Hydrodynamic effects in the electrical conductivity of impure metals

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Normal electron-phonon and electron-electron collisions, if they occur considerably less frequently than electron-impurity scattering, have a qualitative effect on the mechanisms for momentum transfer from electrons to the surface of a metal. Under certain conditions, they give rise to a minimum on the temperature dependence of the electrical resistivity. The nature of these mechanisms depends substantially on the relation between the electron-impurity mean free path l_i and the sample thickness d. In a thick sample ($l_i \ll d$), a high degree of isotropy of the Fermi surface and of electron-impurity scattering will give rise to a minimum on the temperature dependence of the resistivity. In a thin wire with a cross section of arbitrary shape, this minimum is possible only if electron-phonon scattering outweighs electron-electron scattering. In a thin plate, such a minimum is not possible. All the expressions derived here differ substantially from previous theoretical results both in order of magnitude and in terms of the dependence on the kinetic parameters of the problem. The new results provide a qualitative explanation of the minimum in the resistivity of potassium.

1. INTRODUCTION

The observation of a minimum on the temperature dependence of the electrical resistivity of potassium[1] has recently heightened interest in a hydrodynamic mechanism for the electrical conductivity of metals.[2,3] A well-developed hydrodynamic situation is realized under the inequalities

$$(l_N l_V)^{1/2} \gg d \gg l_N. \tag{1}$$

Here d is the thickness of the sample, l_N is the mean free path with respect to normal electron-phonon or electron-electron collisions, and l_v is the mean free path with respect to bulk collisions which do not conserve quasimomentum (umklapp processes or scattering by impurities).

In the temperature interval defined by the inequalities (1), the conductivity has an inverse temperature dependence, $d\sigma/dT > 0$. This result has a simple physical meaning. An electron subjected to frequent normal collisions and moving in the manner of a Brownian particle traverses a distance $\sim d^2/l_N$ before it collides with a boundary. According to (1) this distance is small in comparison with the length l_v. The loss of resultant momentum is thus determined by surface scattering, and the conductivity $\sigma \sim d^2/l_N$, along with the frequency of N-processes increases with increasing temperature until umklapp processes impose the decrease in $\sigma(T)$ which is typical of metals. As the temperature is lowered, and the relationship $l_N \approx d$ becomes satisfied, the derivative $d\sigma/dT$ also changes sign. This Knudsen minimum, well known in the kinetic theory of gases, stems from the circumstance that "grazing" electrons, moving at small angles from the axis of the sample, are important in the conductivity of thin samples: Normal collisions (or any collisions) remove electrons from the group of grazing electrons and thus promote a removal of momentum to the boundary.

The hydrodynamic mechanism was first observed for a phonon gas by Mezhov-Deglin[4] in a study of heat transfer in perfect ^{4}He single crystals (a theory for hydrodynamic heat transfer in insulators has been derived by Sussman and Thellung[5] and Gurzhi[6]). Mezhov-Deglin should also be credited with a calculation on and an experimental observation of a Knudsen minimum in a phonon gas.[7]

In metals, it is considerably more difficult than in insulators to satisfy the basic condition for a fully developed hydrodynamic situation, namely, the condition $l_N \ll l_v$. In other words, N-processes must outweigh collisions involving a loss of quasimomentum. There are two reasons. First, the large Fermi surface of most uncompensated metals means that the probability for umklapp processes in electron-phonon collisions remains substantial down to extremely low temperatures.[8] (In compensated metals, the general carrier drift is not accompanied by an electric current, so hydrodynamic effects are possible during heat transfer.[9]) Second, electrons are scattered much more intensely by impurities than are the long-wave phonons which are important at low temperatures. Specifically because of these factors, it was not until just recently that an exponential temperature dependence was observed for the conductivity of a metal, and then only as a small increment in the residual resistivity.[10,11] On the other hand, in metals one can observe some extremely subtle features in relaxation mechanisms which lead to small effects in the temperature dependence. This would not be possible in insulators, against the background of the strong temperature dependence of the number of phonons. The temperature minimum which was observed in Ref. 1 is smaller by a factor of about 10^5 than the corresponding hydrodynamic effects whch have been observed in the thermal conductivity of insulators by Mezhov-Deglin.

An inverse temperature dependence of the resistivity of potassium[1] was observed at temperatures $T \approx 1$ K and in a region of kinetic parameters completely different from (1) $l_N \gg l_i, d$. Here l_i is the mean free path with respect to collisions with impurities; under the experimental conditions, these collisions are the primary mechanism for momentum relaxation in the interior of a sample. Our purpose in the present paper is to determine the role played by normal collisions in the mechanisms for momentum transfer from electrons to the boundaries of a sample under conditions such that electron-impurity scattering dominates ($l_i \ll l_N$). The nature of these mechanisms and their contributions to the

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resistivity turn out to be qualitatively different in "thick" samples ($l_i \ll d$) and "thin" samples ($l_i \gg d$); furthermore, they depend on the nature of the N processes (electron-phonon or electron-electron collisions). Normal collisions, even if improbable, will give rise to an inverse temperature dependence of the resistivity under certain conditions. We believe that it is time for a study in this direction, since the theoretical papers[12-16] which have appeared in the past four years following the observation of a minimum on the temperature dependence of the resistivity of potassium have been based on assumptions whose validity is difficult to demonstrate (more on this below). In any case, our analysis, which is based on a systematic consideration of normal collisions by perturbation theory, leads to results which are very different from those of these other papers in all cases.

