Gravitational radiation from electromagnetic systems

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It is shown that the spectrum of gravitational radiation of a charge e with mass m, undergoing finite motion in an electromagnetic field, smoothly varying in the neighborhood of the orbit over a region of the order of the radius of curvature, differs in the ultrarelativistic limit from the spectrum of the charge's electromagnetic radiation. The difference consists of the frequency-independent coefficient $4\pi G m^2 \Gamma^2/e^2$, where Γ is of the order of the Lorentz factor of the charge and depends on the direction of the wave vector and on the behavior of the field in the above-indicated region. For a plane-wave external field the gravitational and electromagnetic spectra are strictly proportional to each other for arbitrary velocities of the charge. Localization of the external forces near the orbit violates this proportionality of the spectra and weakens the gravitational radiation by an amount of the order of the square of the Lorentz factor.

1. INTRODUCTION

According to the general theory of relativity the source of gravitationaal radiation (GR) is the conserved total energy-momentum tensor (EMT) of the system. At the same time the EMT of the gravitational field is not an unambiguously defined quanity.[1,2] This is one of the reasons why the problem of radiation of gravitational waves is in general so complicated and far from a definitive solution, in spite of considerable efforts.[1]

Under these circumstances a detailed study of GR (in general—of the gravitational field) of a simple electrodynamic system, namely a charged particle moving in an electromagnetic field, is of considerable interest. In the first place, such GR has some features in common with the GR of a body moving in a gravitational field. In the second place, at the tremendous energies of particles in the accelerators now being planned the possibility arises of testing in the laboratory ultrarelativistic effects of the general theory of relativity.[3-5] There is also interest in the question as to what information about the dynamic properties of the system is transmitted to its GR, in particular to what extent does it exceed the information transmitted by the electromagnetic radiation (EMR) of the system and under what conditions does GR inherit known regularities of the EMR.

As is well known from electrodynamics,[6,7] the spectrum of classical EMR of a charge is fully determined by the Fourier components of the conserved current density $j_\alpha(q)$:

$$d\mathscr{E}_{\mathbf{q}}=|j_\alpha(q)|^2\frac{d^3q}{16\pi^3},\qquad j_\alpha(q)=e\int_{-\infty}^{\infty}d\tau\,\dot{x}_\alpha(\tau)\,e^{-iqx(\tau)},\tag{1}$$

i.e., by its trajectory $x_\alpha(\tau)$, and does not depend on the nature of the forces responsible for the motion of the charge along this trajectory.

On the other hand, the spectrum of classical GR of a body of mass m, moving along the trajectory $x_\alpha(\tau)$, is determined by the Fourier components of the conserved EMT $T_{\alpha\beta}(q)$ of the whole system[7,8]:

$$d\mathscr{E}_{\mathbf{q}}=8\pi G[T_{\alpha\beta}(q)T^{\alpha\beta*}(q)-{}^1/_2|T_\alpha{}^\alpha(q)|^2]\,d^3q/16\pi^3.\tag{2}$$

Since $T_{\alpha\beta}(q)$ is the sum of the EMT of the body under consideration

$$t_{\alpha\beta}(q)=m\int_{-\infty}^{\infty}d\tau\,\dot{x}_\alpha(\tau)\dot{x}_\beta(\tau)\,e^{-iqx(\tau)}\tag{3}$$

and the EMT of the force field responsible for moving the body along the specified trajectory, the GR spectrum depends essentially on the nature of this force field. An exception occurs in the case of nonrelativistic motion of a body (or bodies), forming together with the force field a closed system. In that case, independently of the nature of the forces acting on the body, the GR is quadrupole-like and is described by the familiar formulas[6]

$$\frac{d\mathscr{E}_{\mathbf{q}}}{dt}=\frac{G}{4\pi}\left[\frac{1}{4}(\dddot{D}_{ij}n_in_j)^2+\frac{1}{2}\dddot{D}_{ij}{}^2-\dddot{D}_{ij}\dddot{D}_{ik}n_jn_k\right]d\Omega,$$

$$\frac{d\mathscr{E}}{dt}=\frac{G}{5}\dddot{D}_{ij}{}^2,\quad D_{ij}=\int t_{00}(\mathbf{x},q^0)\left(x_ix_j-\frac{1}{3}r^2\delta_{ij}\right)d^3x,\tag{4}$$

containing the quadrupole moment D_{ij} of the distribution of the moving particle; $\mathbf{n}=\mathbf{q}/q^0$. On the other hand, in the general relativistic case the GR spectrum contains information about the dynamic properties of its source and this circumstance is of considerable interest.

In this paper we study the GR of a body of mass m and charge e, moving under the influence of electromagnetic forces: in a homogeneous magnetic field, in the Coulomb field of a heavy center, in a plane-wave field with circular or linear polarization. Although in each of these cases the GR spectrum has its specific properties, in the ultrarelativistic limit it coincides with the spectrum $|j_\alpha(q)|^2$ of the EMR accurate up to the replacement of the charge squared e^2 by the quantity $4\pi G m^2\Gamma^2$, where Γ is proportional to the effective Lorentz factor of the moving body and depends essentially on the character of the external field. In that limit the radiation wave vector q is pinned to the plane of motion of the body, forming with it the small angle $\alpha\lesssim\gamma^{-1}\ll 1$, and the radiation frequency exceeds by a factor γ^3 the fundamental frequency ω, $q^0\equiv|q|\sim\gamma^3\omega$. In this way, for $\gamma\gg 1$, in the effective region of frequencies and angles of radiation one has the relation

$$8\pi G[T_{\mu\nu}(q)T^{\mu\nu*}(q)-{}^1/_2|T_\mu{}^\mu(q)|^2]\approx\frac{4\pi G m^2\Gamma^2}{e^2}|j_\mu(q)|^2.\tag{5}$$

Since in the rest frame of the body in ultrarelativistic motion any external electromagnetic field looks like a plane wave, one might expect Eq. (1) to be not approximate but exact for the GR spectrum of a body moving in a plane-wave electromagnetic field. Indeed, as will be shown in Sec. 4, for the GR spectrum of such a motion one has the strict relation

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$$T_{\mu\nu}(q)T^{\mu\nu*}(q)-\frac{1}{2}|T_\mu{}^\mu(q)|^2=\frac{m^2\Gamma^2}{2e^2}|j_\mu(q)|^2,$$

$$\Gamma=\frac{m_* q_\perp}{mq_-}=\gamma_*\,\mathrm{ctg}\frac{\theta}{2},\qquad(6)$$

where m_* is the effective mass of the charge, equal to its average kinetic energy in the coordinate system where it is at rest on the average, q_3 and $\mathbf{q}_\perp$ are the parallel and perpendicular components of the radiation wave vector $\mathbf{q}$ with respect to the momentum of the wave $\mathbf{k}$, $q_-=q^0-q_3$, and θ is the angle between the vectors $\mathbf{q}$ and $\mathbf{k}$. The effective Lorentz factor γ_* and velocity v_* are defined by the relations $m_*=m\gamma_*=m(1-v_*{}^2)^{-1/2}$.

The value of Γ for ultrarelativistic motion in a circularly polarized wave for the indicated high frequencies of the GR tends to the Lorentz factor γ. This is due to the fact that as a result of orthogonality of the plane of motion to the vector $\mathbf{k}$ the angle θ differs from $\pi/2$ by no more than γ^{-1} and the effective mass coincides with the constant kinetic energy of the body in the system under consideration.

