Disclinations in a vector field of the polarizations of acoustic and optical beams under conical refraction conditions

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It is shown that internal conical refraction of acoustic and electromagnetic wave beams is characterized by distributions of the polarization vectors with disclination-type singularities ("polarization disclinations") representing lines of zero amplitude of the field such that the angular rotation of the polarization vector is a multiple of 2π when it goes around such a line.

INTRODUCTION

The topology of acoustic and electromagnetic wave fields is being investigated extensively at present. The most general concepts which can be used to describe the characteristic features of the spatial distribution of the polarization and phase of a large class of wave processes can be found in Refs. 1–7. Theoretical and experimental investigations of various specific physical situations leading to the occurrence of such characteristics are reported in Refs. 8–14. We shall show that under the conditions of the classical effect of internal conical refraction the fields of the polarization vectors of acoustic and optical teams exhibits disclination-type singularities.

Plane elastic and electromagnetic waves in crystals are characterized by a singular dependence of the orientation of the unit polarization vectors $\mathbf{a} = \mathbf{u}/u$ and $\mathbf{d} = \mathbf{D}/D$ ($\mathbf{u}$ and $\mathbf{D}$ are the elastic displacement and electrical induction vectors) on the direction of propagation near the acoustic or optic axis, respectively, i.e., close to the direction of degeneracy of the phase velocity of isonormal elastic or electromagnetic waves.[1)] If the relevant polarization vector $\mathbf{a}$ or $\mathbf{d}$ goes round the wave normal $\mathbf{m} = \mathbf{k}/k$ along such a degeneracy direction $\mathbf{m}$ following a cone with an infinitesimally small vertex angle, this vector is rotated by an angle $2\pi n$, where n is the Poincaré index of a singular point $\mathbf{m}_0$ of the vector field $\mathbf{a}(\mathbf{m})$ or $\mathbf{d}(\mathbf{m})$. In the case of the waves that propagate exactly along an acoustic or optic axis $\mathbf{m}_0$, the polarization vectors ($\mathbf{a}$ and $\mathbf{d}$, respectively) can have any orientation in certain planes. The optic axis corresponds to $n = 1$ if the medium is uniaxial, $n = 1/2$ if the medium is biaxial. In the case of acoustic axes we can have the values $n = 0$, $\pm 1/2$, and ± 1. A classification of the polarization singularities, corresponding to acoustic axes of different types in crystals of arbitrary anisotropy, is constructed in Ref. 11. A similar investigation of the vector characteristics of a quasistatic electric field, which accompanies the propagation of sound in piezoelectrics, is reported in Ref. 12.

When we replace the plane-wave approximation with packets of elastic or electromagnetic waves, we have to answer the following natural question: how do such polarization singularities in the $\mathbf{k}$ space affect the wave fields in $\mathbf{r}$ space for beams propagating along the acoustic and optic axes? We investigate from this point of view the phenomenon of internal conical refraction of ultrasonic and optical beams. It is known that such refraction occurs in the directions of acoustic axes of the conical type and optic axes of biaxial crystals. Either type of axis is characterized by a junction ("contact") of isofrequency sheets where the splitting of the sheets is linear as a function of the tilt away from the degeneracy direction. In this case the polarization singularities of weakly diverging wave packets should be manifested particularly strikingly.

It should be pointed out that in solving the problem in question the initial relationships are the results of Refs. 15 and 16, where analytic expressions are obtained for the wave fields of the beams propagating along the acoustic and optic axes of different types.

ACOUSTIC BEAMS

We consider a monochromatic ultrasonic beam of frequency ω propagating along an acoustic axis of the conical type, parallel to a threefold symmetry axis,

$$\mathbf{u}(\mathbf{r}, t) = C\mathbf{A}(\mathbf{r})\, e^{i(Kz-\omega t)}, \tag{1}$$

where z is directed along the threefold symmetry axis; $K = \omega(d/c_{44})^{1/2}$; d is the density of the investigated crystal; c_{44} is an elastic modulus. We assume that at the crystal boundary ($z = 0$) the displacement field is given by the Lorentz distribution

$$\mathbf{A}(\mathbf{r})\,|_{z=0} = \mathbf{a}_0 \frac{\sigma}{2\pi} (\sigma^2 + x^2 + y^2)^{-3/2}, \tag{2}$$