Clearly pertinent here is Black's numerical calculation,[17] by the Monte Carlo method, of the conductivity of a wire with normal electron-electron collisions and scattering by impurities. Black showed in particular that a slight inverse temperature dependence would be retained outside the scope of the hydrodynamic approach under certain conditions. Unfortunately, the numerical data reported in Ref. 17 do not span the region corresponding to the experiments of Ref. 1 or the limiting cases which we have studied, in which qualitatively new mechanisms for the effect of normal collisions are manifested.

2. PERTURBATION THEORY

If electron-impurity collisions are considerably more frequent than electron-phonon and electron-electron collisions, the contribution of the latter can be found by perturbation theory. We begin by writing a kinetic equation for the nonequilibrium increment $-\chi\partial f_0/\partial\varepsilon$ in the electron distribution function:

$$\mathbf{n}\partial\chi/\partial\mathbf{r}+\chi/l_i+\hat{I}\chi=e\mathbf{E}\mathbf{n}, \qquad (2)$$

where $\mathbf{n}=\mathbf{v}/|\mathbf{v}|$ is a unit vector along the velocity, $\hat{I}$ is an operator representing electron-phonon and electron-electron collisions (after multiplication by v^{-1} and an integration of the enegy), and $f_0(\varepsilon)$ is the Fermi distribution.

The increment of first order in the term $\hat{I}\chi$ in the current, averaged over the cross section of the sample, can be written as $\delta j=\mathbf{E}\delta\sigma$, where

$$\delta\sigma=-\frac{2e^2l_i^2}{h^3}\int n_x n_x' I_{\mathbf{pp}'}K_{\mathbf{pp}'}\,dS_{\mathbf{p}}\,dS_{\mathbf{p}'}, \qquad (3)$$

$$K_{\mathbf{pp}'}=\Sigma^{-1}\int(1-\exp(-\tilde{s}/l_i))(1-\exp(-s'/l_i))\,d\Sigma; \qquad (4)$$

$$\hat{I}\chi=\int I_{\mathbf{pp}'}\chi_{\mathbf{p}'}\,dS_{\mathbf{p}'}.$$

Here $I_{\mathbf{pp}'}$ is the kernel of collision operator $\hat{I}$, $n_x=\mathbf{En}/E$, s' is the path length traversed by an electron with $\mathbf{p}'$ from the surface to the point of the collision described by the operator $\hat{I}$, $\tilde{s}$ is the path traversed by an electron with a momentum $\mathbf{p}$ from the point of the collision to the surface (s and $\tilde{s}$ depend on the momentum of the electron and on the position of the collision point in the cross section of the sample), $d\Sigma$ is an area element of the cross section, and $dS_{\mathbf{p}}$ is an area element of the Fermi surface. We are assuming that the reflection from the surface of the sample is diffuse: $\chi=0$ for $s=0$.

If the sample is a plate, the quantity $K_{\mathbf{pp}'}$ can easily be calculated completely:

$$K_{\mathbf{pp}'}=K_1+K_2, \quad K_1=\exp(-|t|-|t'|)\frac{\operatorname{sh}(t-t')}{t-t'},$$

$$K_2=1-\exp(-|t|)\frac{\operatorname{sh}t}{t}-\exp(-|t'|)\frac{\operatorname{sh}t'}{t'}, \qquad (5)$$

where $t=d/l_i n_z$, $2d$ is the thickness of the plate, and the z axis runs perpendicular to the surface of the plate.

3. THICK SAMPLES ($l_i \ll d$)

In a macroscopically homogeneous bulk sample, the additional incorporation of any scattering processes leads to an increase in the resistivity. This is a direct consequence of the variational principle (or of the positivity of entropy production). Normal collisions constitute an exceptional case. Incorporating normal collisions does not affect the bulk resistivity under the condition that other scattering mechanisms themselves lead to a drift distribution function for the electrons in an external electric field:

$$f_0(\varepsilon-\mathbf{pu})\approx f_0(\varepsilon)-\mathbf{up}\partial f_0/\partial\varepsilon,$$

where $\mathbf{u}$ is the velocity of the overall carrier drift. This is the situation when the energy spectrum and the electron scattering mechanisms are isotropic. Numerous results suggest that these conditions are satisfied better in potassium. In particular, the data of Refs. 10 and 11 reveal an exponential temperature dependence of the electron-phonon component of the resistivity of bulk metals in the dirty limit. (Umklapp processes during electron-electron collisions, whose contribution would be proportional to T^2, are not manifested in this temperature range, apparently because the pseudopotential is small.) If either the energy spectrum or the interaction with impurities were anisotropic, the asymptotic behavior of the small increment in the residual resistivity would be a power law[3]: $\delta\rho\sim T^5$ as a result of electron-phonon processes or $\delta\rho\sim T^2$ as a result of electron-electron N processes.