For ultrarelativistic motion in a linearly polarized wave the quantity Γ is given as before by Eq. (6), the ultrarelativism simply leads to a lower bound on the range of effective angles θ: $2\,\mathrm{arccot}\,\sqrt{2}\leqslant\theta\leqslant\pi$. That the θ-dependence of Γ is preserved in the ultrarelativistic case is connected with the fact that in the motion of the charge along the figure-eight trajectory lying in the plane containing the vector $\mathbf{k}$, the angle between its velocity $\mathbf{v}$ and the vector $\mathbf{k}$ transverses all values between $\theta_0=\mathrm{arccot}(2\sqrt{2}/v_*)$ and π four times (see Ref. 6). In the ultrarelativisitic case this range of angles widens ($\theta_0\to2\,\mathrm{arccot}\,\sqrt{2}$ for $v_*\to1$) and restricts the effective angle of radiation θ, since the radiation becomes pinned to the direction of the velocity.

It should be emphasized that $\Gamma(\theta)$ does not coincide with the Lorentz factor of the body at the point where its velocity $\mathbf{v}$ forms the angle θ with the vector $\mathbf{k}$. This is connected with the fact that, in contrast to EMR, which in the ultrarelativistic limit is emitted along the velocity vector of the charge and is formed along a segment of a trajectory γ times smaller than the local radius of curvature, the GR—although emitted by the ultrarelativistically moving charged body along its velocity vector—is formed in a region of the order of the average radius of curvature of the trajectory.

The preservation of the extended region of formation of GR even for ultrarelativistic motion of the body is connected with the fact that along with emission of GR by the local source—the EMT of the body $t_{\mu\nu}$—there occurs emission of GR by the extended source—the EMT $\theta_{\mu\nu}$ of the external and self electromagnetic fields. The latter mechanism consists of emission by the local source—the current j_μ—of a virtual photon, which due to the gravitational interaction with a quantum of the external electromagnetic field creates a real graviton. In the ultrarelativistic limit the frequency of the virtual photon exceeds by γ^3 times the fundamental frequency ω (defined by the radius of curvature r of the trajectory, $\omega\approx c/r$), while its "mass" is of the order of $\gamma^{3/2}\omega$, i.e., small compared to its frequency. Therefore such a photon is emitted almost like a real one along the direction of the velocity vector of the charge and is formed in a small ($\sim c/\gamma\omega$) segment of the charge's trajectory, but its gravitational interaction with the quantum of the external field proceeds along a length of the order of the wavelength of the latter ($\sim c/\omega$). Clearly, the energy and momentum of the graviton coincide in essence with the energy and momentum of the virtual photon, but the probability of the appearance of the graviton is affected by the state of the external field over the length of formation of the graviton. The extension of the range of formation of GR from values $\sim c/\omega\gamma$ to values $\sim c/\omega$ results in Γ being different from the Lorentz factor γ at the moment of emission of the photon, and in its dependence on the structure of the external electromagnetic field. It should also be emphasized that the two GR mechanisms mentioned above are coherent. For the motion of a charge in a plane-wave field their interference results in the extinguishing of GR at an angle $\theta=\pi$, although EMR at this angle is not forbidden.

On the other hand, $\Gamma(\theta)$ diverges like θ^{-1} as $\theta\to0$, giving rise to the logarithmic singularity $d\theta/\theta$ in the GR spectrum since the current density at $\theta=0$ is finite and nonzero. The appearance of this singularity is connected with the fact that as $\theta\to0$ the main role in the emission of gravitons is played by the second, nonlocal mechanism. This amplitude is given by the sum of two amplitudes that are proportional to the propagation functions $(q\pm k)^{-2}$ of the virtual photons contained in the source of the gravitons—the EMT of the field of these photons and the plane electromagnetic wave. For $\theta\to0$ the photon propagators become infinite since $(q\pm k)^2=\mp(2\omega q^0)^{-1}(1-\omega\theta)^{-1}\approx\mp1/\omega q^0\theta^2$, while the remaining factor of the transverse components of EMT goes to zero like θ (the electromagnetic field of the photons and the field of the plane electromagnetic wave propagating in the same direction again constitute a plane electromagnetic wave, which, as is well-known, cannot be a source of gravitons because its EMT has no transverse components[6]). As a result the amplitude, and therefore Γ, diverges like θ^{-1} for $\theta\to0$.

Since the current density at the point $\theta=0$ is different from zero only the fundamental frequency $q^0=\omega$, the singularity in the GR spectrum at this point could be connected with the anomalously small mass ($\sim\omega\theta$) of the virtual photons and with, consequently, the very large ($\sim1/\omega\theta$) region of formation of GR emitted at such small angles.

A similar enhancement of the nonlocal mechanism also arises in the GR process of a charge moving in a constant homogeneous magnetic field (Sec. 3), if the field is homogeneous over a distance l significantly exceeding the wavelength λ of the radiation, i.e., for $l\gg\lambda$. In that case the external field is characterized by a wave vector k_α such that $|\mathbf{k}|\sim l^{-1}$ and $k^0=0$, so that the virtual photons emitted by the current j_α have very small mass, $|(q\pm k)^2|\sim q^0l^{-1}$. If one assumes $l=\infty$ then $T_{\mu\nu}(q)$ will have a pole at the point $q^2=0$. Then the GR spectrum can be represented as an expansion in powers of q^2/ω^2 (ω is the angular frequency in a circular orbit) as follows:

$$T_{\mu\nu}(q)T^{\mu\nu*}(q)-\frac{1}{2}|T_\mu{}^\mu(q)|^2=\left(\frac{\omega^2}{q^2}\right)^2a_{-2}+\frac{\omega^2}{q^2}a_{-1}+a_0+\ldots,\qquad(7)$$

where the leading term is proportional to the EMR spectrum

$$a_{-2}=\frac{2m^2\gamma^2q_\perp{}^2}{e^2\omega^2}|j_\mu(q)|^2.\qquad(8)$$

The proportionality coefficient in fact coincides with the corresponding coefficient in Eq. (6) for the circularly polarized wave. Indeed, the latter is obtained from Eqs. (7) and (8) by the replacement $q \to q \pm k$, where k_α is the wave vector with components $k_3 = k^0 = \omega$ and $k_1 = k_2 = 0$:

$$\frac{2m^2\gamma^2\omega^2 q_\perp^2}{e^2 q^4} \to \frac{m^2\gamma^2 q_\perp^2}{2e^2 q_-^2}. \tag{9}$$

For $l \gg \lambda$ but finite, there will appear in Eq. (7) a dependence on the character of the falloff of the magnetic field at distances $\sim l$. For example, for a Gaussian falloff of the field in the plane of motion $H(x) = H_0 \exp(-x_\perp^2/l^2)$ one must replace q^{-4} in the first term in Eq. (7) by $\pi l^2/16_\perp^2$, since in this case the field is characterized by the wave vector $k_{1,2} \sim l^{-1}$, $k_3 = k^0 = 0$. The second term vanishes in this case.

Equations (6) and (7), (8) clearly demonstrate the difference between the GR spectra for different electromagnetic force fields responsible for moving the massive charge along the same orbit. Thus, in the formation of the GR of a massive charge moving in an external electromagnetic field an essential role is played by the nonlocal mechanism with the participation of virtual photons, which in a number of cases (motion in a plane-wave field, in a constant homogeneous magnetic field, ultrarelativistic motion) results in proportionality of the GR and EMR spectra, with the coefficient of proportionality carrying information about the nonlocal mechanism and the form of the external field.

Such a connection between the GR and EMR spectra disappears if the external electromagnetic field is replaced by a local force field. We will show this using as an example the GR of a body elastically colliding with very massive but small balls distributed uniformly around a circle, so that in the limit of a large number of balls the resultant motion of the body will be uniform along the circumference (or motion along a ring-like trough).