where $\mathbf{a}_0$ is perpendicular to the z axis. The last condition relating to the orientation of the polarization $\mathbf{a}_0$ makes it possible to represent a weakly diverging wave packet ($\sigma K \gg 1$) by a superposition of transverse plane waves characterized by a polarization singularity with a Poincaré index $n = -1/2$ near the acoustic axis (see Ref. 11). The expression for the vector amplitude of the beam $\mathbf{A}(\mathbf{r})$, perpendicular to the z axis, is obtained in Ref. 16 without allowance for the diffraction divergence and can be represented in the form

$$\mathbf{A}(z, \rho, \varphi) = \frac{1}{2\pi} p^{-3/4} \left\{ \left[\sigma \cos\frac{3\psi}{2} + \beta z \sin\frac{3\psi}{2} \right] \hat{I} + \rho \sin\frac{3\psi}{2} \hat{O}(\varphi) \right\} \mathbf{a}_0, \tag{3}$$

where the radius vector $\mathbf{r}$ is defined in a cylindrical coordinate system $[\rho = (x^2 + y^2)^{1/2},\ \tan\varphi = y/x]$,

$$\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{O}(\varphi) = \begin{pmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix}, \tag{4}$$

$$p = (\rho^2 + \sigma^2 - \beta^2 z^2)^2 + 4\sigma^2\beta^2 z^2, \tag{5}$$

$$\psi = \operatorname{arcctg}[(\rho^2 + \sigma^2 - \beta^2 z^2)/2\sigma\beta z], \tag{6}$$

$$\beta = (c_{14}^2 + c_{25}^2)^{1/2}/c_{44} = \operatorname{tg}\Phi, \tag{7}$$

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and Φ is the angle at the base of a circular refraction cone with its axis parallel to the z axis.

We shall determine whether there is a planar vector field $\mathbf{A}(\mathbf{r})$ described by Eq. (2) or any orientational singularities [i.e., singular points of the field of unit vectors with the polarization $\mathbf{a}(\mathbf{r}) = \mathbf{A}/|\mathbf{A}|$]. Clearly, if we allow for the requirements of continuity of the function $\mathbf{A}(\mathbf{r})$, we find that such singularities can exist only in the vicinity of the points where $\mathbf{A}(\mathbf{r}) = 0$. We shall write down the beam polarization of the crystal boundary $z = 0$ in the form $\mathbf{a}_0 = (\cos\Omega, \sin\Omega, 0)$. Analysis shows that the equation $\mathbf{A}(\mathbf{r}) = 0$ may be satisfied only in the $x'z$ plane, where the x' axis is obtained by rotating the x axis about the z axis by an angle $\varphi_0 = \pi/2 - 2\Omega$. In this plane the equation $\mathbf{A}(\mathbf{r}) = 0$ can be written in the form

$$\sigma\cos\frac{\psi}{2} + (x' - \beta z)\sin\frac{\psi}{2} = 0. \tag{8}$$

The relationship (8) reduces to the cubic equation

$$2\left(\frac{\beta z}{\sigma}\right)^3 - 3\left(\frac{\beta z}{\sigma}\right)^2\frac{x'}{\sigma} - 2\frac{\beta z}{\sigma} + \frac{x'}{\sigma} + \left(\frac{x'}{\sigma}\right)^3 = 0, \tag{9}$$

which has only one root satisfying the condition $z > 0$:

$$\frac{\beta z}{\sigma} = \frac{x'}{2\sigma} + \left(\frac{x'^2}{\sigma^2} + \frac{4}{3}\right)^{1/2}\cos\left\{\frac{1}{3}\arccos\frac{(x'/\sigma)^3}{[(x'/\sigma)^2 + 4/3]^{3/2}}\right\}. \tag{10}$$

The solution (10) defines in the $x'z$ plane the line $z = z(x')$ (Fig. 1) on which the amplitude of the beam $\mathbf{A}(\mathbf{r})$ vanishes. Additional analysis of Eq. (3) for $\mathbf{A}(\mathbf{r})$ shows that a displacement field in any plane characterized by $z = \text{const} > z_0$ (Fig. 1) has orientational singularities with the Poincaré indices $n = \pm 1$ in the vicinity of the points where the $z = z(x')$ line found above intersects this plane (Fig. 2a). As z approaches z_0, the singularities come closer together and at $z = z_0$ they merge to form a point with the index $n = 0$ (Fig. 2b). In the case of sections defined by $z = \text{const} < z_0$ the amplitude of the displacement field does not vanish in the beam and the field does not have orientational singularities.

