Self-similar hydromagnetic dynamo

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A simple axisymmetric-dynamo mechanism is proposed. It may be pertinent to the generation of the magnetic fields of stars and sunspots. The dynamo effect occurs in flows of a jet nature when the velocity and magnetic induction are inversely proportional to the distance from the origin of coordinates. There is no contradiction of Cowling's result, since the conditions of the theorem stating that an axisymmetric dynamo is impossible require a stronger dissipation. In swirling jets, the bifurcation of a regime with a magnetic field occurs in the direction of smaller Reynolds numbers, so there is a hysteresis in the transition between a purely hydrodynamic regime and an MHD regime. The self-generation of a magnetic field may cause a turbulent flow to become laminar at large Reynolds numbers.

1. INTRODUCTION

The magnetic fields of the planets and stars are basically axisymmetric. It is thus natural to analyze the problem in which these fields are generated through a dynamo effect on the basis of axisymmetric solutions of the MHD equations. However, Cowling showed[1] back in 1934 that an axisymmetric hydromagnetic dynamo would be impossible. The proof was based on the assumption of closed magnetic field lines, and Cowling stressed that the proof did not apply to fields with open lines. A different proof, proposed by Braginskiĭ,[2] contained the condition that the induction falls off no more slowly than in inverse proportion to the cube of the distance. In the present paper we examine an axisymmetric problem in which those conditions do not hold.

We seek a solution in a conical self-similar class in which the velocity and the magnetic induction are inversely proportional to the distance from the origin of coordinates. The MHD equations allow solutions of this type. For the Navier-Stokes equations, analytic solutions found by Landau[3] and Squire[4] to describe jets of a viscous and incompressible fluid are well known and fall in this class. In the case of a conducting liquid, the solution of the problem of the flow caused by a spherically symmetric current flowing out of a point electrode at the boundary of a half-space and several other problems also belong to this class.[5]

The nature of the motion of the continuous medium in which we are interested here is quite typical. Kinematic structures of the vortex-ring type are elements of many specific flows, in particular, thermal convection (Bénard cells). If the motion is sufficiently intense, the velocity field in toroidal structures can be approximated fairly well by distributions from the conical class, except in small regions adjacent to immobile points.

According to the results found here, the self-similar cores of such convection cells can serve as magnetic-field generators in the case of a conducting fluid.

Intense jets have been observed comparatively recently near young stars and galactic nuclei.[6,7] A hydrodynamic model for these flows based on Squire's solution was proposed in Refs. 8 and 9. In jets of this type, conditions favor the self-generation of a magnetic field. Preliminary results on nonswirling jets were reported in Ref. 10, where the possible appearance of a poloidal induction was demonstrated. Observations show that astrophysical jets are usually swirling[11] and contain an azimuthal magnetic field component.[12]

In the present paper we will see that allowance for swirling in the dynamo problem leads to some important new effects. First, a sufficiently intense rotation will alter the nature of the magnetic-field bifurcation, from forward to inverse. The transition between the hydrodynamic and MHD regimes becomes hysteretic. Second, while intense rotation in the hydrodynamic regime leads to the development of a return flow near the axis of the jet, the self-generation of a magnetic field quenches that effect, localizes the rotational motion near the symmetry axis and plane, and sets the stage for a pronounced swirling without an expansion of the jet. These conclusions agree with observations of high rotation velocities and highly collimated jets.[11]

The mechanism proposed here for the generation of a magnetic field appears to be the simplest mechanism which has been proposed to date. At the same time, it may be characteristic of a wide class of flows.

2. SELF-SIMILAR CLASS

In this section of the paper we consider steady-state flows of a viscous, incompressible, conducting fluid with constant physical properties. We seek a solution of the MHD equations[1] in the self-similar class for which the velocity field and the field of the magnetic induction can be represented as follows in the spherical coordinate system (r,θ,φ):

$$v_r=-\frac{\nu y'(x)}{r},\quad v_\theta=-\frac{\nu y(x)}{r\sin\theta},\quad v_\varphi=\frac{\nu\Gamma(x)}{r\sin\theta},$$
$$x=\cos\theta, \tag{1}$$
$$B_r=-\frac{B\Phi'(x)}{r},\quad B_\theta=-\frac{B\Phi(x)}{r\sin\theta},\quad B_\varphi=\frac{BL(x)}{r\sin\theta}.$$

The prime means differentiation with respect to x, ν is the kinematic viscosity, and B is a normalization constant which is to be determined. The condition that the velocity and induction fields be solenoidal is satisfied automatically in class (1). When we substitute (1) into the other MHD equations, cancel out the common factor r^{-3}, and carry out some simple manipulations, we find the following system of ordinary differential equations:

$$(1-x^2)y'+2xy-y^2/2=F-S\Phi^2/2, \tag{2}$$

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$$(1-x^2)\Phi''=\mathrm{Bt}(y\Phi'-y'\Phi), \quad (3)$$

$$(1-x^2)\Gamma''=y\Gamma'-S\Phi L', \quad (4)$$

$$(1-x^2)L''=\mathrm{Bt}\{yL'-\Phi\Gamma'+2(y'L-\Phi'\Gamma)+[2x/(1-x^2)](yL-\Phi\Gamma)\}, \quad (5)$$

$$(1-x^2)F'''=2\Gamma\Gamma'-2SLL'. \quad (6)$$

The auxiliary function F is introduced in the course of the transformations associated with the elimination of the pressure and the triple integration of the equation for the angular part of the meridional stream function $y(x)$.[13] The dimensionless Batchelor number Bt characterizes the properties of the medium and is the ratio of the kinematic viscosity to the magnetic viscosity ν_m. The parameter S is a measure of the intensity of the magnetic induction.

We will be discussing a bipolar jet generated by a vortex-sink motion of the medium in the $x=0$ plane. In this formulation of the problem,[8] it is assumed that the following values are given:

$$y'(0)=\mathrm{Re},\ \Gamma(0)=\Gamma_0. \quad (7)$$

These quantities characterize the intensity of the motion. In addition, we impose the symmetry conditions

$$y(0)=\Phi'(0)=L(0)=0 \quad (8)$$

in the plane, and we will consider only the half-space $0\leqslant x\leqslant 1$.

By virtue of (1), the requirement that the velocity and induction be bounded at the symmetry axis means

$$y(1)=\Gamma(1)=\Phi(1)=L(1)=0. \quad (9)$$

It then follows from (2) that we have $F(1)=0$. Differentiating (2), and using (9), we find

$$F'(1)=0. \quad (10)$$

Since system (2)–(6) is of tenth order, the set of boundary conditions in (7)–(10) closes the problem. Since the equations and the boundry conditions for the magnetic induction are homogeneous, a purely hydrodynamic solution with $\Phi=L\equiv 0$ is possible. The dynamo problem is one of seeking a nontrivial solution for Φ and L. We begin with the particular case $\Gamma_0=0$.