2. THE GR SPECTRUM OF A BODY MOVING ALONG A RING-LIKE TROUGH

We consider the trough as a system of very massive small balls of mass M, much bigger than the mass m of the body elastically colliding with them, and energy-momentum tensors of the type of Eq. (3), where $m \to M$. It is not hard to see that the space components of these tensors are smaller by a factor of the order of $M/m\gamma$ than the corresponding components of the EMT of the moving body, and therefore in the limit $(M/m\gamma) \to \infty$ they can be ignored and one may take for the space components of the EMT for the whole system the space components of the tensor (3), i.e., $T_{ij}(q) = t_{ij}(q)$. The remaining four components of the $T_{\mu\nu}$ tensor can be found from the four conservation laws:

$$q^\mu T_{\mu\nu}(q) = 0, \quad \nu = 1, 2, 3, 0. \tag{10}$$

Then

$$T_{0j}(q) = -\frac{q^i}{q^0} t_{ij}(q), \qquad T_{00}(q) = \frac{q^i q^j}{q^{02}} t_{ij}(q). \tag{11}$$

For uniform motion with velocity $v = \omega r$ on a circle of radius r in the 1,2 plane the following space components are nonzero:

$$t_{11,22}(q) = \frac{mv^2\gamma}{2} \sum_n 2\pi\delta(q^0 - n\omega) \times \left[J_n \pm \frac{1}{2} J_{n+2} e^{-i2\varphi} \pm \frac{1}{2} J_{n-2} e^{i2\varphi} \right] e^{in\varphi},$$

$$t_{12}(q) = \frac{mv^2\gamma}{4i} \sum_n 2\pi\delta(q^0 - n\omega) \left[J_{n-2} e^{i2\varphi} - J_{n+2} e^{-i2\varphi} \right] e^{-in\varphi}, \tag{12}$$

where $J_n = J_n(z)$ is a Bessel function, $z = q_\perp r = |n| v \sin\theta$, θ and φ are the polar and azimuthal angles of the vector $\mathbf{q}$, and the sum is taken over integer $n \lessgtr 0$. We obtain hence and from Eq. (11) the following expression for the GR spectrum

$$T_{\mu\nu}(q) T^{\mu\nu *}(q) - \frac{1}{2} |T_\mu{}^\mu(q)|^2 = t \sum_n 2\pi\delta(q^0 - n\omega) \frac{m^2 v^4 \gamma^2}{4} \times \Big[J_{n+2}^2 + J_{n-2}^2 - \sin^2\theta (J_{n+2}^2 + J_{n-2}^2 + J_n J_{n+2} + J_n J_{n-2}) + \frac{1}{2} \sin^4\theta \Big(J_n + \frac{1}{2} J_{n+2} + \frac{1}{2} J_{n-2} \Big)^2 \Big]. \tag{13}$$

This is substantially different from the spectrum of EMR of a charge moving around a circle:

$$|j_\mu(q)|^2 = t \sum_n 2\pi\delta(q^0 - n\omega) e^2 [\mathrm{ctg}^2\theta J_n^2 + v^2 J_n'^2]. \tag{14}$$

This difference persists in the ultrarelativistic limit $\gamma \gg 1$, where effectively $z \approx n \sim \gamma^3$, $\alpha \equiv (\theta - \pi/2) \sim \gamma^{-1}$, and the square brackets in Eqs. (13) and (14) become

$$[\ldots]_{\mathrm{GR}} \approx 8 \Big[\Big(\alpha^2 + \frac{1}{2\gamma^2} \Big)^2 J_n^2 + \alpha^2 J_n'^2 \Big], \tag{13'}$$

$$[\ldots]_{\mathrm{EMR}} \approx \alpha^2 J_n^2 + J_n'^2, \tag{14'}$$

and instead of J_n and J_n' one should use their asymptotic representations in terms of the Airy function $\Phi(y)$:

$$J_n(z) \approx \frac{1}{\pi} \Big(\frac{2}{n} \Big)^{1/3} \Phi(y), \qquad J_n'(z) \approx -\frac{1}{\pi} \Big(\frac{2}{n} \Big)^{2/3} \Phi'(y),$$

$$y = (n/2)^{2/3} (1 - z^2/n^2). \tag{15}$$

It is easily seen that Eq. (5) does not hold, and if it is viewed as an order-of-magnitude estimate then one must take $\Gamma \sim 1$ and not $\Gamma \sim \gamma$, as it would be for a system with a nonlocal EMT.

In the nonrelativistic limit only the quadrupole terms $n = \pm 2$ remain in Eq. (13). According to Eq. (2) the term $n = 2$ gives rise to GR of intensity

$$\frac{d\mathscr{E}}{dt} = \frac{Gm^2 v^4 \omega^2}{8\pi} \int d\Omega (1 + 6\cos^2\theta + \cos^4\theta) = {}^8/_5 G m^2 v^4 \omega^2, \tag{16}$$

which is $\frac{1}{4}$ the intensity of the GR of that same body rotating in an orbit in a force field with extended EMT (see Ref. 6, §110).

This means that for the nonrelativistic system, for which the force field has an extended EMT, the contributions of the local and nonlocal channels to the GR amplitude coincide, so that the full amplitude is twice the amplitude of the local channel, and the corresponding intensities differ by a factor 4. We shall demonstrate this in Sec. 5 by a direct calculation on the example of the GR of a charge held on a circular orbit by a Coulomb force center.

In contrast to Eq. (6) the differential distribution (13) [and (16)] does not vanish for $\theta=\pi$.

3. GRAVITATIONAL RADIATION BY A CHARGE MOVING ON A CIRCLE IN A CONSTANT HOMOGENEOUS MAGNETIC FIELD

The source of GR of a charge moving in an electromagnetic field is the conserved tensor $T_{\mu\nu}=t_{\mu\nu}+\theta_{\mu\nu}$, consisting of the EMT $t_{\mu\nu}$ of the point charge, Eq. (3), and the EMT $\theta_{\mu\nu}$ of the external ($\varphi_{\alpha\beta}$) and self ($f_{\alpha\beta}$) electromagnetic fields[6,8]:

$$\theta_{\mu\nu}=-F_{\mu\lambda}F^{\lambda}{}_{\nu}-{}^1/_4 g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},\qquad F_{\alpha\beta}=\varphi_{\alpha\beta}+f_{\alpha\beta}. \tag{17}$$

The terms in the tensor $\theta_{\mu\nu}$ quadratic in $f_{\alpha\beta}$ may be omitted since we are not taking into account the action of the self field of the charge on itself. At the quantum level this corresponds to ignoring radiative corrections. In that approximation the tensor $T_{\mu\nu}$ is strictly conserved, and in the expression for the Lorentz force only the external field $\varphi_{\alpha\beta}$ appears, and not $\varphi_{\alpha\beta}+f_{\alpha\beta}$.

For the external fields considered below the terms in $\theta_{\mu\nu}$ quadratic in $\varphi_{\alpha\beta}$ are not a source of GR and they too may be ignored. Thus we shall use for the Fourier transform $\theta_{\mu\nu}(q)$ the expression

$$\theta_{\mu\nu}(q)=-\int\frac{d^4k}{(2\pi)^4}\Big[\varphi_{\mu\alpha}(k)f^{\alpha}{}_{\nu}(q-k)+\varphi_{\nu\alpha}(k)f^{\alpha}{}_{\mu}(q-k)+\frac{1}{2}g_{\mu\nu}\varphi_{\alpha\beta}(k)f^{\alpha\beta}(q-k)\Big], \tag{18}$$

where according to Maxwell's equations the self field may be expressed in terms of the current density

$$f_{\alpha\beta}(q)=\frac{i}{q^2}[q_\alpha j_\beta(q)-q_\beta j_\alpha(q)]. \tag{19}$$