It therefore follows that the line $z = z(x')$ can be interpreted as a polarization disclination. The wave field $\mathbf{u}(\mathbf{r})$ vanishes at any point on such a disclination[2)] and it is characterized by an orientational singularity with the index $n = 1$ or $n = -1$ in each $z = \text{const} > z_0$ plane. We recall that the sign of the index n is regarded as positive if the directions of rotation of the vectors $\mathbf{r}$ and $\mathbf{A}(\mathbf{r})$ coincide and it is assumed that the orientations of the normals (the ends of which determine the directions of such rotation) meet at an acute angle. It therefore follows that the sign of n depends on the selection rule applicable to a plane R in which the radius vector $\mathbf{r}$ goes round a singularity of the field $\mathbf{A}(\mathbf{r})$. The plane R is selected above to coincide with the rotation plane $\mathbf{A}(\mathbf{r})$, i.e., it is assumed to be orthogonal to the z axis. In this case the point $z = z_0$ separates a disclination line into two branches with different signs of the index $n = \pm 1$. An alternative rule would specify orientation of this rotate plane R relative to the $z(x')$ line: for example, the R plane may be selected to be always orthogonal to the $z(x')$ line. We can easily see that in this case the sign of the index n along the whole $z(x')$ line remains constant (positive or negative depending on the accepted convention relating to the "direction" of a disclination line), with the exception of the point

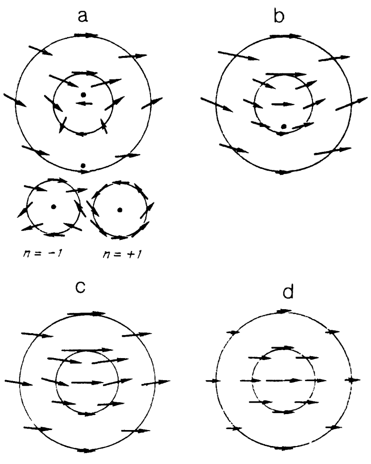

FIG. 2. Displacement field in an acoustic beam in transverse sections shown in Fig. 1. Figures 2a and 2b identify the points where the field vanishes; Fig. 2a shows separately the images of the polarization singularities on an enlarged scale.

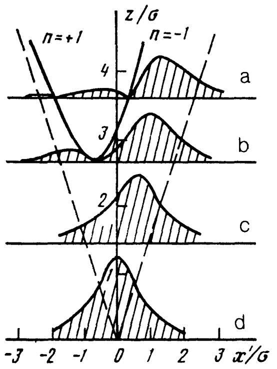

FIG. 1. Shape of a polarization disclination line in an acoustic beam. The figure shows profiles of the beam amplitude in the zx' plane in different z = const sections (a, b, c, d). The dashed lines are the generators of an internal refraction cone. Here and in Figs. 2 and 3 it is assumed that the value of $\beta = \tan\Phi$ is $\beta = 0.31$, which represents quartz.

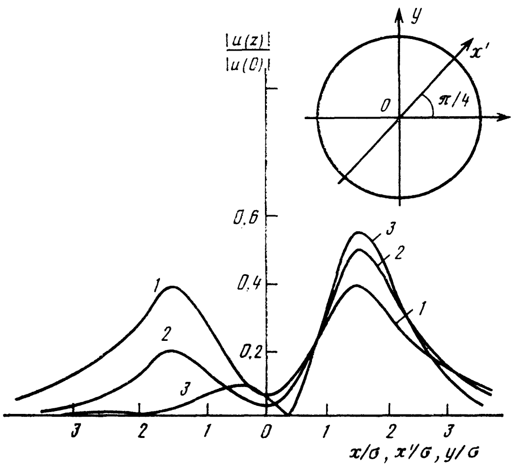

FIG. 3. Profiles of an acoustic beam obtained in the $z/\sigma = 3.6$ case (section a in Fig. 1) along the directions shown in the inset: 1) $0x$; 2) $0x'$; 3) $0y$.

z_0 where the sign of n can be found from the condition of continuity.