3. NONSWIRLING JET

3.1. Collapse in a Squire flow

In the absence of rotation, the problem simplifies greatly. Equations (4) and (5) have solutions $\Gamma=L\equiv 0$. It follows from (6) that we have $F'''=0$; using the boundary conditions we find

$$F=(1-x)^2[\mathrm{Re}+S\Phi^2(0)/2].$$

With $S=0$, Eq. (2) has the analytic solution[4]

$$y=\mathrm{Re}\cdot(1-x)\{\chi\,\mathrm{ctg}[\chi\ln(1+x)]-{}^1/_2\}^{-1},\quad \chi={}^1/_2(2\,\mathrm{Re}-1)^{1/2}. \quad (11)$$

Squire assumed that this solution describes a submerged jet issuing from a small aperture in a wall, but that interpretation is not correct, since the adhesion condition does not hold at the wall. The solution is more suitable for an air jet above a water sink in a tank and constitutes a very simple model for the astrophysical jets which are observed near young stars and galactic nuclei.[8,9]

From this point of view, solution (11) acquires an unexpected property: At a finite and indeed quite moderate Reynolds number $\mathrm{Re}=\mathrm{Re}_*=7.67$ (Ref. 13) the velocity at the axis of the jet becomes infinite. The mathematical reason is that the root of the function in braces in (11), which lies outside the interval [0,1] at small values of Re, approaches the interval boundary $x=1$ as Re approaches its critical value Re_*. If we formally let Re become larger than Re_*, we find that the function $y(x)$ acquires a pole in the interval [0,1], and solution (11) loses physical meaning.

The effect which arises as $\mathrm{Re}\to\mathrm{Re}_*$ is analogous to a convergence. The convective transport of momentum toward the symmetry axis begins to outweigh the viscous diffusion, and a self-focusing occurs. The tendency of the velocity of a jet to become infinite at a rather moderate accretion rate agrees well with the high velocities observed for astrophysical jets. On the other hand, the appearance of infinite velocities is evidence that the particular employed model of the medium is failing. As we will show below, however, we can retain our model and even the self-similar class if we deal with the possible onset of turbulence in the flow in an appropriate way. First, however, we will discuss the laminar case.

3.2. Linear stage of a dynamo

If we analyze the dynamo problem in the linear approximation, ignoring the effect of the generated magnetic field on the original flow, we find that the problem reduces to one of finding nontrivial solutions of Eq. (3) for a given function $y(x)$. Substituting (11) into (3), we find an equation in which the coefficients depend on the two parameters Re and Bt. Since the solution is defined within an arbitrary factor (because the problem is linear), we choose the normalization condition $\Phi'(1)=-1$. Integrating (3) as a Cauchy problem, starting at $x=1$, we then find $\Phi'(0)$. With $\mathrm{Bt}=0$ we have $\Phi=1-x$ and $\Phi'(0)=-1$. In the limit $\mathrm{Bt}\to\infty$, in the case $y(x)\geqslant 0$, the frozen-in condition holds in the limit:

$$\Phi=-y(x)/y'(1),\ \Phi'(0)=-y'(0)/y'(1).$$

Let us assume that the function $y(x)$ has a single maximum, which is reached in the interval [0,1]; this assumption is valid for (11). We then have $\Phi'(0)>0$ as $\mathrm{Bt}\to\infty$. Since the functional dependence on the parameter Bt is continuous, we find that value of this parameter which corresponds to the condition $\Phi'(0)=0$. Numerical calculations confirm that such a value exists and is unique.

The results of the calculations are shown by curve *1* in Fig. 1. Let us assume that the properties of the medium (i.e., Bt) are fixed and that the intensity of the motion is characterized by a magnetic Reynolds number $\mathrm{Re}_m=\mathrm{Bt}\cdot\mathrm{Re}$. What happens as Re_m is increased? In region I there is only the purely hydrodynamic regime; on ray *2* ($\mathrm{Re}_m=7.67\mathrm{Bt}$), this regime disappears; and on curve *1* it loses stability as a result of a bifurcation of the MHD regime. The critical values of Re_m lie in the narrow interval between $\mathrm{Re}_m^*=3.5$ as $\mathrm{Bt}\to\infty$ (the dashed asymptote) and $\mathrm{Re}_m^*=1.74$ at $\mathrm{Bt}=0.226$ (point K).

3.3. Nonlinear stage; asymptotic behavior

In the nonlinear case, Eqs. (2) and (3) must be integrated jointly. We introduce the normalization $\Phi(0)=1$. We can then write S as $\mathrm{Al}\cdot\mathrm{Re}^2$, where the Alfvén number

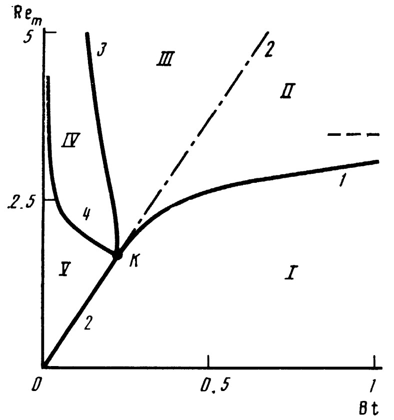

FIG. 1. Map of regimes for a nonswirling jet. H) Purely hydrodynamic regime; M) MHD regime; L) laminar regime, T) turbulent regime. Stable regimes are in boldface. Region I—HL only; II—HL and **MHL**; III—HT and **MHL**; IV—HT and **MHT**; V—**HT**.

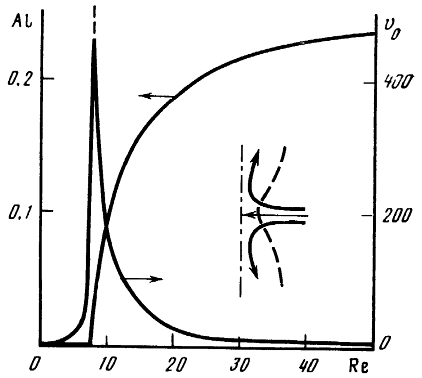

FIG. 2. The Alfvén number and the axial velocity $v_0 = -y'(1)$ versus the Reynolds number ($\Gamma_0 = 0$, Bt $= 0.25$). The inset is a diagram of (solid arrows) streamlines and (dashed lines) magnetic field lines.

$$\mathrm{Al} = B_\theta^2(0)/[4\pi\rho v_r^2(0)]$$

characterizes the ratio of the magnetic energy to the kinetic energy at the $x = 0$ plane. Equation (2) can be rewritten as

$$(1-x^2)y' + 2xy - y^2/2 = (1-x)^2\,\mathrm{Re} - {}^1/_2\,\mathrm{Al}\,\mathrm{Re}^2[\Phi^2 - (1-x)^2]. \tag{12}$$

We assume Re $\to\infty$, while $y(x)$ remains a bounded function. We then have $\Phi \to \Phi = 1 - x$, and from (3) we find $y \to y_p = C\Phi_p$. This external solution does not satisfy the condition $y(0) = 0$, so a boundary layer is formed near the plane. We introduce the internal variables

$$\eta = x\,\mathrm{Re}, \quad \Phi = 1 - x + \varphi(\eta)/\mathrm{Re}, \quad y = y_1(\eta).$$

Using them in (3) and (12), and letting Re go to infinity, we find the equations of the boundary layer:

$$y_1' = 1 - \mathrm{Al}\,\varphi, \quad \varphi'' + \mathrm{Bt}\,y_1' = 0, \quad y(0) = \varphi(0) = 0, \quad \varphi'(0) = 1.$$

These equations have the solution

$$\varphi = y_1 = \mathrm{Bt}^{-1}[1 - \exp(-\mathrm{Bt}\,\eta)], \quad \mathrm{Al} = \mathrm{Bt}.$$

The last equality follows from the requirement that φ remain bounded as $\eta \to \infty$.