For a constant homogeneous magnetic field **H**, directed along the 3-axis, only the following components are nonzero

$$\varphi_{12}(k)=-\varphi_{21}(k)=H(2\pi)^4\delta(k). \tag{20}$$

For a charge moving in such a field along a circular trajectory

$$x_1(\tau)=r\sin\Omega\tau,\ x_2(\tau)=r\cos\Omega\tau,\ x_3=0,\ x^0(\tau)=\gamma\tau, \tag{21}$$

with proper frequency $\Omega=\omega\gamma$, fixed by the field ($\Omega=eH/m$), the space components of the tensor $t_{\mu\nu}$ were given in Eq. (12). The nonzero space components of the tensor $\theta_{\mu\nu}$ are given by the equations

$$\theta_{11}(q)=\theta_{22}(q)=-\theta_{33}(q)=\frac{iH}{q^2}[q_1j_2(q)-q_2j_1(q)],\qquad \theta_{13}(q)=\frac{iH}{q^2}q_3j_2(q),\quad \theta_{23}(q)=-\frac{iH}{q^2}q_3j_1(q), \tag{22}$$

in terms of the space components of the current density, which equal

$$j_1(q)\pm ij_2(q)=ev\sum_n 2\pi\delta(q^0-n\omega)J_{n\mp1}(z)\exp[-i(n\mp1)\varphi]. \tag{23}$$

The remaining mixed and time components of the tensors $t_{\mu\nu}$ and $\theta_{\mu\nu}$ can be obtained from the same Eqs. (3) and (18)–(20); on the other hand, the corresponding components of the conserved tensor $T_{\mu\nu}$ can be reconstructed from its space components by Eq. (11). Both methods lead to the same result, Eq. (7), for the spectrum of the GR.

The photon propagator becoming infinite means that we have a cascade process: the current emits first a real photon, which is then transformed as it moves in the constant field into a graviton.[9–11] Interestingly, the constant field transfers zero 4-momentum to the graviton.

Since a real magnetic field will sooner or later cease to be homogeneous, one may imagine that its sources do not contribute to the Fourier components of the EMT under consideration. If such a magnetic field falls off at distances $\sim l\gg\lambda=2\pi\omega^{-1}$, for example like $H(x)=H\exp(-x_\perp{}^2/l^2)$ or $H(x)=H\exp(-|x_3|/l)$, then in Eq. (22) for θ_{ik} the factor $1/q^2$ will be replaced by $i\sqrt{\pi}l/4q$ or $il/2|q_3|$, leading to a corresponding change in Eq. (7) for the spectrum.

4. GRAVITATIONAL RADIATION BY A CHARGE MOVING IN THE FIELD OF A PLANE ELECTROMAGNETIC WAVE

The GR of a charge in a plane-wave external field is due to the same sources $t_{\mu\nu}$ and $\theta_{\mu\nu}$, the contribution of the latter proceeding via a virtual photon from the self field of the charge in order that the graviton be real.

We consider first the GR of a charge in the field of a circularly polarized wave

$$\varphi_{\alpha\beta}(x)=-\overset{(1)}{\varphi}_{\alpha\beta}\sin(kx)+\overset{(2)}{\varphi}_{\alpha\beta}\cos(kx),\qquad \overset{(i)}{\varphi}_{\alpha\beta}=k_\alpha \overset{i}{a}_\beta-k_\beta \overset{i}{a}_\alpha,\quad \overset{i}{a}_\alpha a^{j\alpha}=a^2\delta_{ij},\quad k_\alpha a^{i\alpha}=k^2=0. \tag{24}$$

We choose a coordinate system where the charge is at rest on the average, and the wave propagates along the 3-axis with wave vector $k_1=k_2=0$, $k_3=k^0=\omega$ and potential amplitudes $a^i_\alpha=a\delta^i_\alpha$. Then the particle's trajectory will be a circle in the plane $x_3=$ const, along which the particle will move with velocity $v=\xi,\xi=ea/m_*$, $m_*=(m^2+e^2a^2)^{1/2}$ and a phase as in Eq. (21) if $x_3=\pi/\omega$ is chosen in place of $x_3=0$. Therefore the components of $t_{\mu\nu}$ are the same as in Eq. (12), but with the phase factor $p\equiv\exp(-iq_3\pi/\omega)$. We list in addition the mixed and time components

$$t_1{}^0\pm it_2{}^0=mv\gamma\sum_n 2\pi\delta(q^0-n\omega)J_{n\mp1}\exp[-i(n\mp1)\varphi],\qquad t_{00}=m\gamma\sum_n 2\pi\delta(q^0-n\omega)J_ne^{-in\varphi}, \tag{25}$$

which must be equipped with the same factor p.

Since the Fourier components reduce to two δ-functions:

$$\varphi_{\alpha\beta}(q)=(2\pi)^4\frac{i}{2}[\Phi_{\alpha\beta}\delta(q-k)-\Phi_{\alpha\beta}{}^*\delta(q+k)],\qquad \Phi_{\alpha\beta}=\overset{(1)}{\varphi}_{\alpha\beta}-i\overset{(2)}{\varphi}_{\alpha\beta}, \tag{26}$$

the tensor $\theta_{\mu\nu}(q)$ is easily found in terms of the current components $j_\alpha(q\mp k)$, which differ from the components of the current for the motion (21) by the factor $\exp[-i(q_3\mp\omega)\pi/\omega]=-p$. Omitting the phase factor p common to all the tensors $t_{\mu\nu}$ and $\theta_{\mu\nu}$, we obtain

$$\theta_{11}=-\theta_{22}=\frac{mv^2}{4}\gamma\sum_n 2\pi\delta(q^0-n\omega)\left\{\left(\frac{q_\perp}{vq_-}J_{n-1}-J_{n-2}\right)\right.$$
$$\left.\times e^{-i(n-2)\varphi}+\left(\frac{q_\perp}{vq_-}J_{n+1}-J_{n+2}\right)e^{-i(n+2)\varphi}\right\},$$
$$\theta_{12}=\frac{mv^2\gamma}{4i}\sum_n 2\pi\delta(q^0-n\omega)\left\{\left(\frac{q_\perp}{vq_-}J_{n-1}-J_{n-2}\right)e^{-i(n-2)\varphi}\right.$$
$$\left.-\left(\frac{q_\perp}{vq_-}J_{n+1}-J_{n+2}\right)e^{-i(n+2)\varphi}\right\},$$
$$\theta_{13}=\theta_1{}^0=\frac{mv^2\gamma}{8}\sum_n 2\pi\delta(q^0-n\omega)$$
$$\times\left\{\left[2\frac{q_3-\omega}{vq_-}J_{n-1}-\frac{q_\perp}{q_-}(J_{n-2}-J_n)\right]e^{-i(n-1)\varphi}\right.$$
$$\left.+\left[2\frac{q_3+\omega}{vq_-}J_{n+1}-\frac{q_\perp}{q_-}(J_{n+2}-J_n)\right]e^{-i(n+1)\varphi}\right\},$$
$$\theta_{23}=\theta_2{}^0=\frac{mv^2\gamma}{8i}\sum_n 2\pi\delta(q^0-n\omega)$$
$$\times\left\{\left[2\frac{q_3-\omega}{vq_-}J_{n-1}-\frac{q_\perp}{q_-}(J_{n-2}-J_n)\right]e^{-i(n-1)\varphi}\right.$$
$$\left.-\left[2\frac{q_3+\omega}{vq_-}J_{n+1}-\frac{q_\perp}{q_-}(J_{n+2}-J_n)\right]e^{-i(n+1)\varphi}\right\},$$
$$\theta_{33}=\theta_3{}^0=\theta_{00}=\frac{mv^2\gamma}{2}\sum_n 2\pi\delta(q^0-n\omega)\frac{q^0+q_3-q^0v^{-2}}{q_-}J_ne^{-in\varphi}. \quad (27)$$

Here $q_-=q^0-q_3$; the Bessel functions depend on $z=q_\perp r$.