Figure 3 shows distributions of the scalar amplitude of the beam $|\mathbf{u}(\mathbf{r})|$ at $z = \text{const}$ for different values of the azimuth φ. An amplitude maximum occurs in the vicinity of the direction of the group velocity corresponding to the polarization $\mathbf{a}_0$ specified on the entry plane of the crystal (see Ref. 17).

In a piezoelectric an induction wave of a quasistatic electric field, which accompanies the propagation of sound, is related to a field of mechanical displacements of Eq. (1) in the vicinity of the threefold axis:

$$D_i(\mathbf{r}) \approx -iKe_{ij3}u_j(\mathbf{r}),\ i,\ j=1,\ 2 \tag{11}$$

($\hat{e}$ is the tensor of the piezoelectric coefficients). Therefore, the relevant vector field $\mathbf{D}(\mathbf{r})$ also includes a polarization disclination $z(x')$ defined by Eq. (10) and singularities of the field $\mathbf{D}(\mathbf{r})$ located in its vicinity are characterized by the same values of the index n as the singularities of the field $\mathbf{u}(\mathbf{r})$ (see Ref. 12) because of the condition $\det e_{ij3} > 0 (i, j = 1,2)$.

OPTICAL BEAMS

We consider the distribution of the electrical induction in a beam of electromagnetic waves,

$$\mathbf{D}(\mathbf{r}) = C\mathbf{D}_0(\mathbf{r})\, e^{i(Kz-\omega t)}, \tag{12}$$

which propagates along an optic axis lying in a symmetry plane $x_1 = 0$ of a transparent nongyrotropic nonmagnetic crystal with the monoclinic symmetry. The direction z representing the optic axis makes an angle θ_0 with the x_0 crystallographic axis; $K = \omega\varepsilon_{11}^{2}/c$, where c is the velocity of light in vacuum; ε_{ij} represents here and later the components of the permittivity tensor expressed in terms of the crystallographic coordinate system x_1, x_2, and x_3. In this case the direction of the axis of a circular refraction cone (cone of rays) lying in the symmetry plane is tilted, relative to the z axis which is one of the generators of the cone, at an angle $\phi = \tan^{-1}\beta$, where $\beta = [(\varepsilon_{22}^{-1} - \varepsilon_{33}^{-1})\sin 2\theta_0 + 2\varepsilon_{23}^{-1}\cos 2\theta_0]/4\varepsilon_{11}^{-1}$ (Ref. 15). We shall assume that the wave field of the beam along the $z = 0$ boundary of a crystal is described by a Lorentz distribution of the form (2) with an initial polarization $\mathbf{d}_0 = (\cos\Omega, \sin\Omega, 0)$. The expression for $\mathbf{D}_0(\mathbf{r})$ obtained in Ref. 15 can be represented in a form similar to Eq. (3). Then, instead of Eq. (4) we have

$$\hat{O}(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}, \tag{13}$$

and, moreover, there is a change in the meaning of the parameters ρ and φ, which should now be measured from the direction of the refraction cone axis, i.e. (see Fig. 4)

$$\rho = [(y+z\beta)^2 + x^2]^{1/2},\ \operatorname{tg}\varphi = -x/(y+z\beta). \tag{14}$$

Using this relationship between the expressions for the acoustic and optical beams, we can readily find the shape of a disclination in an optical beam [i.e., we can find the solution of the equation $\mathbf{D}_0(\mathbf{r}) = 0$] using the results of the preceding section. We can easily see that $\mathbf{D}_0(\mathbf{r}) = 0$ when $\varphi = 2\Omega$, i.e., in the $\tilde{z}\tilde{x}$ plane (Fig. 4), where the $\tilde{z}$ axis is directed along the refraction cone and the $\tilde{x}$ axis is obtained by rotation of the x axis about $\tilde{z}$ by an angle χ corresponding to the condition

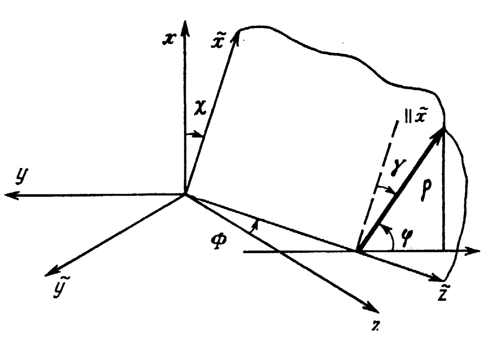

FIG. 4. Geometry of the problem in the case of an optical beam.