An inverse temperature dependence of the conductivity could arise as a result of normal collisions in a thin boundary layer with a thickness on the order of l_i (or near any macroscopic distortions which scatter electrons e.g., dislocations). In accordance with the discussion above, we will restrict the discussion to the isotropic model, and we will ignore umklapp processes. The basic component of $\delta\sigma$, which results from the violation of this approximation, is incorporated in (17). For the normal-collision operator $\hat{I}$ we have

$$\int n_x' I_{\mathbf{pp}'}\,dS_{\mathbf{p}}=\int n_x' I_{\mathbf{pp}'}\,dS_{\mathbf{p}'}=0.$$

It follows that the only nonvanishing component in (3) comes from those terms $K_{\mathbf{pp}'}$, which depend simultaneously on $\mathbf{p}$ and $\mathbf{p}'$. Omitting the exponentially small terms of order $\exp(-d/l_i)$, we find, after some simple calculations,

$$\delta\sigma=-\frac{2e^2l_i^3}{h^3}\frac{L}{\Sigma}\int\limits_{\substack{n_z>0\\ n_z'<0}} dS_{\mathbf{p}}dS\, n_x n_x'\frac{n_z n_z'}{n_z-n_z'}I_{\mathbf{pp}'}, \qquad (6)$$

where Σ is the cross-sectional area of the sample, and L is the perimeter of the cross section. Expression (6) holds for a sample with a cross section of any shape, provided only that its dimensions are considerably greater than l_i.

It is a simple matter to find from (6) an estimate of the

temperature-dependent increment in the average conductivity (through the use of the definition $\int dS'_{\mathbf{p}} I_{\mathbf{pp}'} \approx l_N^{-1}$). For a wire of diameter d, for example, we have

$$|\delta\sigma| \approx ne^2 p_F^{-1} l_i^3 / d l_N, \quad d \gg l_i,$$

where $n \approx (p_F/h)^3$ is the density of electrons. To derive the latter expression it is sufficient to assume that N collisions are manifested only in a boundary layer of thickness $\sim l_i$, where the conductivity is

$$\sigma \sim (l_i^{-1} + l_N^{-1})^{-1} \approx l_i - l_i^2/l_N,$$

while outside this layer we have $\sigma \sim l_i$. An incorrect result

$$\delta\sigma \approx ne^2 p_F^{-1} l_i^4 / d^2 l_N$$

was obtained in Refs. 16. In those studies, where the Callaway method $I^N \chi = (\chi_{\mathbf{p}} - \mathbf{up})\tau_N^{-1}$ was used, it was assumed that the carrier drift velocity $\mathbf{u}$ does not change near the surface. It is easy to verify that incorporating spatial variation in the velocity $\mathbf{u}$ will eliminate this discrepancy (as well as several other puzzling results of Refs. 16). The sign of $\delta\sigma$, however, as we now see, depends on the structural details of the collision operator, which are impossible in principle to deal with in the Callaway method.

We will begin with some physical arguments in favor of a "hydrodynamic" role of normal collisions in a thin layer of thickness l_i near a surface. Without altering the sum of the electron velocities (by virtue of $\mathbf{p} = m\mathbf{v}$) in the isotropic approximation, normal collisions do affect the current, causing a redistribution of nonequilibrium momentum between electrons which are incident on the surface and electrons which are reflected from it [it is no coincidence that only transitions in which the sign of n_z changes are involved in (6)]. The longitudinal momentum $\mathscr{P}_x$ is transferred primarily from the incident electrons to the reflected electrons, since the former are obviously further from equilibrium. Normal collisions, by hindering the transport of nonequilibrium momentum to the surface, thus increase the conductivity. The most graphic case in this regard is that of collisions involving a small change in the x projection of the electron velocity, indicated by the vertical arrows in Fig. 1. As a result of transitions of type (a) a certain fraction of the nonequilibrium electrons which make a positive contribution to the current ($n_x > 0$) avoids collisions with the surface. On the other hand, for the electrons which make a negative contribution to the current ($n_x < 0$) transitions of type (b) cause the number of collisions with the surface to obviously increase. Consequently, normal collisions have the consequence that a certain fraction of the momentum $\mathscr{P}_x$ will be transferred from the incident electrons to the outgoing electrons and will thus not be absorbed at the surface.

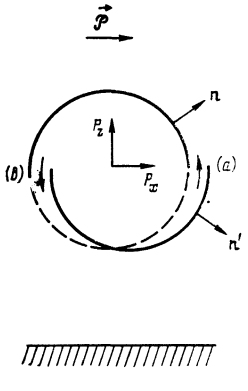

FIG. 1. Electron distribution function at a boundary. The incident electrons drift along the x axis, while the reflected electrons are at equilibrium (in the case of diffuse scattering). The dashed line is the Fermi surface.