With the help of Eqs. (12), (24), and (27) we evaluate the GR spectrum to be

$$T_{\mu\nu}T^{\mu\nu*}-\frac{1}{2}|T_\mu{}^\mu|^2$$
$$=t\sum_n 2\pi\delta(q^0-n\omega)\frac{m^2\gamma^2q_\perp{}^2}{2q_-{}^2}\left[\frac{q_3{}^2}{q_\perp{}^2}J_n{}^2+v^2J_n'^2\right]. \quad (28)$$

A comparison with the EMR spectrum (14) of a charge moving on a circle shows that the GR and EMR spectra are connected by the simple relation, Eq. (6), with a proportionality coefficient independent of the frequency of the radiation (or the order of the harmonic).

We pass now to the GR of a charge in the field of a linearly polarized wave

$$\varphi_{\alpha\beta}(x)=-\Phi_{\alpha\beta}\sin(kx),\ \Phi_{\alpha\beta}=k_\alpha a_\beta-k_\beta a_\alpha,\quad k^2=ak=0. \quad (29)$$

In the coordinate system where the charge is at rest on the average, and the wave vector is in the directioin of the 3-axis, $k_1=k_2=0$, $k_3=k^0=\omega$, the trajectory of the charge forms a figure-eight lying in the plane of the wave vector **k** and the amplitude of the electric field $\mathbf{E}=\omega\mathbf{a}$. It is described by

$$x_1(\tau)=-\frac{\xi}{\omega}\sin\Omega_*\tau,\qquad x_2(\tau)=0,$$
$$x_3(\tau)=\frac{\xi^2}{8\omega}\sin 2\Omega_*\tau,\qquad x^0(\tau)=\gamma_*\tau+\frac{\xi^2}{8\omega}\sin 2\Omega_*\tau, \quad (30)$$

where

$$\Omega_*=\omega\gamma_*,\quad \gamma_*=m_*/m,\quad \xi=ea/m_*;$$

$m_*=(m^2+\frac{1}{2}e^2a^2)^{1/2}$ is the effective mass of the charge, equal to its average kinetic energy in the system under consideration.

In this case the nonzero components of the EMT of the charge are

$$t_{11}(q)=m_*\xi^2\sum_s 2\pi\delta(q^0-s\omega)A_2(s\alpha\beta),$$
$$t_{13}(q)=-\frac{m_*\xi^3}{4}\sum_s 2\pi\delta(q^0-s\omega)(2A_3-A_1),$$
$$t_{33}(q)=\frac{m_*\xi^4}{16}\sum_s 2\pi\delta(q^0-s\omega)(4A_4-4A_2+A_0),$$
$$t_1{}^0(q)=-m_*\xi\sum_s 2\pi\delta(q^0-s\omega)\left[\left(1-\frac{\xi^2}{4}\right)A_1+\frac{\xi^2}{2}A_3\right],$$
$$t_3{}^0(q)=\frac{m_*\xi^2}{4}\sum_s 2\pi\delta(q^0-s\omega)$$
$$\times\left[\xi^2A_4+(2-\xi^2)A_2-\left(1-\frac{\xi^2}{4}\right)A_0\right],$$
$$t_{00}(q)=m_*\sum_s 2\pi\delta(q^0-s\omega)$$
$$\times\left[\frac{\xi^4}{4}A_4+\xi^2\left(1-\frac{\xi^2}{4}\right)A_2+\left(1-\frac{\xi^2}{4}\right)^2A_0\right]. \quad (31)$$

Here $A_n(s\alpha\beta)$ are the functions introduced in Ref. 12:

$$A_n(s\alpha\beta)=\frac{1}{2\pi}\int^{\pi}d\varphi\cos^n\varphi\exp[i(\alpha\sin\varphi-\beta\sin 2\varphi-s\varphi)], \quad (32)$$

with arguments $\alpha=-\xi q_1/\omega$ and $\beta=\xi^2q_-/8\omega$.

Since the Fourier components of the field are now equal to

$$\varphi_{\alpha\beta}(q)=(2\pi)^4\frac{i}{2}\Phi_{\alpha\beta}[\delta(q-k)-\delta(q+k)], \quad (33)$$

we obtain for the EMT of the field, according to Eqs. (18) and (19),

$$\theta_{11}(q)=-\theta_{22}(q)=\frac{a}{4}\left[j_1+j_1-\frac{q_1}{q_-}(j_-+j_-)\right],$$
$$\theta_{12}(q)=-\frac{aq_2}{4q_-}(j_-+j_-),$$
$$\theta_{13}(q)=\theta_1{}^0(q)$$
$$=\frac{a}{4q_-}[(q^0-k^0)j_3-(q_3-k_3)j^0+(q^0+k^0)j_3-(q_3+k_3)j^0], \quad (34)$$
$$\theta_{23}(q)=\theta_2{}^0(q)=-\frac{aq_2}{4q_-}(j_1+j_1),$$
$$\theta_{33}(q)=\theta_3{}^0(q)=\theta_{00}(q)=\frac{a}{4q_-}[q_1(j^0+j_3+j^0+j_3)$$
$$-(q^0-k^0+q_3-k_3)j_1-(q^0+k^0+q_3+k_3)j_1].$$

Here j_α are the Fourier components of the current density; like components of that appear first in the square brackets depend on $q-k$, and those written second depend on $q+k$. The expressions for these components differ from those for $j_\alpha(q)$ by the replacement of the functions $A_n(s)$ by the functions $A_n(s\mp 1)$:

$$j_1(q\mp k)=-e\xi\sum_s 2\pi\delta(q^0-s\omega)A_1(s\mp1,\alpha\beta),$$

$$j_3(q\mp k)=\frac{e\xi^2}{4}\sum_s 2\pi\delta(q^0-s\omega)[2A_2(s\mp1,\alpha\beta)$$

$$-A_0(s\mp1,\alpha\beta)]\quad j^0(q\mp k)=e\sum_s 2\pi\delta(q^0-s\omega)$$

$$\times\left[\left(1-\frac{\xi^2}{4}\right)A_0(s\mp1,\alpha\beta)+\frac{\xi^2}{2}A_2(s\mp1,\alpha\beta)\right]. \quad (35)$$

We now form the conserved EMT $T_{\mu\nu}(q)$ and obtain for the spectrum of the GR the expression

$$T_{\mu\nu}(q)T^{\mu\nu*}(q)-\frac{1}{2}|T_\mu{}^\mu(q)|^2$$

$$=t\sum_s 2\pi\delta(q^0-s\omega)\frac{m^2q_\perp{}^2}{2q_-{}^2}[-A_0{}^2+x^2(A_1{}^2-A_0A_2)], \quad (36)$$

where $x=ea/m$, $A_n\equiv A_n(s\alpha\beta)$. In the derivation we used certain relations valid for the functions A_n, namely

$$(s-2\beta)A_0-\alpha A_1+4\beta A_2=0, \quad (37)$$

$$A_n(s-1)-A_n(s+1)$$
$$=\frac{2}{n+1}[4\beta A_{n+3}(s)-\alpha A_{n+2}(s)+(s-2\beta)A_{n+1}(s)], \quad (38)$$

$$A_n(s-1)+A_n(s+1)=2A_{n+1}(s). \quad (39)$$

The spectrum (36) is connected to the EMR spectrum of a charge in the field of a linearly polarized wave

$$|j_\mu(q)|^2=t\sum 2\pi\delta(q^0-s\omega)\frac{e^2}{\gamma_*{}^2}[-A_0{}^2+x^2(A_1{}^2-A_0A_2)] \quad (40)$$

by Eq. (6).

