1}(e^2-e_0^2)=1-\beta_2e_0^4\ln\frac{\Lambda^2}{k^2}, \qquad \text{(A37)}$$

where

$$\beta(e^2)=e^4(\beta_1+\beta_2e^2+\ldots),\quad \beta_1=1/8\pi^2,\ \beta_2=1/32\pi^4$$

(see, for example, Ref. 15).

[1] S. L. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969).
[2] M. T. Grisaru, in *Recent Developments in Gravitation*, eds. S. Deser and M. Levy, Plenum Press, New York, 1979.
[3] T. E. Clark, O. Piguet, and K. Sibold, Nucl. Phys. **B143**, 445 (1978), **B172**, 201 (1980).
[4] O. Piguet and K. Sibold, Nucl. Phys. **B196**, 428, 447 (1982).
[5] D. R. T. Jones, Phys. Lett. **123B**, 45 (1983).
[6] M. T. Grisaru and P. West, Nucl. Phys. **B254**, 249 (1985).
[7] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov, and M. A. Shifman, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 161 (1984) [JETP Lett. **40**, 920 (1984)].
[8] D. R. T. Jones and L. Mezincescu, Phys. Lett. **136B**, 283 (1984), **138B**, 242 (1984).
[9] P. Breitenlohner, D. Maison, and K. Stelle, Phys. Lett. **134B**, 63 (1984).
[10] D. Espriu, Oxford preprint 57/84, 1984.
[11] D. I. Kazakov, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 272 (1985) [JETP Lett. **41**, 335 (1985)].
[12] D. R. T. Jones, L. Mezincescu, and P. West, Phys. Lett. **151B**, 219 (1985).
[13] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakarov, Phys. Lett. **157B**, 169 (1985).
[14] M. T. Grisaru, B. Milewski, and D. Zanon, Phys. Lett. **157B**, 174 (1985); Nucl. Phys. **B266**, 589 (1986).
[15] M. A. Shifman and A. I. Vainshtein, Nucl. Phys. **B277**, 456 (1986); A. I. Vainshtein and V. I. Zakharov, in *A Search of Symmetry Behind Topological Chiral Current*, preprint CTP 1643, MIT, October 1988.
[16] A. A. Ansel'm and A. A. Ioganson, Pis'ma Zh. Eksp. Teor. Fiz. **49**, 185 (1989) [JETP Lett. **49**, 214 (1989)].
[17] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[18] J. Kodaira, Nucl. Phys. **B165**, 129 (1980).
[19] J. C. Collins, *Renormalization*, Cambridge Univ. Press, 1984.
[20] A. D. Dolgov and V. I. Zakharov, Nucl. Phys. **B12**, 68 (1971).
[21] M. T. Grisaru and D. Zanon, Nucl. Phys. **B252**, 578 (1985).
[22] J. C. Collins, A. Duncan, and S. D. Joglekar, Phys. Rev. **D16**, 438 (1977).

Translated by Adam M. Bincer