It should be noted, however, that these arguments are based on the intuitive assumption that in acting to bring the system to a state of internal equilibrium normal collisions lead to equalization of the average quasimomentum of the incident and reflected electrons. This is indeed always the case under conditions such that normal collisions outweigh impurity collisions ($l_N \ll l_i$), but it is not necessarily the case in the opposite limit of arbitrarily infrequent normal collisions, which are acting on electrons whose distribution function is determined entirely by the interaction between impurities and the surface. It is easy to see that collisions with a relatively large momentum transfer, for which there is a change in the sign of not only n_z but also n_x, increase the differences between the x components of the incident and reflected electrons [correspondingly, under the conditon $n_x n'_x < 0$ we find $\delta\sigma < 0$ from (6) if $I_{\mathbf{pp}'} > 0$]. It is clear from this discussion that we have $\delta\sigma > 0$ if transitions through small angles dominate,

$$\frac{d}{d\alpha} I_{\mathbf{pp}'} < 0, \tag{7}$$

where α is the angle between the vectors $\mathbf{p}$ and $\mathbf{p}'$, and $I_{\mathbf{pp}'}$ is a matrix element of the transition between states $\mathbf{p}$ and $\mathbf{p}'$:

$$I_{\mathbf{pp}'} = I_{\mathbf{p}'} \delta_{\mathbf{pp}'} - I_{\mathbf{pp}'}.$$

It is not difficult to show that inequality (7) is a sufficient (but by no means necessary) condition for the positivity of expression (6) for the quantity $\delta\sigma$. Specifically, by virtue of the isotropy we have (φ, φ' are the aximuthal angles of the unit vectors $\mathbf{n}$,$\mathbf{n}'$ in the xy plane)

$$n_x \int n_x' I_{\mathbf{pp}'} \, d\varphi' = 2\left(\frac{1-n_z'^2}{1-n_z^2}\right)^{1/2} n_z^2 \int_0^{\pi} I(\alpha) \cos(\varphi - \varphi') \, d(\varphi - \varphi'),$$

and the integral in this expression is positive, since the angle α increases monotonically with $\varphi - \varphi'$ between 0 and π.

In the case of *electron-electron collisions* the kernel of the collision operators can be written

$$I^{ee}_{\mathbf{pp}'} = \frac{2\pi^3 T^2}{3\hbar h^6 v^4} \int (2P_{\mathbf{pp}_1,\mathbf{p}_2\mathbf{p}'} - P_{\mathbf{pp}',\mathbf{p}_1\mathbf{p}_2}) \, dS_{\mathbf{p}_1} dS_{\mathbf{p}_2}, \tag{8}$$

$$P_{\mathbf{p}_1\mathbf{p}_2,\mathbf{p}_3\mathbf{p}_4} = [(V_{\mathbf{p}_3-\mathbf{p}_2} - V_{\mathbf{p}_4-\mathbf{p}_2})^2 + V^2_{\mathbf{p}_3-\mathbf{p}_2} + V^2_{\mathbf{p}_4-\mathbf{p}_2}] \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4),$$

where T is the temperature.

Describing the screened electron-electron interaction in the standard approximation

$$V_{\mathbf{p}} = 4\pi e^2 \hbar^2 (p^2 + \varkappa^2)^{-1},$$

where $\hbar\varkappa^{-1}$ is the screening radius, we find from (8)

$$\hat{I}^{ee}(\alpha)=\frac{2e^2p_FT^2}{3\hbar^3v^4\varkappa^4}\left[\frac{1+\gamma/2}{(1+\gamma)^{3/2}\sin\alpha/2}-\frac{1+\delta/2}{(1+\delta)^{3/2}\cos\alpha/2}\right.$$
$$-\frac{1}{(1+\gamma)^{1/2}(1+\delta)\sin\alpha/2}+\frac{1}{(2+\delta)(1+\delta)^{1/2}\cos\alpha/2}$$
$$\left.+\frac{1}{(1+\delta)^2\sin\alpha/2}\right], \qquad (9)$$
$$\delta=\left(\frac{2p_F\sin\alpha/2}{\varkappa}\right)^2, \quad \gamma=\left(\frac{2p_F\cos\alpha/2}{\varkappa}\right)^2.$$

In the case of extremely strong screening, $\varkappa\gg p_F$, with

$$\hat{I}^{ee}(\alpha)\sim 2(\sin\alpha/2)^{-1}-(\cos\alpha/2)^{-1},$$

condition (7) obviously holds, and we have $\delta\sigma>0$. In order of magnitude here we have

$$\delta\sigma\approx\delta\sigma_1(p_F/\varkappa)^4,$$

where

$$\delta\sigma_1=ne^2p_F^{-1}l_i^3/dl_{ee},$$

and $l_{ee}\sim T^{-2}$ is the Landau-Pomeranchuk electron-electron mean free path. In the opposite case, of weak screening, $\varkappa\ll p_F$, the behavior $\hat{I}^{ee}(\alpha)$ is extremely complicated. Condition (7) does not hold over the greater part of the interval $0<\alpha<\pi$. Calculations show that we have $\delta\sigma>0$, although a nonzero result arises only in the cubic approximation in the small parameter $\varkappa/p_F$:

$$\delta\sigma\approx-\delta\sigma_1p_F/\varkappa.$$

Numerical calculations based on (6) and (9) show that $\delta\sigma$ changes sign at $\varkappa/p_F\approx 1/4$, and condition (7) is eventually violated as $\varkappa$ decreases, at $\varkappa/p_F\approx 1$. Using the relation

$$\varkappa^2=4\pi e^2\hbar^2N(\varepsilon_F)$$

and the known values of the state density $N(\varepsilon_F)$, we find $\varkappa/p_F\approx 1.7$ for potassium. This value is considerably higher than the critical value $\varkappa/p_F\approx 1/4$. Because of the large margin involved here, the errors introduced by our approximations are probably inconsequential, and we can assert that infrequent electron-electron collisions in a thick sample lead to an inverse temperature dependence in potassium: $\delta\sigma>0$ under the conditions $d\gg l_i$ and $l_{ee}\gg l_i$.