In this way, the GR and EMR spectra of a charge in the field of a plane wave differ only by the frequency-independent coefficient

$$\frac{4\pi Gm^2}{e^2}\gamma_*{}^2\operatorname{ctg}^2\frac{\theta}{2} \quad (41)$$

[we include here the factor $8\pi G$, omitted in Eqs. (28) and (36), see Eq. (2)]. Although the relation, Eq. (6), was already discussed in the Introduction, we remark that it is a consequence of the simultaneous action of the local and nonlocal mechanisms of GR. As a result the final answer depends on $j_\alpha(q)$ while $\theta_{\mu\nu}(q)$ depends on $j_\alpha(q\pm k)$. In the ultrarelativistic limit the nonlocal mechanism becomes dominating. We clarify this using Eq. (36). If we were to take on the left side in place of $T_{\mu\nu}$ just the field source $\theta_{\mu\nu}$, we would obtain for it ($\theta^\mu_\mu=0$)

$$\theta_{\mu\nu}(q)\theta^{\mu\nu*}(q)=t\sum_s 2\pi\delta(q^0-s\omega)m^2$$

$$\times\left[-\frac{x^2}{2+x^2}A_0A_2+\frac{q_\perp{}^2}{2q_-{}^2}x^2(A_1{}^2-A_0A_2)\right], \quad (42)$$

i.e., Eqs. (36) and (42) differ in essence in the first term. But in the ultrarelativistic case, when $x\gg1$, the main contribution to the integral for A_n in Eq. (32) comes from the saddle point $\varphi=\psi$, where

$$\cos\psi|_{x\gg1}\approx-q_1/\sqrt{2}q_-,$$

see Ref. 12. Since the azimuth angle is pinned to the value 0 or π, we have $q_1\approx\pm q_\perp$. Therefore $A_2\approx\cos^2\psi A_0\approx(q_\perp^2/2q_-^2)A_0$ and Eq. (42) goes into (36), where, naturally, one should use the asymptotic expression (see Refs. 12, 13) for the functions A_0^2 and $A_1^2-A_0A_2$.

We note that Eqs. (28) and (36) permit passage to the limit of an infinitely heavy charge and describe in that case the angular distribution of GR produced when a plane electromagnetic wave is incident on a fixed Coulomb center:

$$T_{\mu\nu}T^{\mu\nu*}-{}^1/{}_2|T_\mu{}^\mu|^2|_{m\to\infty}$$

$$=t\sum_{n=\pm1}2\pi\delta(q^0-n\omega)\frac{e^2a^2}{8}\operatorname{ctg}^2\frac{\theta}{2}\begin{cases}1+\cos^2\theta,\\1-\sin^2\theta\cos^2\varphi\end{cases}$$

(here the top and bottom lines refer to circular and linear polarizations, respectively). This result is in agreement with Ref. 14.

5. GRAVITATIONAL RADIATION BY A CHARGE ROTATING IN THE FIELD OF A COULOMB CENTER

We consider the motion of a charge e on a circle, Eq. (21), in the attractive Coulomb field of a fixed charge e'. In contrast to the magnetic and plane-wave cases this field has a continuous spectrum of wave vectors

$$\varphi_{\alpha\beta}(k)=-i\frac{e'}{k^2}(k_\alpha\delta_\beta{}^0-k_\beta\delta_\alpha{}^0)2\pi\delta(k^0). \quad (43)$$

Using in the integral (18) the causal proper-time representation for the propagators k^{-2} and $(q-k)^{-2}$, one may perform the Gaussian integration over $\mathbf{k}$ and the integration over one of the proper times. Then the tensor $\theta_{\mu\nu}$ is represented as an integral over the dimensionless variable u (the ratio of one of the proper times to their sum):

$$\theta_{\mu\nu}(q)=-\frac{iee'}{8\pi|\mathbf{q}|}\int_0^1du\int_{-\infty}^{\infty}d\tau e^{if}a_{\mu\nu},$$
$$f=-q_\alpha x^\alpha(\tau)+u(\mathbf{qx}(\tau)+|\mathbf{q}|r), \quad (44)$$

where $a_{\mu\nu}$ are polynomials of degree no higher than second in the coordinates $x_{1,2}(\tau)$ with coefficients quadratic in the components q_α and linear in u. They are unwieldy and will be omitted. The integral over τ in Eq. (44) can be represented in the form of a series over Bessel functions with argument $z_1=(1-u)z$, $z=q_\perp r$. The product of charges can be eliminated using the equation of motion $mv\gamma\omega=-ee'/4\pi r^2$. Beside the EMT $\theta_{\mu\nu}(q)$ of the external and self fields, the conserved EMT of the whole system contains the EMT $t_{\mu\nu}(q)$ of the charge e [see Eqs. (12), (25)], as well as the EMT $\tau_{\mu\nu}(q)$ of the Coulomb center with vanishing space and nonvanishing mixed and time components of harmonic $n=\pm1$:

$$\tau_1{}^0(q)\pm i\tau_2{}^0(q)=-mv\gamma(1-v^2-iv)e^{iv}\cdot2\pi\delta(q^0\mp\omega),$$

$$\tau_{00}(q)=-\frac{mv\gamma q_\perp}{2\omega}(1-v^2-iv)e^{iv} \quad (45)$$

$$\times[e^{-i\varphi}\cdot2\pi\delta(q^0-\omega)-e^{i\varphi}\cdot2\pi\delta(q^0+\omega)].$$

The divergence $iq_\alpha\tau^{\alpha\beta}(q)$ of this tensor equals the force density exerted on the fixed center e' by the charge e rotating around it.

Thus the spectrum of the GR consists of the following six terms:

$$T_{\mu\nu}(q)T^{\mu\nu*}(q)-{}^1/_2|T_\mu{}^\mu(q)|^2=\theta_{\mu\nu}\theta^{\mu\nu*}+2\,\mathrm{Re}\,\theta_{\mu\nu}t^{\mu\nu*}+2\,\mathrm{Re}\,\theta_{\mu\nu}\tau^{\mu\nu*}+(t_{\mu\nu}t^{\mu\nu*}-{}^1/_2|t_\mu{}^\mu|^2)+(\tau_{\mu\nu}\tau^{\mu\nu*}-{}^1/_2|\tau_\mu{}^\mu|^2)+\mathrm{Re}(2t_{\mu\nu}\tau^{\mu\nu*}-t_\mu{}^\mu\tau_\nu{}^{\nu*}). \quad (46)$$

The first, pure-field, contribution to the spectrum is the most complicated. It can be represented in the form

$$\theta_{\mu\nu}\theta^{\mu\nu*}=t\sum_n 2\pi\delta(q^0-n\omega)\frac{m^2v^4\zeta^2}{2}\int_0^1\int du\,du'\exp[i(u-u')\zeta]R_n, \quad (47)$$

where $\zeta=|q|r=|n|v$ and

$$R_n=b_1J_n(z_1)J_n(z_2)+b_2J_n'(z_1)J_n(z_2)+b_3J_n(z_1)J_n'(z_2)+b_4J_n'(z_1)J_n'(z_2); \quad (48)$$

the arguments of the Bessel functions are $z_1=(1-u)z$, $z_2=(1-u')z$. The coefficients b_i are functions of the variables u,u' and have the properties

$$b_{1,4}(u,u')=b^*_{1,4}(u',u),\qquad b_2(u,u')=b_3^*(u',u), \quad (49)$$

resulting in the contribution of each harmonic in Eq. (47) being real. These coefficients are given in the Appendix.