Working from the external and internal solutions, we can construct a uniform approximation:

$$\begin{aligned} y &= \mathrm{Bt}^{-1}[1 - x - \exp(-\mathrm{Re}_m x)], \\ \Phi &= 1 - x - \mathrm{Re}_m^{-1}[1 - \exp(-\mathrm{Re}_m x)]. \end{aligned} \tag{13}$$

The electric current density j is proportional to $(1-x^2)^{1/2}\mathrm{Re}_m \exp(-\mathrm{Re}_m x)$, so there is no current outside a small neighborhood of the plane, and a current sheet forms near the plane. The fact that the limiting value of Al is equal to Bt means that energy is lost to equal degrees through viscous and Joule dissipation.

3.4. Nonlinear stage; calculations

Equations (12) and (13) were integrated from $x = 1$ to $x = 0$. At $x = 1$ we imposed the values $\Phi(1) = (y)1 = 0, \Phi'(1) = -1, y'(1)$. The latter quantity cannot be found from Eq. (12) (since the point $x = 1$ is singular) and is a parameter to be determined. We found it, along with Al, by trial and error, working from the conditions $y(0) = \Phi'(0) = 0$. We then renormalized $\Phi(x)$ and Al in order to satisfy the condition $\Phi(0) = 1$. We left the parameters Bt and Re adjustable. Figure 2 shows the results of a calculation for a fixed value of the Batchelor number. At small values of Re, the regime is purely hydrodynamic. As Re is increased, the axial velocity increases rapidly, and in the absence of a field it becomes infinite at Re $= 7.67$. At the smaller value Re $= 7.56$, however, a forward fork bifurcation of the MHD regime occurs. According to the general theory,[14] if the initial regime is stable, then in this situation it will lose the stability, which the new regime will inherit. Actually, there are two new regimes, which differ only in the sign of the magnetic induction. The bifurcation remains soft at all values of Bt. As Re is increased, the fraction of the kinetic energy which converts into magnetic energy increases rapidly. The axial velocity accordingly falls off sharply. At Re $\gg 1$, the numerical results agree with the asymptotic behavior described above. Accordingly, if a laminar hydrodynamic regime does not exist at sufficiently large values of Re (above line *2* in Fig. 1), the MHD solution can be continued into the region of arbitrarily large Reynolds numbers.

3.5. Elementary mechanism

The observed self-generation of a magnetic field in a convergent jet flow can be explained at a qualitative level. A converging flow causes the field lines to move closer together near the axis. The result, however, is not a dynamo but simply a transport, since the number of field lines does not change. The dynamo effect itself arises from the following circumstance. We assume that we are given an initial perturbation of the magnetic field with an induction which is directed along the symmetry axis. We consider a fluid conductor in the form of a torus near the equatorial plane. The flow transports and squeezes the torus towards the axis. As a result of crossing field lines, an aximuthal electric current is excited in the torus. This current induces a magnetic field, which is in the same direction as the original field near the axis. There is accordingly a positive feedback. Furthermore, the velocity increases toward the axis; this increase intensifies the effect.

If the conductivity of the medium is low, however, the intensification of the induction is suppressed by Joule dissipation and by magnetic field diffusion. The dissipation mechanisms may be dominant at an arbitrary conductivity if the induction falls off sufficiently rapidly with increasing distance, e.g., $\propto r^{-3}$; this is the content of the Cowling theorem. If the decrease is slow, however, as in the case at hand, the gradients (and hence the diffusion flux) are small, so a dynamo effect becomes possible even at magnetic Reynolds numbers on the order of unity. Once a significant fraction of the kinetic energy has converted into magnetic energy, and the flow has slowed down, a balance is struck, and the steady-state regime in which we are interested here is established. At $\mathrm{Re}_m \gg 1$, the field generation is localized in the current sheet near the plane; outside this sheet, the flow and the magnetic field are potential, and the streamlines coincide with field lines.

A convergent flow is an important condition for the dynamo effect. When the velocity field is reversed, and the flow becomes a descending, spreading flow, no bifurcation of the MHD solution is observed. There is no generation of a magnetic field, even in a Landau jet.

4. TURBULENT JET

4.1. Approximation of a narrow jet

If $\mathrm{Bt} > 0.226$, the laminar hydrodynamic regime is lost at a value of Re lower than that at which the MHD solution undergoes bifurcation. Since the molecular Batchelor numbers are usually extremely small ($\sim 10^{-6}$) under both technological conditions[5] and astrophysical conditions,[15] this case deserves special study. We know[16] that jets lose stability even at small Reynolds numbers, so it is reasonable to suggest that the loss of existence is preceded by a loss of stability and the onset of turbulence in the jet. The experiments of Refs. 17 and 18 show that in submerged jets the region of turbulent motion is inside a cone with a small vertex angle. A model for turbulence with a vortex viscosity which is piecewise-constant along the angular scale was used in Refs. 8 and 9. In the turbulent region near the axis, the viscosity is high, while outside this region the viscosity is the molecular viscosity. This model revealed an unexpected phenomenon: the spontaneous onset of a rotational motion. However, empirical data on the value of the vortex viscosity were used in those studies. In the present paper we are taking a different approach, which is conceptually similar to one proposed by Schneider.[19] Since the turbulent cone is quite narrow, we use the simplification that the vertex angle of the cone is zero. We know that a jet, particularly if turbulent, will expel the surrounding fluid, which flows into the turbulent region. Accordingly, in assuming the vertex angle of the turbulent cone to be zero we should assume that there are sinks for the fluid at the axis.

At $\mathrm{Re} = \mathrm{Re}_*$, at which the laminar solution ceases to exist, the root and pole in the function $y(x)$ coincide at the point $x = 1$. In this case, $y(1)$ acquires a finite value $y(1) = 4$, as follows from Eq. (2) with the conditions $F(1) = \Phi(1) = 0$ and directly from solution (11). This situation corresponds to that in which the expulsion per unit axis length is $2\pi\nu y(1) = 8\pi\nu$ for a Schlichting jet. The expulsion of a turbulent jet is greater than that of a laminar jet, and it increases with the Reynolds number.[20] For a turbulent jet we thus assume

$$y(1) = q \geqslant 4. \tag{14}$$

As is shown below, the quantity q is determined from the conditions of the problem and thus does not require any empirical information (in contrast with Refs. 8 and 19). This circumstance is an advantage of the present approach.