Small-angle *electron-phonon scattering* is described by an integrodifferential operator, which can be written in the isotropic case as[18]

$$\hat{I}^{ep}\chi=-\mathscr{D}\Delta\chi_{\mathbf{p}}+\frac{\mathscr{D}}{\pi^2p_F}\int \operatorname{ctg}\alpha\Delta\chi_{\mathbf{p}'}\,dS_{\mathbf{p}'}. \qquad (10)$$

Here $\mathscr{D}\approx p_F^2l_{ep}^{-1}$ is the electron diffusion coefficient over the Fermi surface, $l_{ep}\sim T^{-5}$ is the electron-phonon transport mean free path, and Δ is the two-dimensional Laplacian acting in the plane tangent to the Fermi surface. The second term in (10) incorporates the nonequilibrium nature (the drag) of the phonons. Examining the corresponding part of the exact collision integral, we can show that this part does not satisfy condition (7) (an exchange of phonons is possible between any electron states on the Fermi surface and therefore corresponds to large-angle scattering). Nevertheless, we have an increment $\delta\sigma>0$ even in the case of electron-phonon collisions. In the calculation of $\delta\sigma$ we need to recall that the diffusion approximation breaks down in a small region near $n_z=n_z'=0$: The substitution of (10) into (6) leads to some poorly defined functions (because of the rapid variation in the electron distribution at small $|n_z|$; Fig. 1). In order to resolve this uncertainty for the first term in (10) we should go back to our original integral expression

$$\int W_{\mathbf{pp}'}(\chi_{\mathbf{p}}-\chi_{\mathbf{p}'})\,dS_{\mathbf{p}'},$$

where the kernel W falls off exponentially at $|\mathbf{p}-\mathbf{p}'|\gg q$. Also using

$$(2\hbar^3)^{-1}\int W_{\mathbf{p},\mathbf{p}+\mathbf{q}}q_z^2\,d^2q=\mathscr{D},$$

we find

$$\delta\sigma=\frac{2e^2l_i^3\mathscr{D}L}{\hbar^3\Sigma}\left(\frac{\pi}{2}-\psi\right), \qquad (11)$$

where the number $\psi>0$ is the contribution from the second term in (10), i.e., from the nonequilibrium nature of the phonons. A numerical calculation reveals $\pi/2-\psi>0.2$.

4. THIN SAMPLES ($d\ll l_i$)

We know that "grazing" electrons, which are moving at a small angle with respect to the axis of the sample, play an important role in the electrical conductivity of thin samples. In the relaxation-time approximation, which is fairly successful in describing large-angle scattering, the electrical conductivity of a sheet is $\sigma\sim d\ln l/d$, and that of a wire is $\sigma\sim d-d^2l^{-1}\ln(l/d), d\ll l$. Grazing electrons dominate the conductivity of a sheet and determine an increment in the conductivity of a wire which depends on bulk collisions. In each case we naturally have $d\sigma/dT<0$, since large-angle scattering removes an electron from the group of grazing electrons and thus promotes momentum relaxation. The latter assertion is always true for electron-impurity and electron-electron collisions. It is also true for scattering by phonons in the "dirty" limit, provided that the scattering angle $\delta\vartheta\approx g/p_F$ is much larger than the characteristic angle of the dispersal of the grazing electrons, d/l_i. In the clean limit ($l_i\to\infty$) the mechanisms for electron-phonon relaxation in thin samples are much more complicated, but an inverse temperature dependence does not arise.[19,20] We are left with only a single possibility, which is the subject of the discussion below. This is the case in which electron-phonon collisions in the dirty limit are manifested as a small-angle process:

$$\delta\vartheta l_i\ll d\ll l_i\ll l_{ep}^{eff}. \qquad (12)$$

Here $l_{ep}^{\text{eff}}=l_{ep}(d/l_i)^2$ is the mean free path with respect to the removal of an electron from a grazing positon as a result of diffusion. As the temperature is lowered, inequality (12) unavoidably becomes satisfied (if $d\ll l_i$).