Next in complexity is the contribution of the interference between the field EMT and the EMT of the light particle:

$$2\,\mathrm{Re}\,\theta_{\mu\nu}t^{\mu\nu*}=t\sum_n 2\pi\delta(q^0-n\omega)m^2v^2\zeta\,\mathrm{Re}\,i\int_0^1 du\,e^{iu\zeta}r_n, \quad (50)$$

where r_n differs from R_n in Eq. (48) by the replacement of z_2 by z and the coefficients b_i by the coefficients c_i given by

$$c_1=\gamma^2\left(1-\frac{z}{z_1}+\frac{2i}{\zeta}\right)+\gamma^2v^2\left[1-2u-\frac{z}{z_1}-2i\zeta\left(\frac{4n^2}{z_1^2z^2}-\frac{1}{z^2}-\frac{1}{z_1^2}\right)+\frac{2n^2}{z_1z}\left(2u+\frac{i}{\zeta}(1-2u)\right)-\left(u+\frac{i}{\zeta}\right)\left(\frac{4n^2(n^2+1)}{z_1^2z^2}-\frac{2n^2}{z^2}-\frac{2n^2}{z_1^2}\right)\right], \quad (51)$$

$$c_2=i\gamma^2\left[(1-2u)\frac{z}{\zeta}-\frac{2\zeta}{z}\right]+\gamma^2v^2\left[-\frac{2}{z}+i\frac{2\zeta}{z_1}\left(2\frac{n^2+1}{z^2}-1\right)-i(1-2u)\left(2\frac{n^2}{z^2}-1\right)\frac{z}{\zeta}+\frac{2}{z_1}\left(u+\frac{i}{\zeta}\right)\left(\frac{4n^2}{z^2}-1\right)\right], \quad (52)$$

$$c_3=-\frac{2i\gamma^2\zeta}{z_1}+\gamma^2v^2\left[-\frac{2}{z_1}+i\frac{2\zeta}{z}\left(2\frac{n^2+1}{z_1^2}-1\right)+\frac{2z_1}{\zeta^2}+2i(1-2u)\left(\frac{z}{\zeta}-\frac{n^2}{\zeta z_1}\right)+\frac{2}{z}\left(u+\frac{i}{\zeta}\right)\left(\frac{4n^2}{z_1^2}-1\right)\right], \quad (53)$$

$$c_4=2\gamma^2v^2\left[2u-\frac{4i\zeta}{z_1z}+\frac{i}{\zeta}(1-2u)-\frac{2(n^2+1)}{z_1z}\left(u+\frac{i}{\zeta}\right)\right]. \quad (54)$$

The contribution of the interference between the field EMT and the EMT of heavy particles can be reduced to the following expression:

$$2\,\mathrm{Re}\,\theta_{\mu\nu}\tau^{\mu\nu*}=t\sum_{n=\pm1}2\pi\delta(q^0-n\omega)m^2v^2\gamma^2\,\mathrm{Re}\left\{(1-v^2+iv)e^{-iv}\left[\left(-iv-\frac{v^2}{2}\right)e^{iv}+\left(iv-\frac{3}{2}v^2\right)J_0(z)+\frac{1}{2}(1-iv)zJ_1(z)-\frac{i}{2}vq_3{}^2r^2\int_0^1 du\,e^{iuv}J_0(z_1)\right]\right\}. \quad (55)$$

The contribution to the spectrum of the EMT of a light particle follows from Eqs. (12), (25):

$$t_{\mu\nu}t^{\mu\nu*}-\frac{1}{2}|t_\mu{}^\mu|^2=t\sum_n 2\pi\delta(q^0-n\omega)m^2\left\{\left(\frac{1}{2\gamma^2}+\frac{2\gamma^2v^4}{z^2}\right)J_n^2+2\gamma^2v^2\left(v^2\frac{n^2+1}{z^2}-1\right)\left[\left(\frac{n^2}{z^2}-1\right)J_n^2+J_n'^2\right]-\frac{2\gamma^2v^4}{z}\left(\frac{4n^2}{z^2}-1\right)J_nJ_n'\right\}. \quad (56)$$

The contribution to the spectrum of the EMT of a heavy particle is much simpler:

$$\tau_{\mu\nu}\tau^{\mu\nu*}-{}^1/_2|\tau_\mu{}^\mu|^2=t\sum_{n=\pm1}2\pi\delta(q^0-n\omega)m^2v^2\gamma^2(1-v^2+v^4)\times\left(-1+\frac{q_\perp{}^2}{8\omega^2}\right). \quad (57)$$

Last, interference between the tensors of the light and heavy bodies gives the contribution

$$\mathrm{Re}(2t_{\mu\nu}\tau^{\mu\nu*}-t_\mu{}^\mu\tau_\nu{}^{\nu*})=t\sum_{n=\pm1}2\pi\delta(q^0-n\omega)m^2\times(2v^2\gamma^2J_0-\gamma^2zJ_1+{}^1/_2zJ_1)\,\mathrm{Re}(1-v^2+iv)e^{-iv}. \quad (58)$$

As can be seen from these expressions, the GR spectrum is quite complicated and contains nonlocal terms due to the field tensor $\theta_{\mu\nu}$; this makes difficult a comparison with the simple spectrum of EMR by a charge undergoing the same motion, see Eq. (14). We consider therefore the nonrelativistic and ultrarelativistic cases. In the nonrelativistic limit we shall give the differential and total intensities of the GR for the 1st and 2nd harmonics:

$$(T_{\mu\nu}T^{\mu\nu*}-{}^1/_2|T_\mu{}^\mu|^2)_{n=1,\,v\to0}\approx t\cdot2\pi\delta(q^0-\omega)\,{}^9/_{64}m^2v^6\sin^2\theta(1-{}^2/_3\sin^2\theta+{}^1/_{72}\sin^4\theta), \quad (59)$$

$$d\mathscr{E}/dt|_{n=1,\,v\to0}\approx{}^5/_{28}Gm^2\omega^2v^6, \quad (60)$$

$$(T_{\mu\nu}T^{\mu\nu*}-{}^1/_2|T_\mu{}^\mu|^2)_{n=2,\,v\to0}\approx t\cdot2\pi\delta(q^0-2\omega)m^2v^4(1-\sin^2\theta+{}^1/_8\sin^4\theta), \quad (61)$$

$$d\mathscr{E}/dt|_{n=2,\,v\to0}\approx{}^{32}/_5Gm^2\omega^2v^4. \quad (62)$$

As was to be expected, the last two equations for the quadrupole radiation determine the main contribution and coincide with the familiar results obtained from the Eisntein formula (see Ref. 6, §110). They exceed by a factor of 4 the differential and total intensities of GR of a body with localized EMT, see Eq. (16).

In the ultrarelativistic limit the main contribution comes from the pure field source $\theta_{\mu\nu}$, i.e., Eq. (47):

$$\theta_{\mu\nu}\theta^{\mu\nu*}|_{v\to 1}\approx t\sum_n 2\pi\delta(q^0-n\omega)\,\frac{m^2}{2\pi^2}\left(\frac{2}{n}\right)^{2/3}$$
$$\times\left[-\Phi^2+\gamma^2\left(\frac{2}{n}\right)^{2/3}(y\Phi^2+\Phi'^2)\right]. \qquad (63)$$

Here we used the asymptotic expressions, Eq. (15), for the Bessel functions when $z_{1,2}\approx n\sim\gamma^3\gg 1$, as well as the effective value of the integration variables $u,u'\sim n^{-1}\sim\gamma^{-3}$. We note that the contribution, Eq. (56), of the local tensor $t_{\mu\nu}$, just like the contribution of Eq. (13), is γ^2 times smaller than the field contribution, Eq. (63). Since in the same limit the expression for $|j_\alpha(q)|^2$ differs from Eq. (63) by the replacement $m^2\to 2e^2/\gamma^2$, the connection between the spectra of GR and EMR is determined in this limit by Eq. (5) in which $\Gamma=\gamma$. This agrees with the result of Sushkov and Khriplovich.[15]

6. DISCUSSION AND CONCLUSION

For the electromagnetic systems considered the motion of the charge is due to external fields whose EMT $\theta_{\mu\nu}$ is nonlocal and makes an essential contribution to the GR of the system. Therefore the classical GR of the system, in contrast to its EMR, may serve as a source of information about its internal structure. Moreover, for ultrarelativistic systems the contribution of the nonlocal source $\theta_{\mu\nu}$ exceeds the contribution of the local source by the factor γ^2.