Condition (14) is thus adopted as the condition under which the jet is turbulent. If, as the parameters are varied, the value found for q falls below four, we need to return to the laminar formulation. Actually, the onset of turbulence should occur at Reynolds numbers lower than those at which the laminar regime is lost, but we will ignore this distinction in the present model.

In our interpretation of Squire solution (11) and of its generalizations to the case of swirling and MHD flows, the part of the jet near the axis is assumed to be induced by a convergent external flow. Accordingly, when the turbulent cone collapses, the axis should not be the source of an axial component of the momentum for the external flow. The total flux of the axial component of the momentum through a unit area of the lateral surface of the cone can be written as follows after some simple but lengthy calculations (similar to those which were carried out in Ref. 8; see also Ref. 21):

$$\Pi_{z\theta} = \frac{\rho\nu^2}{r^2 \sin\theta}\left[F - xF' - \frac{1-x^2}{2}F'' + \frac{\Gamma^2 - L^2}{2}\right].$$

Requiring that the expression in square brackets vanish at $x = 1$, as it must if $\Pi_{z\theta}$ is to be bounded, we find

$$F(1) = F'(1) - {}^1/_2\Gamma^2(1) + {}^1/_2L^2(1). \tag{15}$$

It follows from Eq. (3) that with $y(1) = q \neq 0$ the solution near $x = 1$ is

$$\Phi = \Phi_1(1-x)^{1-\gamma} + o(1-x), \quad \gamma = {}^1/_2\,\mathrm{Bt}\,q \tag{16}$$

under the assumption $\gamma < 1$ (in the case at hand, this condition holds, as we will see below). We then have $\Phi(1) = 0$. As a consequence of (2) we have the further condition

$$2q - {}^1/_2 q^2 = F(1). \tag{17}$$

From Eq. (4) we find $\Gamma'(1) = 0$, but the value of $\Gamma(1)$ for the external solution may be nonzero. The function $L(x)$ can be written as follows near $x = 1$, according to (5):

$$L = L_1(1-x) - (2/q)\Gamma(1)\Phi_1(1-x)^{1-\gamma} + o(1-x).$$

We thus have $L(1) = 0$, and the last term in (15) can be discarded.

4.2. Expulsion of turbulent jet

We first consider the purely hydrodynamic problem with $\Phi = L = 0$. Under the condition $\Gamma'(1) = 0$, Eq. (4) has the solution $\Gamma \equiv \mathrm{const}$, so we have $\Gamma(1) = \Gamma_0$. Equation (6) then reduces to $F''' = 0$. By virtue of (2) and (7) we have $F(0) = \mathrm{Re}$; two more conditions follow from (15) and (17), so we can write an explicit expression for F. As a result, Eq. (2) takes the form

$$(1-x^2)y' + 2xy - y^2/2 = (1-x)^2\,\mathrm{Re} - {}^1/_2\Gamma_0^2 x(1-x) - (q/2)(q-4)x. \tag{18}$$

In this case the function $y(x)$ must satisfy the conditions

$y(1) = q$, $y(0) = 0$. Differentiating (18), and substituting $x = 1, y = q$, we find

$$y'(1) = q/2 - \Gamma_0^2/2q. \tag{19}$$

In integrating (18) from $x = 1$ to $x = 0$, we must satisfy the condition $y(0) = 0$; for this purpose, a special choice of q is required. The rate at which the surrounding fluid is expelled by the turbulent jet is thus found as a function of the parameters Re and Γ_0.

At Re $\gg 1$, the functional dependence $q(\mathrm{Re}, \Gamma_0)$ can be determined explicitly through an asymptotic analysis. As Re increases, we would expect an increase in q and thus in the function $y(x)$. Ignoring the terms which are linear in y on the left side of (18) and also the first term on the right side [as we must in order to keep $y(x)$ real at $x \ll 1$], we find $y(x) \approx y_p(x)$ in the core of the flow, where

$$y_p = g[x(1-\delta x)]^{1/2}, \quad g^2 = q^2 + \Gamma_0^2, \quad \delta = \Gamma_0^2/g^2. \tag{20}$$

The function $y_p(x)$ satisfies the conditions $y_p(1) = q$ and (19) at the axis; it also satisfies the condition $y_p(0) = 0$. The function $y'_p(x)$, however, has a square-root singularity at $x = 0$, telling us that there is an infinite radial velocity. Since we must have $y'(0) = \mathrm{Re}$, a boundary layer forms near the $x = 0$ plane.

Introducing the internal variables $\eta = \mathrm{Re}^{1/2} x, y = \mathrm{Re}^{1/2} w$, using them in Eq. (18), letting Re go to infinity, and assuming that the quantity $(1/2) g^2 \mathrm{Re}^{-3/2} = b$ remains bounded in the limit (this condition is necessary in order to match the internal and external solutions), we find an equation for the boundary layer near the axis:

$$w' = 1 + w^2/2 - b\eta, \quad w(0) = 0. \tag{21}$$

In the limit $\eta \to \infty$, the solution tends toward the function $w = \pm (2b\eta)^{1/2}$. It is easy to see that for arbitrary b the solution asymptotically approaches the branch $w = -(2b\eta)^{1/3}$, while in order to reconcile the results with (20) we must have $w \to (2b\eta)^{1/2}$. We can arrange this only through a special choice of b. A calculation yields $b = 0.6876$; we then find the asymptotic dependence

$$g = 1.73\,\mathrm{Re}^{3/4}. \tag{22}$$

It follows from (22) and (20) that if q is to remain larger than four the circulation Γ_0 must not be too large. Specifically, we know[8] that a sufficiently large circulation will cause an expansion of a turbulent cone and even a vanishing of the flux near the axis. Under these conditions, the approximation of a narrow turbulent jet is not appropriate.

4.3. Turbulent dynamo

To determine the boundary of the bifurcation of the MHD regime, we must choose one of the parameters Bt, Γ_0, Re in such a way that there exists a nontrivial solution of Eq. (3). We fix Bt and Γ_0 in some arbitrary way, and we seek a critical value of Re. The function $y(x)$ is determined independently as a solution of Eq. (18), and (3) is integrated starting at the point $x = x_1 = 1 - \varepsilon (0 < \varepsilon \ll 1)$, at which $\Phi(x_1)$ and $\Phi'(x_1)$ are specified in accordance with asymptotic representation (16). By virture of the linearity and homogeneity of the problem for $\Phi(x)$, we can set $\Phi_1 = 1$. As a result, an integration of (3) to $x = 0$ should be carried out: $\Phi'(0) = 0$. This result serves as a condition for determining Re^*. The results calculated with $\Gamma_0 = 0$ correspond to curve *4* in Fig. 1. The critical value of Re_m thus increases as we move away from point K in the direction of either increasing or decreasing Bt. However, while Re_m^* remains finite as Bt $\to \infty$, it increases without bound as Bt $\to 0$. Let us find the asymptotic $\mathrm{Re}_m^*(\mathrm{Bt})$ at Bt $\ll 1$. For this purpose we use the potential solution (20) in Eq. (3). The coefficients here now depend on the two parameters δ and $\alpha = \mathrm{Bt} g$. The relationship between them is found from the requirement that (3) have a nontrivial solution. By virtue of its meaning, the parameter δ varies between 0 and 1. The nature of the function $\alpha(\delta)$ is shown by the following data:

δ	0	0.2	0.4	0.6	0.8	1
α	0.872	0.943	1.033	1.152	1.318	1.571

In particular, with $\Gamma_0 = 0$ we find, using (22), the asymptotic behavior for curve *4* in Fig. 1:

$$\mathrm{Re}_m^* = 0.674\,\mathrm{Bt}^{-1/3}, \tag{23}$$

in complete agreement with the results of the calculation. The existence of a circulation increases the coefficient in (23), but not beyond 1.477.