Kaveh and Wiser[12] have predicted an inverse temperature dependence of the conductivity, specifically for the case described by the inequalities (12). Their arguments run as follows: As a result of a collision with a phonon, the angle ϑ betwen the electron velocity and the surface of the sample can decrease or increase, with equal probabilities, by an amount on the order of $\delta\vartheta=q/p_F$. However, the elongation of the path traversed by an electron before relaxation at the surface is greater in the first case than the shortening of this

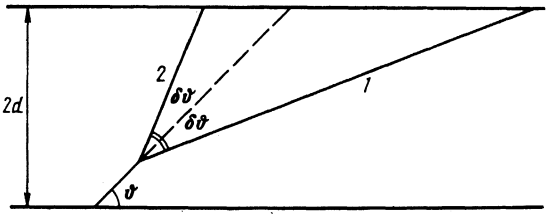

FIG. 2. The elongation of a path of type 1 is greater than the shortening of a path of type 2.

path in the second (Fig. 2). The lengths are on the order of $d(\delta\vartheta)^2\delta^{-3}$, so there is an increase in the conductivity averaged over the cross section of the wire, by an amount

$$\delta\sigma \approx ne^2 p_F^{-1} l_i^2 / l_{ep}.$$

In the case of a sheet, a result greater by a factor of l_i/d is found, as is easily verified. These arguments are incorrect, however, since they ignore the fact that the grazing electrons ($\vartheta \approx d/l_i$) dominate the value of $\delta\sigma$. For these electrons, an elongation of the path is not possible (the mean free path cannot exceed l_i), and the substantial shortening of the path of these electrons is the dominant effects, as we will see below. In other words, if, following Ref. 12, we restrict the discussion to electron-phonon scattering accompanied by a change in the angle ϑ alone, then the increment in the conductivity turns out to be negative.

In case (12), of small-angle electron-phonon scattering, it is convenient to rewrite expression (3) in a form corresponding to the diffusion approximation, (10):

$$\delta\sigma = -\frac{2e^2}{h^3} l_i^2 \mathscr{D} \frac{1}{\Sigma} \int \nabla[n_x(e^{-\tilde{s}/l_i}-1)]\nabla[n_x(e^{-s/l_i}-1)] \times dS_p\, d\Sigma + \delta\sigma_{dr}. \quad (13)$$

The term $\delta\sigma_{dr}$ here stems from the second term in (10). Expression (13) holds only under the condition $\delta\vartheta \ll d/l_i$, of course; in the opposite limit, the relaxation-time approximation, with a characteristic mean free path $l_{ep}(\delta\vartheta)^2$, holds. If the sample is a sheet, then only the "fast" dependence of s and $\tilde{s}$ on the angle ϑ need be taken into account in the differentiation in (13), in the leading approximation in the small parameter d/l_i:

$$s = s_z/\sin\vartheta,\ d/d\vartheta \exp(-s/l_i) \approx s\vartheta^{-1} l_i^{-1} \exp(-s/l_i)$$

under the conditon $\vartheta \ll 1$ (s_z is the distance from the point at which the collision occurred to the surface of the sheet). We have thus shown that the first term in (13) is negative (by definition, $s, \tilde{s} > 0$), so the conclusion of Ref. 12 is incorrect.

There are, on the other hand, some processes which are ignored both in (13) and in Ref. 12, although their contributions are on the same order of magnitude as those which have been considered. These are transitions in a small neighborhood (of width $\sim\delta\vartheta$) of the point $\vartheta = 0$, in which the diffusion approximation breaks down (see the preceding section of this paper). Their positive contribution to $\delta\sigma$ is, by virtue of the inequality $\delta\vartheta \ll d/l_i$, the same as in a thick sample and is thus described by the first term in (11). The contribution from the drag, $\delta\sigma_{dr}$, on the other hand, is small, in proportion to the parameter d/l_i, since as the nonequilibrium phonons are reabsorbed most of their momentum is transferred to nongrazing electrons and quickly undergoes relaxation at the surface. As a result of simple calculations we find the following result for a sheet, under the inequalities (12):

$$\delta\sigma = -\frac{\pi e^2 l_i^3 \mathscr{D}}{3h^3 d} \approx -\frac{ne^2}{p_F}\frac{l_i^3}{l_{ep} d} < 0. \quad (14)$$

The situation in a wire is more complicated since a substantial change in the path of the grazing electrons results from changes in two angles during electron-phonon collisions: the angle ϑ, between the velocity of the electron and the wire axis, and the azimuthal angle φ. As in the case of a sheet, a change in the angle ϑ leads to a net shortening of the path, while a change in φ increases the path, on the average, regardless of the cross-sectional shape of the wire. Specifically, let us look at the scalar product which figures in (13) in a spherical coordinate system $s_\perp = s\sin\vartheta$, $b = s_\perp + \tilde{s}_\perp$, $\vartheta \ll 1$:

$$p_F^2 \nabla \exp(-sl_i^{-1}) \nabla \exp(-\tilde{s}l_i^{-1}) \approx \vartheta^{-4}[s_\perp \tilde{s}_\perp + (\partial s_\perp/\partial\varphi)(\partial\tilde{s}_\perp/\partial\varphi)]\exp(-bl_i^{-1}\vartheta^{-1}).$$

The first term in square brackets, which corresponds to a change in ϑ, is positive. The second term, in contrast, is negative if the angles β and $\tilde{\beta}$ in Fig. 3 are either both acute or both obtuse (this is always the case for a circular cross section):

$$(\partial s_\perp/\partial\varphi)(\partial\tilde{s}_\perp/\partial\varphi) = -s_\perp \tilde{s}_\perp \operatorname{ctg}\beta \operatorname{ctg}\tilde{\beta}.$$