In this connection we note that in view of nonconservation of the tensors $t_{\mu\nu}$ and $\theta_{\mu\nu}$, their separate contributions to the GR discussed in a relativistically invariant form—as was done, for example, in Eq. (46)—can be assigned independent significance with difficulty only. Thus, the quantity $t_{\mu\nu}t^{\mu\nu*}-\frac{1}{2}|t^{\mu}_{\mu}|^2$ turns out to be negative for the motion of a charge on a circular orbit.[16] However if use is made of the conservation law to express the mixed and time components of the full EMT $T_{\mu\nu}$ in terms of the space components [see Eq. (10.4.2) in Ref. 8], then the terms in the GR intensity quadratic in the space components t_{ij} will give a positive contribution for the matter tensor, those quadratic in the θ_{ij} will give a positive contribution for the field tensor, and those bilinear in t_{ij} and θ_{kl} will give the interference contribution.

From that point of view Eq. (13) determines the contribution of the matter tensor of the body in uniform motion on the circle independently of the force field. In the ultrarelativistic limit and in the effective region of **q** this contribution, like EMR, is formed on a trajectory segment small compared to the local radius of curvature r, and is described on that segment by Eqs. (13′) and (15). For contact forces, considered in Sec. 2, the GR is determined by the contribution of the matter tensor only.

For the electromagnetic systems considered in Secs. 3–5 the contribution of the field tensor θ_{ij} to the spectrum of GR is comparable to the contribution from t_{ij}, and for $\gamma\gg 1$ it exceeds the latter by the factor γ^2 and is proportional to the spectrum of EMR.

We consider as the most interesting result this work the representation of the spectrum of the GR by a charge in the plane-wave field in the form of a product of its EMR spectrum and the factor $4\pi Gm^2\Gamma^2/e^2$, independent of the frequency of radiation. While the first factor is local, the second factor is always nonlocal, increases with increasing γ like γ^2, and depends on the direction of the wave vector and the behavior of the external field outside the orbit of the charge—see the end of Sec. 3 and all of Sec. 4. This result permits the assertion that for ultrarelativistic motion of a charge in an arbitrary electromagnetic field (which under these conditions looks like a plane wave in the proper frame of the charge) the spectrum of the GR is also proportional to the local spectrum of the EMR and to a nonlocal factor, which is of order γ^2 or larger, provided the external field does not change abruptly over a distance of the order of the radius of curvature of the orbit or larger.

APPENDIX

We list here the coefficients b_i entering Eq. (48), with Eq. (49) taken into account:

$$b_1=-1+\gamma^2\left[2-2u-2u'+4uu'-\frac{z^2}{\zeta^2}+\frac{4}{\zeta^2}+i\frac{2}{\zeta}\left(1-\frac{z^2}{\zeta^2}\right)(u-u')\right]$$
$$+\frac{z^2}{\zeta^2}(1-2u-2u'+4uu')+\gamma^2\frac{z}{z_1}(1+2u'-4u)$$
$$+\gamma^2\frac{z}{z_2}(1+2u-4u')$$
$$+\frac{n^2}{z_1z_2}\left(-2u-2u'+4uu'+\frac{\gamma^2v^2z^2}{\zeta^2}\right)+\frac{2\gamma^2n^2zz_2}{\zeta^2z_1^{\,2}}\left(u+\frac{i}{\zeta}\right)$$
$$+\frac{2\gamma^2n^2zz_1}{\zeta^2z_2^{\,2}}\left(u'-\frac{i}{\zeta}\right)+i\frac{2\gamma^2zz_2}{\zeta z_1^{\,2}}$$
$$-i\frac{2\gamma^2zz_1}{\zeta z_2^{\,2}}-\frac{2\gamma^2n^2z}{z_1^{\,2}z_2}\left(1+u-2u'+\frac{i}{\zeta}\right)$$
$$-\frac{2\gamma^2n^2z}{z_1z_2^{\,2}}\left(1-2u+u'-\frac{i}{\zeta}\right)$$
$$-i\frac{2\gamma^2n^2z}{\zeta z_1^{\,2}z_2}\left[v^2-\left(u+\frac{i}{\zeta}\right)(1-2u')\right]$$
$$+i\frac{2\gamma^2n^2z}{\zeta z_1z_2^{\,2}}\left[v^2-\left(u'-\frac{i}{\zeta}\right)(1-2u)\right]$$
$$+2\gamma^2n^2\left(\frac{2n^2+2}{z_1^{\,2}z_2^{\,2}}-\frac{1}{z_1^{\,2}}-\frac{1}{z_2^{\,2}}\right)\left[v^2+\left(u+\frac{i}{\zeta}\right)\left(u'-\frac{i}{\zeta}\right)\right]$$
$$-2i\gamma^2\zeta\left(\frac{4n^2}{z_1^{\,2}z_2^{\,2}}-\frac{1}{z_1^{\,2}}-\frac{1}{z_2^{\,2}}\right)\left(u-u'+\frac{2i}{\zeta}\right),$$

$$b_2=i\frac{z}{\zeta}\left[1-2u-2u'+4uu'+\gamma^2(2-2u-u'+2uu')-i\frac{\gamma^2}{\zeta}(1-2u)\right]-i\frac{\gamma^2z^2}{\zeta z_2}(1-2u)-\frac{2\gamma^2zz_2}{\zeta^2z_1}\left(u+\frac{i}{\zeta}\right)$$
$$-i\frac{2\gamma^2zz_2}{\zeta z_1}+\frac{2\gamma^2z}{z_1z_2}\left(1+u-2u'+\frac{i}{\zeta}\right)$$
$$+\frac{2\gamma^2z}{z_2^{\,2}}\left(1-2u+u'-\frac{i}{\zeta}\right)$$
$$+i\frac{2\gamma^2zn^2}{\zeta z_1z_2}\left[v^2-\left(u+\frac{i}{\zeta}\right)(1-2u')\right]-i\frac{\gamma^2z}{\zeta}\left(\frac{2n^2}{z_2^{\,2}}-1\right)$$
$$\times\left[v^2-\left(u'-\frac{i}{\zeta}\right)(1-2u)\right]-\frac{2\gamma^2}{z_1}\left(\frac{4n^2}{z_2^{\,2}}-1\right)$$
$$\times\left[v^2+\left(u+\frac{i}{\zeta}\right)\left(u'-\frac{i}{\zeta}\right)\right]$$
$$+i\frac{2\gamma^2\zeta}{z_1}\left(\frac{2(n^2+1)}{z_2^{\,2}}-1\right)\left(u-u'+\frac{2i}{\zeta}\right),$$

$$
\begin{aligned}
b_4 = {} & -2u-2u'+4uu' + \frac{\gamma^2 z^2}{\zeta^2}(1-2u-2u'+4uu') \\
& - \frac{2\gamma^2 z}{z_1}\left(1+u-2u' + \frac{i}{\zeta}\right) - \frac{2\gamma^2 z}{z^2}\left(1-2u+u' - \frac{i}{\zeta}\right) \\
& -i\frac{2\gamma^2 z}{\zeta z_1}\left[v^2 - \left(u+\frac{i}{\zeta}\right)(1-2u')\right] \\
& +i\frac{2\gamma^2 z}{\zeta z_2}\left[v^2 - \left(u'-\frac{i}{\zeta}\right)(1-2u)\right] \\
& +\frac{4(n^2+1)\gamma^2}{z_1 z_2}\left[v^2 + \left(u+\frac{i}{\zeta}\right)\right. \\
& \left.\times\left(u'-\frac{i}{\zeta}\right)\right] - i\frac{8\gamma^2\zeta}{z_1 z_2}\left(u-u'+\frac{2i}{\zeta}\right).
\end{aligned}
$$

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