4.4. Return to laminar flow

The turbulent MHD regime with $\Gamma_0 = 0$ is found as a solution of Eq. (3) and Eq. (2), written in the form

$$(1-x^2)y' + 2xy - y^2/2 = (1-x)^2\,\mathrm{Re} - (q/2)(q-4)x - {}^1/_2 \mathrm{Al}\cdot\mathrm{Re}^2[\Phi^2 - (1-x)^2]. \tag{24}$$

Since $\Phi'(x)$ is not bounded at the point $x = 1$, the integration is again carried out starting at the point x_1, where Φ and Φ' are given by (16), but now the constant Φ_1 is an unknown and is to be found from the condition $\Phi(0) = 1$. It follows from (24) that we have $y'(1) = 2 + (q-4)/2$. At x_1 the quantity $y(x)$ is calculated from the expression $y(x_1) = q - y'(1)(1 - x_1)$. After integrating to $x = 0$, we need $y(0) = 0$ and $\Phi'(0) = 0$; these conditions are arranged through a suitable choice of q and Al. This algorithm is applicable under the condition $\gamma < 1$. At a fixed value of Bt the maximum of γ is reached along with the maximum of q [see (16)]. As Re increases, the quantity q also increases asymptotically, according to (22) (at $\Gamma_0 = 0$ we have $g = q$). After the transition to the MHD regime, however, the induction weakens the jet, and the value of q in fact decreases as Re increases further. The maximum value of q is therefore reached at the bifurcation point (curve *4* in Fig. 1). On this curve, the value of γ decreases from 0.464 at point K to the asymptotic value 0.436 determined from (16), (22), and (23). The condition $\gamma < 1$ thus holds.

The decrease in q with increasing Re with Bt = const has the consequence that at a certain Re the value $q = 4$ is reached. For large values of Re there exists a laminar solution which satisfies the condition $y(1) = 0$. The lower boundary on the existence of the laminar MHD regime, determined by the condition $q = 4$, corresponds to curve *3* in Fig. 1. If this boundary is approached from above, the results are the same as when line *2* is approached from below, except that the induction is nonzero and becomes infinite at the axis along with the longitudinal velocity. The scheme for the onset of turbulence described in Subsection 4.1 is thus equally applicable to the transition from region III to IV in Fig. 1.

According to the model which we have adopted, we can draw the following scenario of transitions with increasing Re at a fixed Bt < 0.226. At small Reynolds numbers the flow is purely hydrodynamic and laminar. When Re passes through 7.67 (line *2* in Fig. 1) the flow becomes turbulent but remains purely hydrodynamic. With a further increase in Re (on line *4*), there is a bifurcation of the magnetic field, and the MHD regime which arises is turbulent. At even larger Reynolds numbers (on curve *3*), however, the turbulence in the axial part of the jet is suppressed, and the MHD regime becomes laminar.

According to the calculations, curve *3* has the asymptote $\mathrm{Re}_m = 2.52\,\mathrm{Bt}^{-1/2}$, so at $\mathrm{Bt} \ll 1$ there exists a fairly wide range of Reynolds numbers [cf. (23)] at which the MHD regime is turbulent.

In the case of a swirling jet, the turbulent MHD regime is calculated by integrating system (2)–(6) starting at the point x_1 and by using relations (16) and (18) to determine the initial values of Φ, L, and their derivatives. The quantities F and F' are found from (15) and (17); and we use $y(x_1) = q\text{-}y'(1)(1-x)$, where $y'(1) = 2 - F'(1)/q$. The values of Φ_1, L_1, $\Gamma(1)$, q, and Al are found from the conditions $\Phi(0) = 1$, $L(0) = 0$, $\Gamma(0) = \Gamma_0$, $y(0) = 0$, $\Phi'(0) = 0$. The boundary at which there is a transition back to the laminar regime is identified by the value $q = 4$. The results of these calculations are presented below.

The fact that a magnetic field stabilizes the flow and suppresses turbulence is itself well known.[22] An unusual aspect of the effect observed here is that it is achieved not by an external magnetic field but by an intrinsic self-induced magnetic field.

5. SWIRLING JET

5.1. Collapse of a swirling jet

In the case $\Gamma_0 \neq 0$ [see (7)], qualitative changes occur both in the original hydrodynamic regime and in the nature of the bifurcation of the magnetic field. The presence of swirling leads, in particular, to an increase in the critical Reynolds number Re_*, at which the laminar solution is lost. Curve T in Fig. 3 shows the $\mathrm{Re}_*(\Gamma_0)$ behavior. (Since the sign of Γ_0 does not affect the nature of the meridional motion, the line $\Gamma_0 = 0$ serves as a symmetry axis.) Laminar solutions exist in the region to the left of curve T. Extending the approach of the preceding section to the case of swirling jets, we assume that a turbulent regime with $y(1) = q \geqslant 4$ and $\Gamma(x) \equiv \Gamma_0$ prevails in the region to the right of T. In the limit $\mathrm{Re} \to \infty$, curve T has (22) as an asymptote; in view of the value $q = 4$, we can assume $g = \Gamma_0$.

5.2. Linear dynamo

In the case at hand, it is convenient to formulate this problem in the following way: For each given hydrodynamic regime, characterized by the parameters Re and Γ_0, i.e., for each point in the plane of Fig. 3, we are to find a value $\mathrm{Bt} = \mathrm{Bt}^*$ at which Eq. (3) with boundary conditions (8), (9) has a nontrivial solution. Calculations show that the function $\mathrm{Bt}^*(\mathrm{Re},\Gamma_0)$ is a monotonically decreasing function as R increases, for any $\Gamma_0 = \mathrm{const}$, and it is a monotonically increasing function with increasing $|\Gamma_0|$ under the condition $\mathrm{Re} = \mathrm{const}$. When projected onto the $\{\mathrm{Re},\Gamma_0\}$ plane, the contour lines $\mathrm{Bt}^* = \mathrm{const}$ correspond to curves B_1 and B_2 ($\mathrm{Bt}^* = 1$ and 0.12).