$$\operatorname{tg}\chi = \cos\Phi \operatorname{ctg} 2\Omega. \tag{15}$$

In this plane a disclination line $\tilde{z} = \tilde{z}(\tilde{x})$ is described by an equation which is obtained from Eq. (10) by the substitution (Fig. 4)

$$x' \to \tilde{x}/\cos\gamma,\ z \to (\tilde{z} - \tilde{x}\operatorname{tg}\gamma)\cos\Phi, \tag{16}$$

where

$$\sin\gamma = \sin\Phi\cos 2\Omega. \tag{17}$$

CONCLUSIONS

Results show that internal conical refraction of ultrasonic and optical beams creates polarization disclinations, which are lines of orientational singularities in the fields of the vector amplitudes $\mathbf{A}(\mathbf{r})$ and $\mathbf{D}_0(\mathbf{r})$. We considered beams with a Lorentzian profile for which an answer can be obtained in an analytic form. On the other hand, we can expect that in the case of "topologically continuous" changes in the beam profile these disclinations are suitably deformed but are not destroyed. It should also be mentioned that the geometric locus of singularities of the planar vector field $\mathbf{A}(\mathbf{r})$ or $\mathbf{D}_0(\mathbf{r})$, defined in three-dimensional space by the equation $\mathbf{A}(\mathbf{r}) = 0$ or $\mathbf{D}_0(\mathbf{r}) = 0$, is generally a line,[3] i.e., under the conditions considered here the wave field of any linearly polarized acoustic or optical beam should have a linear (and not point or planar) stationary "defect" in the vector amplitude distribution.

We shall point out one general difference between the orientational singularities of the wave fields in the $\mathbf{k}$ and $\mathbf{r}$ spaces. In the $\mathbf{k}$ space each direction of the wave normal $\mathbf{m} = \mathbf{k}/k$ corresponds to its "own" plane wave with a unit polarization vector $\mathbf{a}$ or $\mathbf{d}$, defined apart from the sign. Consequently, in the field of the polarizations of plane waves the vectors $\mathbf{a}(\mathbf{m})$ or $\mathbf{d}(\mathbf{m})$ can rotate by an angle which is a multiple of π when $\mathbf{m}$ follows a path around a singularity $\mathbf{m}_0$ where the orientation of the polarization is arbitrary in the degeneracy plane, i.e., the Poincaré index of a polarization singularity in the $\mathbf{k}$ space can be a half-integer. In the $\mathbf{r}$ space the wave fields of beams $\mathbf{u}(\mathbf{r})$ or $\mathbf{D}(\mathbf{r})$ should be continuous, because in this case the orientational singularities appear in the vicinity of the points $\mathbf{A}(\mathbf{r}) = 0$ and $\mathbf{D}_0(\mathbf{r}) = 0$, and can be represented only by integral (or half-integral) Poincaré indices (see Refs. 4–7). In particular, in the situation considered in the present paper the singularities in $\mathbf{k}$ space characterized by the indices $n = -1/2$ in the case of acoustic beams and by $n = 1/2$ in the case of optical beams "generate" in **r** space singularities with the indices $n = \pm 1$ in sections of wave beams.

In the case of absorbing and gyrotropic media the topological samples described above should be modified in the spirit of Ref. 6, bearing in mind that in such media the wave polarization is generally not characterized by vectors but by ellipses.

The first experimental attempt to analyze the distribution of the polarization field in a section of an acoustic beam undergoing conical refraction was made in Ref. 17. The results obtained suggested that it should be possible to detect experimentally the disclinations described above in acoustic beams. And even more promising is the possibility of observing such polarization singularities in optical beams.

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[1)]For simplicity, we ignore the spatial dispersion and absorption, i.e., we consider the case of the linear polarization waves.

[2)]It should be noted that such a disclination is stationary (time-independent), in contrast to a topological model of a disclination of an elliptically polarized wave field discussed in Refs. 4, 5, 7.

[3)]The codimensionality of a manifold of zero vectors on a plane is 2 and, consequently, the dimensionality of the corresponding inverse image in the three-dimensional **r** space is unity.

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