It is convenient to carry out the integration in (13) by transforming from the variables $\vartheta, \beta, \mathbf{r}$ to $\vartheta, \beta, \mathscr{L}, s_\perp$, where $\mathbf{r}$ is the coordinate of the point of the electron-phonon collision in the cross section, and $d\mathscr{L}$ is an element of length along the perimeter. The Jacobian of this transformation is $\sin\beta$. We have $(0 < \beta < \pi)$

$$\delta\sigma \sim -\int d\vartheta\, d\beta\, ds_\perp\, d\mathscr{L}\, \vartheta^{-3} s_\perp (b - s_\perp) e^{-b/\vartheta l_i} (1 - \operatorname{ctg}\beta \operatorname{ctg}\tilde{\beta}) \sin\beta.$$

After integration over $s_\perp$ and ϑ (b and ϑ depend on $\mathscr{L}$ and β), we find

$$\delta\sigma \sim -\int d\mathscr{L}\, d\beta\, b(1 - \operatorname{ctg}\beta \operatorname{ctg}\tilde{\beta})\sin\beta = 0.$$

It is straightforward to verify the latter equality, by integrating by parts in the integral corresponding to the second term in parentheses and by using the relation

$$b \operatorname{ctg}\tilde{\beta} = \partial b/\partial\beta.$$

Accordingly, to first order in the small parameter d_{max}/l_i (d_{max} is the largest of the dimensions of the wire cross section), the corrections to the conductivity which result from the changes in the angles ϑ and φ cancel each other out. In

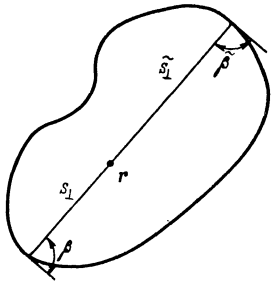

FIG. 3. Projection of an electron path onto the plane perpendicular to the wire axis.

other words, there is no contribution $|\delta\sigma| \sim l_i^2/l_{ep}$ in a wire with a cross section of any shape. It is not difficult to show that a nonzero result arises in the first term in expression (13) only when terms $\sim (d/l_i)^2$, i.e., $\delta\sigma \sim d^2/l_{ep}$, are taken into account. Consequently, in a wire, in contrast with a sheet, drag plays the leading role in (13), and it leads to a nonzero result in the first approximation in d/l_i. The sign of $\delta\sigma_{dr}$ becomes obvious when we write it in the form [see (3), (4), (10)]

$$\delta\sigma_{dr} \approx \frac{e^2 l_i^2 \mathcal{D}}{\pi^2 \hbar^3 p_F^4} \frac{1}{\Sigma} \int\limits_{0<\vartheta,\vartheta'<\pi} dS_{\mathbf{p}}\, dS_{\mathbf{p}'}\, d\Sigma \cdot \ldots \cdot (\mathbf{r}')]^2$$

$$\approx \frac{ne^2}{p_F^{-1}} \frac{l_i d}{l_{ep}} > 0, \qquad (15)$$

$$\psi_{\mathbf{p}}(\mathbf{r}) = e^{-s/l_t} + e^{-\tilde{s}/l_t}.$$

Here we have used

$$p_F^2 \Delta \operatorname{ctg}\alpha = \sin^{-3}\alpha \cos\alpha,\ s(-\mathbf{p}, \mathbf{r}) = \tilde{s}(\mathbf{p}, \mathbf{r}),$$

$$\int \Delta \operatorname{ctg}\alpha\, dS_{\mathbf{p}'} = \int \Delta \operatorname{ctg}\alpha\, dS_{\mathbf{p}} = 0,$$

and the circumstance that the integrand falls off quite rapidly under the conditons $\vartheta, \vartheta' \gg d/l_i$.

Finally, we need to take into account the transitions near the point $\vartheta = 0$, which are not described in the diffusion approximation. Their contribution is also positive, of course, and it is given in order of magnitude by

$$\delta\sigma \approx \frac{ne^2}{p_F} \frac{l_i^3 \delta\vartheta}{d l_{ep}}. \qquad (16)$$

For a wire of arbitrary cross-sectional shape (but with $d_{\max} \ll l_i$), we thus have an inverse temperature dependence of the conductivity. At temperatures $\delta\vartheta \ll (d/l_i)^2$ ($\delta\vartheta \approx T/\Theta$, where Θ is the Debye temperature), we have $\delta\sigma \sim T^5$ according to (15), while in the region $(d/l_i)^2 \ll \delta\vartheta \ll d/l_i$ we have $\delta\sigma \sim T^6$ according to (16).

5. CONCLUSION

Let us formulate the condition under which a minimum exists on the temperature dependence of the electrical resistivity in the dirty limit ($l_N \gg l_i$), i.e., in which the situation which we have called "weak hydrodynamics" prevails.

In thick samples, with $l_i \ll d$, for an arbitrary relation between l_N and d, we have the following expression for the temperature-dependent increment in the conductivity (we are omitting a common factor of $ne^2 p_F^{-1}$ and numerical coefficients ~ 1):

$$\delta\sigma \sim l_i^3/l_N d - a l_i^2/l_N - l_i^2/l_U. \qquad (17)$$

Here d is the radius of the wire or the thickness of the sheet (more precisely, the ratio of the cross-sectional area of the sample to its perimeter). The coefficient a is a measure of the anisotropy of the Fermi surface or of the scattering by impurities (if the deviation from isotropy is slight, a is proportional to the square of the deviation). The lengths l_N and l_U can refer to either electron-phonon or electron-electron collisions.