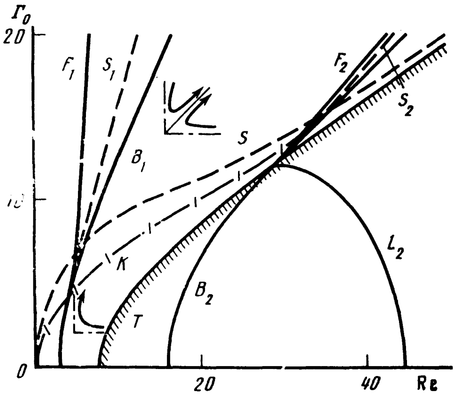

FIG. 3. Map of regimes for vortex jets. To the right of T are turbulent regimes, and to the left are laminar regimes. S—Boundary for the appearance of a return flow near the axis; B_1, B_2—lines of bifurcation of the MHD regime with Bt = 1 and 0.12; K—boundary between a forward bifurcation (on the right) and an inverse bifurcation; F_1, F_2—boundaries of the projection of the Al (Re,Γ_0) surfaces for Bt = 1 and 0.12, respectively; S_1, S_2—projections of lines onto these surfaces, which separate multicell and ascending regimes; L_2—projection of the boundary for the conversion of the MHD regime into a laminar regime for Bt = 0.12. The insets show diagrams of the flow regimes.

As Re is reduced at $\Gamma_0 = \mathrm{const}$, the flow goes from an ascending regime (in the region between curves T and S) into a two-cell regime (between S and the axis Re = 0; see the insets in Fig. 3). In the limit $\mathrm{Re} \to 0$, the eigenfunction $\Phi(x)$ becomes a δ-function (Fig. 4), and Bt^* tends toward infinity. Under the condition $\Gamma_0 \neq 0$, the quantity Re_m^* also tends toward infinity. At Re < 0 the flow regime is descending, and no bifurcation of the MHD regime is observed.

5.3. Asymptotic behavior for a slightly swirling MHD regime

Let us consider the situation as $\mathrm{Re} \to \infty$, but $\Gamma_0 \ll \mathrm{Re}$. Under these conditions, the asymptotic analysis can be extended to swirling flows. Since the rotation is assumed to be slight, the results for y,Φ, and Al derived in Subsection 3.3 remain in force. Working from Eq. (4), assuming that the circulation and its derivatives in the core of the flow are bounded, and using $S = \mathrm{Al}\cdot\mathrm{Re}^2$, we find $L = L_p = \mathrm{const}$. Making use of the constancy of L in the core of the flow and relations (13), we can derive from (5) (under the assumption $\mathrm{Re} \gg 1$) an equation for the circulation: $(1-x^2)\Gamma' = 2(\Gamma - L_p)$. Hence $\Gamma_p = C(1+x)/(1-x) + L_p$.

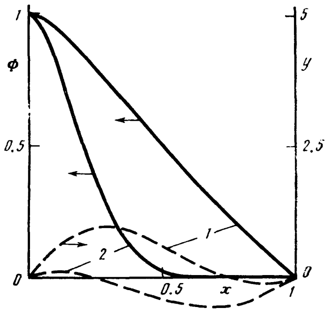

FIG. 4. Changes in the meridional flow and in the eigenfunction $\Phi(x)$ as R is reduced at $\Gamma_0 = 10$. 1—Re = 7.5, Bt* = 1; 2—Re = 2.5, Bt* = 21.3.

We now consider the boundary layer near the plane. Introducing the internal variables $\eta = x\mathrm{Re}$ and $l = L\,\mathrm{Re}$, and taking the limit $\mathrm{Re}\to\infty$, we find from (4) and (5) $\Gamma'' = -l'$, $l'' = -\Gamma'$, or $l''' = l'$.

Since we have $l(0) = 0$, and since l must be bounded in the limit $\eta\to\infty$, we find

$$L=L_p[1-\exp(-\eta)],\quad \Gamma=C_1-L_p\exp(-\eta)\,\mathrm{Re}.$$

From the condition $\Gamma\to 0$ as $\eta\to\infty$ we find $C_1 = 0$, and from $\Gamma(0) = \Gamma_0$ we find $L_p = -\Gamma_0/\mathrm{Re}$. Reconciling solutions $\Gamma(\eta)$ and $\Gamma_p(x)$, we find

$$C=-L_p,\quad \Gamma_p=2x\Gamma_0/[\mathrm{Re}_m(1-x)].$$

The function Γ_p has a pole at $x = 1$, so a boundary layer forms near the axis. Introducing the variables

$$\zeta=\mathrm{Re}_m(1-x)/2,\quad \Gamma_*=\Gamma/\Gamma_0,\quad l_*=L\,\mathrm{Re}/\Gamma_0,$$

and substituting them into (4) and (5), we find, in the limit $\mathrm{Re}\to\infty$,

$$\Gamma_*''=l_*',\quad l_*''=\Gamma_*'+\Gamma_*/\zeta$$

or

$$\Gamma_*'''=\Gamma_*'+\Gamma_*/\zeta,\quad l_*=\Gamma_*'+C_2.$$

Working from the last equation, using the requirement $\Gamma_*'(\zeta)\to 0$ as $\zeta\to\infty$, and reconciling with the potential solution in the core, we find $C_2 = -1$. It then follows, by virtue of $l_*(0) = 0$, that we have $\Gamma_*'(0) = 1$. We integrate the equation for the circulation once:

$$\zeta\Gamma_*''=\Gamma_*'+\zeta\Gamma_*-1,\quad \Gamma_*(0)=0,\quad \Gamma_*'(0)=1.$$

The quantity $\Gamma_*''(0)$ is an independent parameter, which is determined from the requirement that Γ_* be bounded as $\zeta\to\infty$. A calculation yields $\Gamma_*''(0) = 1.57$; Fig. 5 shows the functions Γ_* and l_*. Both the circulation and the azimuthal component of the induction reach maxima at the axis. The maximum value of the circulation is half the value of the circulation at the plane, and it is reached at $\zeta = 1.42$.

On the basis of the distributions found in the core and in the boundary layers, we can construct uniform asymptotic representations:

$$\begin{aligned}\Gamma&=\Gamma_0[\exp(-\mathrm{Re}_m x)+x\Gamma_*(\zeta)],\\ L&=\Gamma_0\,\mathrm{Re}^{-1}[\exp(-\mathrm{Re}_m x)-l_*(\zeta)].\end{aligned}\tag{25}$$

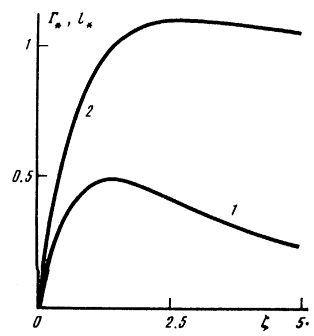

FIG. 5. Distributions of (*1*) the circulation Γ_* and (*2*) the azimuthal induction l_* in the boundary layer near the axis in the case $\mathrm{Re}\gg 1$.