In thin samples, an inverse temperature dependence is possible in a wire with a cross section of arbitrary shape (provided that the maximum transverse dimension satisfies $d_{\max} \ll l_i$), under the condition that small-angle electron-phonon scattering outweighs electron-electron scattering $l_{ep}^{\text{eff}} < l_{ee}$, and under the condition that the scattering angle $\delta\vartheta = q/p_F$ is small in comparison with $d_{\max}/l_i$ (see inequality (12)]. According to (15) and (16), we have

$$\delta\sigma \sim l_i d/l_{ep} + l_i^3 \delta\vartheta/l_{ep} d - (d^2/l_{ee}) \ln(l_{ee}/d) \qquad (18)$$

in this case. The last term here describes the contribution of electron-electron collisions. Expression (18) is based on results derived for an isotropic model. It is easy to see, however, that a sufficient condition for its applicability is that the axis of the wire coincide with a fourfold symmetry axis of a crystal or that the cross section of the wire have the given symmetry. Otherwise, the leading contribution to $\delta\sigma$ will not cancel out,

$$|\delta\sigma| \sim l_i^2/l_{ep},$$

and the sign of the effect will depend on the relative orientation of the anisotropy axes of the crystal and of the wire cross section.

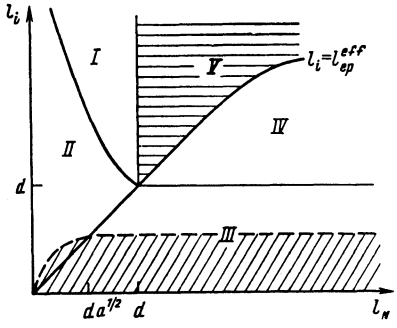

FIG. 4. Regions in which normal collisions act on the electrical conductivity by different mechanisms.

Figure 4 shows schematically the regions in the (l_k, l_N) plane in which normal collisions act on the electrical conductivity by different mechanisms. It is assumed here that umklapp processes are considerably less frequent than electron-impurity and normal collisions. Region I corresponds to a fully developed hydrodynamic situation; in this region we have $\delta\sigma \approx \sigma \sim d^2/l_N$. The boundary of the region on the left is the hyperbola $d^2/l_N = l_i$. A Knudsen minimum occurs along the right boundary. In region II the hydrodynamic situation persists in a surface layer with a thickness on the order of $(l_i l_N)^{1/2}$:

$$\delta\sigma \sim l_i (l_i l_N)^{1/3} d^{-1} - a l_N, \qquad (19)$$

where the second term is the bulk contribution of normal collisions which stems from the anisotropy [a is on the order of the corresponding coefficient in (17)]. Region III corresponds to a weak hydrodynamic situation in thin samples; here $\delta\sigma$ is given by (17). Region IV has the same meaning in the case of a thin wire; here $\delta\sigma$ is given by (18). In region I we have $d\sigma/dT > 0$, of course. In regions II and III this inequality holds for $a = 0$, but for $1 > a > 0$ we have $d\sigma/dT < 0$ in a certain part of the regions (hatched in Fig. 4), as can be seen from (17) and (19). We see from (18) that in region IV we have $d\sigma/dT > 0$ under a relatively lenient requirement on the probability for electron-electron collisions,

$$l_{ee} \gg l_{ep}^{eff} (d/l_i + \delta\vartheta l_i/d)^{-1}$$

and under the restrictions on the symmetry of the sample which we listed above.

In the experiments of Ref. 1 and also in a subsequent study,[21] a minimum on the temperature dependence of the resistance was apparently observed at a value of d comparable to l_i (the boundary between regions III and IV in Fig. 4). The behavior $\delta\rho_{an} \sim d^{-1}$, which holds approximately for the anomalous part of the resistivity in Ref. 21, corresponds to our results both in the case $d \gg l_i$ [expression (17)] and in the case $d \ll l_i$ [expression (18)]. (The results of Ref. 12 at $d \ll l_i$ and of Ref. 16 at $d \gg l_i$ lead to $\delta\rho_{an} \sim d^{-2}$.) A comparison with experimental data for liquid-helium temperatures shows that the mean free paths with respect to normal electron-phonon and electron-electron collisions in potassium at $T \sim 1$ K are not substantially different and correspond in order of magnitude to the observed values at the minimum $\delta\sigma/\sigma \sim 10^{-3} - 10^{-4}$. In the region of the inverse temperature dependence we could expect a temperature dependence of either $|\delta\rho| \sim T^5$ or $|\delta\rho| \sim T^2$ under the condition $d \gg l_i$ and $|\delta\rho| \sim T^5$ under the condition $d \ll l_i$. The experiments of Ref. 21 yield $|\delta\rho| \sim T^{7/3}$. It is possible that the temperature dependence will approach $|\delta\rho| \sim T^5$ with decreasing thickness.

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