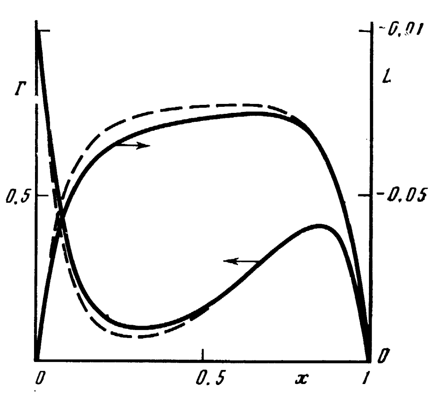

FIG. 6. Comparison of the distributions of the circulation and of the azimuthal induction in the case $\mathrm{Bt} = 1$, $\Gamma_0 = 1$, $\mathrm{Re} = 14$ ($\mathrm{Al} = 0.85$) found by direct calculation (the solid lines) and from asymptotic expressions (25) (the dashed lines).

Figure 6 compares these asymptotic representations with the results of a direct calculation.

This analysis shows that the azimuthal magnetic field is uniformly weak throughout the flow region and is on the order of $\Gamma_0\mathrm{Re}^{-1}$. The circulation is on the same order of magnitude in the core. The rotation of the medium in the plane causes a swirling of only the close-lying layers of liquid, which are transported by the meridional flow to the origin of coordinates and are then carried off by the jet near the axis. The diffusion of the circulation from the plane and from the swirling jet near the axis is suppressed.

5.4. Effect of rotation on the translation back to a laminar regime

The boundary conditions for the transition from a turbulent MHD regime to a laminar one were formulated in Subsection 4.4. Specifically, we have $y(1) = 4$, $y'(1) = 2 - \Gamma_1^2/8$, $\Gamma(1) = \Gamma$, $\Gamma'(1) = F(1) = F'(1) = 0$. The parameters S, Φ_1, L_1, Γ_1, and $F''(1)$ are found from the conditions $\Phi'(0) = 0$, $\Phi(0) = 1$, $L(0) = 0$, $\Gamma(0) = \Gamma_0$, $y(0) = 0$. We can then calculate Re and Al. Figure 3 shows the results of a calculation for $\mathrm{Bt} = 0.12$ (curve L_2).

The swirling facilitates a transition to a laminar regime. With increasing Γ_0, the value of Re and Al at which a laminar solution is restored decrease. If the rotation is sufficiently pronounced, the flow near the axis is laminar. The region of turbulent MHD regimes is bounded by B_2 and L_2, which intersect on line T.

5.5. Inverse bifurcation

If the rotation is weak, the bifurcation of the MHD regime is direct (soft). A new solution exists only for $\mathrm{Re} > \mathrm{Re}^*$ (Fig. 2). If Γ_0 is sufficiently large, however, the bifurcation becomes inverse, and the derivative $d\,\mathrm{Re}/d\,\mathrm{Al}$ changes sign at $\mathrm{Re} = \mathrm{Re}^*$, becoming negative. The change in the sign of $d\,\mathrm{Re}/d\,\mathrm{Al}$ occurs on curve K (Fig. 3), i.e., at a moderate swirling, while the original flow is laminar and ascending. The critical Reynolds numbers Re^* during the inverse (hard) bifurcation cannot be determined by the linear approach.

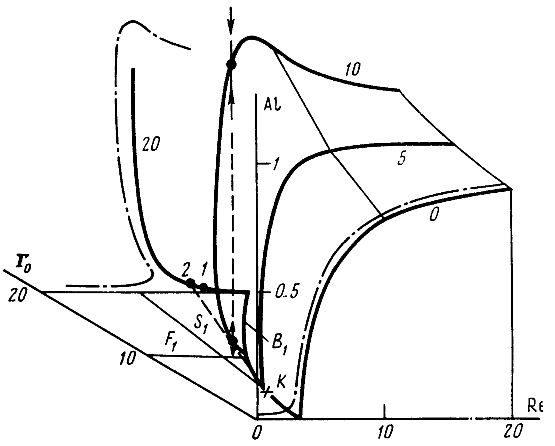

FIG. 7. Transition from a soft bifurcation of the MHD regime to a hard bifurcation with an intensification of the swirling with Bt = 1. Curves 0, 5, 10, 20 correspond to $\Gamma_0 = 0, 5, 10, 20$. Here F_1, B_1, and K have the same meaning as in Fig. 3.

5.6. Hysteresis

Above curve K (Fig. 3) in the $\{\mathrm{Re}, \Gamma_0\}$ plane there exists a region in which the stable solution is not unique for each value of Bt. For Bt = 1, this region lies between curves F_1 and B_1, while for Bt = 0.12 it lies between F_2 and B_2. Figure 7 shows the nature of the $\mathrm{Al}(\mathrm{Re}, \Gamma_0)$ surface. Corresponding to each point (Re, Γ_0) in the nonunique region are three solutions: a purely hydrodynamic solution (Al = 0) and two MHD solutions. According to the general theory,[14] the nature of the transitional trajectories between solutions is as shown by the arrows in Fig. 7 for $\Gamma_0 = 10$ and Re = 7.5. The structure of these three solutions is clear from Fig. 8.

Solution 2, shown by the dashed lines, is unstable. The corresponding point in the phase space belongs to a separatrix which separates regions of attraction toward solutions *1* and *3*. At small values of the Alfvén number, the structure of the solutions undergoes important changes. For the curve $\Gamma_0 = 20$ (Fig. 7), for example, in the region between the point of intersection with the Al = 0 plane and point *1* we have a two-cell regime (see the inset in Fig. 3). Above point *1*, in a small neighborhood of the axis, the flow becomes ascending, and we go into a three-cell regime. As Al is increased, the central cell becomes progressively narrower, and it disappears at point *2*, above which the flow is of a single-cell, ascending nature on the entire $\Gamma_0 = 20$ curve. At Bt = 1, these metamorphoses occur on the lower part of the $\mathrm{Al}(\mathrm{Re}, \Gamma_0)$ surface, that corresponds to unstable solutions. With decreasing Bt, however, the line which separates the ascending and multicell regimes (curve S_1 in Fig. 3) moves to the upper part of the surface (curve S_2 in Fig. 3). The distributions in the jets are more commonly represented in cylindrical coordinates. Figure 9 shows a representative distribution for a stable swirling MHD regime.

6. DISCUSSION

The physical mechanism for the dynamo effect discussed above is quite clear. We might add to the discussion in Subsection 3.5 that in the case of a swirling flow the toroidal "fluid conductor" should be replaced by a "fluid conductor" wound into a cylinder of a certain radius whose axis coincides with the symmetry axis. The flow twists the cylinder and compresses it, to a maximum extent near the equatorial

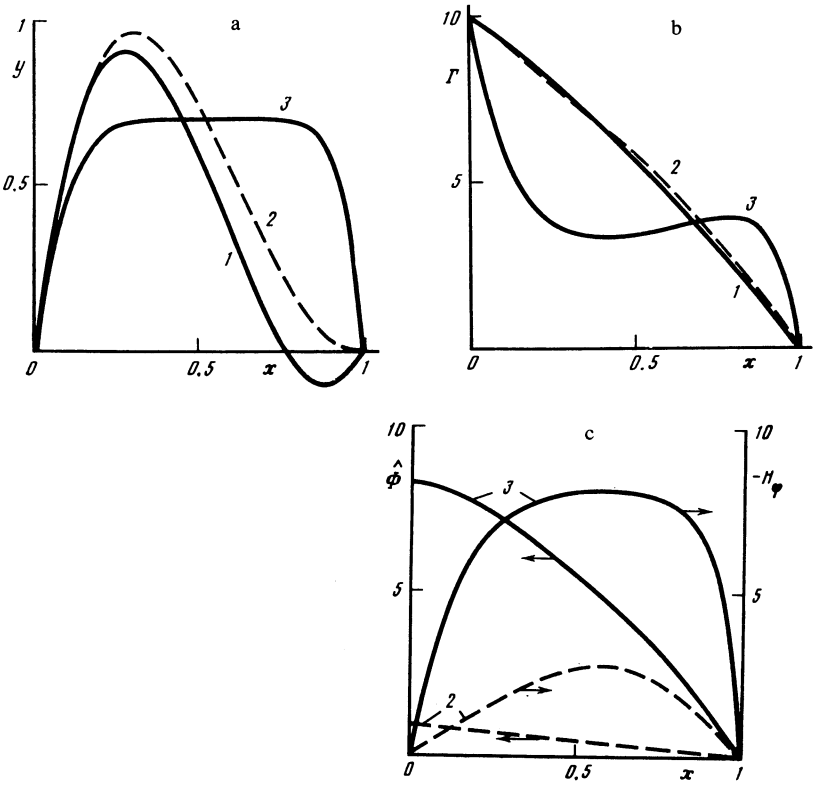

FIG. 8. Distributions of (a,b) the velocity and (c) the induction for Bt = 1, Re = 7.5, and $\Gamma_0 = 10$. *1*—Al = 0; *2*—Al = 0.0124; *3*—Al = 1.22 $H_\varphi rB_\varphi/(4\pi\rho\nu\nu_m)^{1/2}$, $\hat{\Phi} = \Phi\,\mathrm{Re}\cdot\mathrm{Al}^{1/2}$.

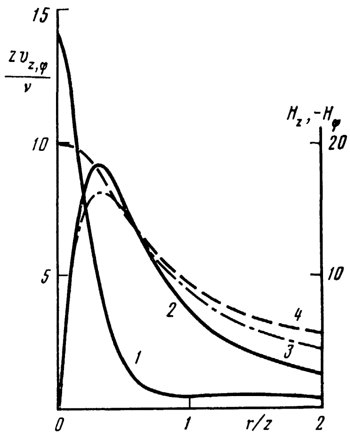

FIG. 9. Profiles of (*1,2*) the velocity and (*3,4*) the induction for an MHD jet near the axis in a cylindrical coordinate system $H_z = rB_z/(4\pi\rho\nu\nu_m)^{1/2}$, Bt = 1, Re = 10, $\Gamma_0 = 10$, Al = 1,22. *1,3*—*z* components; *2,4*—φ components.

plane. The compression of the solenoid intensifies the original magnetic field near the axis, creating a positive feedback.

Although this problem is comparatively simple, and allows analytic solutions in limiting cases, it is nevertheless nontrivial, incorporating such effects as the collapse of dynamic and magnetic jets, a self-focusing of a rotation, soft and hard bifurcations of a magnetic field, hysteresis, and a transition back to laminar flow.

A question which remains open is the extent to which the self-similar solution derived here applies to real, non-self-similar flows. We know[23] that self-similar solutions can serve as intermediate asymptotes for specific flows. In the case of astrophysical jets, for example, such solutions can be used to approximate the velocity and induction fields at distances much greater than the size of the massive central object but much smaller than the distances between massive objects. Just how a deviation from self-similarity will affect the properties of the solution is difficult to predict. The agreement of experimental data with a theoretical analysis of the stability, particularly that of the boundary layers, carried out in the self-similar approximation is evidence that a deviation from self-similarity may be unimportant for an analysis of the bifurcations of secondary regimes. We would expect that those properties of the solution which are not structurally stable would exhibit the greatest changes.

For example, a weak external magnetic field should disrupt the fork nature of bifurcation. The dependence of the induction on the flow intensity in this case may acquire the shape shown by the dot-dashed curves in Fig. 7 near lines $\Gamma_0 = 0$ for nonswirling jets and $\Gamma_0 = 20$ for swirling jets. The induction is now zero at all times, even if there is no motion. A converging flow causes the magnetic lines to crowd together near the axis. This effect, which is not related to an instability, leads to a slight increase in the induction with an increase in velocity. Near critical values of the Reynolds number, however, there is an anomalous intensification of the induction due to the disrupted bifurcation. To get an idea of the possible scales of the anomalous intensification, it is sufficient to compare the magnetic fields of the galactic background ($\sim 10^{-6}$ Oe) and those of stars (~ 1 Oe).[2,15] During the disruption of a fork bifurcation, such properties as the nonuniqueness of the solutions and the hysteresis are retained, since they are coarse properties.

The results derived here may also have some meaning in the case in which there is a pronounced decrease in the induction outside the self-similar region, and Cowling's theorem applies. Let us assume that the motion occurs in a bounded volume $r_i < r < r_0$, $r_0/r_i \gg 1$ and that there exists a subregion $r_i \ll r_1 < r < r_2 \ll r_0$ in which the flow is approximately the self-similar flow described above (this situation is typical of jets). We assume that we are given a perturbation of the induction at $t = 0$ which is on the order of B_0 and which is localized in the region $r < r_0$. If Re > Re*, then the perturbation grows to a size $\sim B_s$ in the self-similar subregion over a time on the order of $t_1 \gtrsim r_1^2/\nu_m$. This magnitude of the perturbation corresponds to a steady-state self-similar solution. This level will persist in a quasisteady fashion until a time on the order of $t_2 = r_2^2/\nu_m$, after which the induction will decay because of dissipation and dispersal, as follows from Cowling's theorem. Astrophysical jets are sometimes three orders of magnitude larger than the size of the region in which they are formed,[6] so t_2 may be six orders of magnitude greater than t_1. However, the lifetime of jets near stars is itself $\sim 10^{-5}$ of the lifetime of the star.[6] It can thus be suggested that in the course of the condensation of the interstellar medium and the formation of stars there will be a generation of a magnetic field by the mechanism described above, and then (if no other dynamo mechanism operates) this field will decay slowly.

Since it turns out that the primary necessary condition for the occurrence of a dynamo is not so much the jet itself as a converging nature of the motion of the medium, we do not rule out the possibility that a similar effect might also be observed in flows of other types near immobile points in whose vicinity the flow is convergent, e.g., in sunspots.

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