Soliton turbulence in nonintegrable wave systems

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The character of the evolution of wave turbulence is studied within the framework of the nonintegrable nonlinear Schroedinger equation in the absence of wave collapse. It is shown that the turbulence separates into two components—weak wave turbulence and a low-density soliton gas whose density decreases with time. The soliton gas is a statistical attractor whose dimension decreases with time. The results of this work were confirmed by numerical experiments.

1. INTRODUCTION

As a rule turbulence in nonlinear continuous media is accompanied by the appearance of localized, extremely nonlinear structures. In those cases when the equations describing the medium have stable soliton solutions the solitons are natural candidates for the role of such structures. Turbulence in this case can be termed soliton turbulence. Soliton turbulence was apparently first studied in 1973 by Kingsep, Rudakov, and Sudan.[1] The purpose of this paper is to give a qualitative description of soliton turbulence within the framework of the nonintegrable nonlinear Schroedinger equation in the absence of wave collapse. It is shown below—based on simple estimates which were confirmed by a numerical experiment (see also Refs. 2 and 3)—that with time two components of the turbulence can be identified: weakly nonlinear wave turbulence and a low-density soliton gas. Over long times the degree of rarefaction of the soliton gas (the ratio of the characteristic distance between solitons to the size of the solitons) increases asymptotically, and in the process almost the entire value of the integral of the "number of quasiparticles" (physically playing the role of energy) is concentrated in the soliton component.

A similar assertion was already made in Refs. 4 and 5, where it was justified by a thermodynamic approach. Such an approach is, however, inadequate, since in reality the turbulence is far from a state of thermodynamic equilibrium. For this reason, in Ref. 6 Protogenov and Fraiman cast doubt on the idea that solitons play a special role in turbulence free of collapse. In this paper we prove the assertion that solitons play a central role in the asymptotic state of turbulence of this type and we thereby show that a soliton gas is a unique statistical attractor, whose dimension decreases with time, in a nonintegrable Hamiltonian system with an infinite number of degrees of freedom.

2. ANALYSIS OF THE CHARACTER OF THE EVOLUTION OF TURBULENCE

A quite universal model of wave turbulence is the nonlinear Schroedinger equation (see, for example, Refs. 2, 5, and 7)

$$i\psi_t+\Delta\psi+f(|\psi|^2)\psi=0, \tag{1}$$

which has a solution in the form of a moving soliton:

$$\psi(\mathbf{r},t)=g(\lambda,\xi)\exp\left[\frac{i}{2}(\mathbf{vr})+i\left(\lambda^2-\frac{v^2}{4}\right)t\right],$$
$$\xi=\mathbf{r}-\mathbf{v}t,\quad \Delta g+f(g^2)g-\lambda^2 g=0,\quad \nabla g|_{\xi=0}=0,$$
$$g\to0 \quad \text{for} \quad \xi\to\infty \tag{2}$$

for $f(u)>0$ and $f'(u)\geqslant Cu^{(2-d)/d}$ (d is the dimension of the space) the soliton (2) is unstable and is not realized in turbulent processes. Conversely, if the soliton is stable (for the important particular case of a power-law nonlinearity $f=u^{s/2}$ the condition for stability has the form $sd<4$), the turbulence is soliton turbulence. The question of the character of the evolution of this turbulence is of fundamental, general physical interest. Equation (1) has the following integrals of motion:

$$N=\int|\psi|^2d\mathbf{r},\quad \mathbf{P}=i\int[\psi\nabla\psi^*-\psi^*\nabla\psi]d\mathbf{r},$$
$$H=\int[|\nabla\psi|^2-\Phi(|\psi|^2)]d\mathbf{r},$$
$$\Phi'(u)=f(u). \tag{3}$$

If quantum effects are neglected, the value of the Hamiltonian H is determined primarily by the short-wavelength range because the energy tends to be distributed uniformly over the degrees of freedom. In this case a condensate—a uniform field accompanied by small-scale fluctuations—could form in the system. But for $f'(u)>0$, the condensate is unstable and in the absence of collapse it decomposes into solitons. In the special integrable case $f(u)=Cu$, $C>0$, and $d=1$ solitons are scattered elastically by one another, and the number of solitons is conserved. In the general nonintegrable case a qualitative thermodynamic analysis of the interaction of solitons with free waves[4,5] shows that the behavior of the system is determined by the accumulation of weak effects which arise owing to the fact that opposing processes are uncompensated. When solitons with a weakly turbulent spectrum interact the processes which increase the amplitudes of the solitons as the number of solitons decreases are thermodynamically advantageous. As the solitons merge the value of the integral H decreases somewhat and the difference is carried away by free waves; the integral corresponding to the number of quasiparticles N (the wave energy) is determined primarily by the soliton. In the process the size of the soliton decreases.

We shall study in greater detail the elementary interactions of solitons with one another and with weakly nonlinear free waves taking into account the integrals of motion (3). For simplicity we shall confine our attention to a power-law nonlinearity $f(u)=u^{s/2}$. Then a soliton of arbitrary size λ^{-1} can be expressed in terms of the universal mode $R(\eta)$:

$$g(\lambda,\xi)=\lambda^{2/s}R(\lambda\xi),\quad \Delta R-R+R^{s+1}=0$$

($R(\eta)=[(1+s/2)^{1/2}/ch(s\eta/2)]^{2/s}$ with $d=1$), which

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makes it possible, in particular, to derive useful relations between the integrals (3) for the soliton solution (2):

$$\mathbf{P}=N\mathbf{v},\quad H=N\frac{v^2}{4}-\frac{\varkappa}{1+q}N^{1+q},\quad N=\lambda^{2/q}N_{cr},\tag{4}$$

where

$$N_{cr}=\int R^2\,d\eta,\quad q=\frac{2s}{4-sd}>0,\quad \varkappa=N_{cr}^{-q}.$$

With the help of Eqs. (4) it is easy to see that in the case of scattering of a weakly nonlinear wave of the form $\propto\exp[i(\mathbf{kr}-k^2t)]$ with δN quanta, momentum $\delta\mathbf{P}=2\mathbf{k}\delta N$, and Hamiltonian $\delta H=k^2\delta N$ by a soliton $(N,\mathbf{v})$ the following relation holds:

$$\delta N_0(|\mathbf{k}_0-\mathbf{v}/2|^2+\varkappa N^q)=\delta N_1(|\mathbf{k}_1-\mathbf{v}/2|^2+\varkappa N^q),\tag{5}$$

where the indices 0 and 1 refer to the starting and scattered waves, respectively.

It follows from (5) that as energy is transferred from the wave into the soliton ($\delta N_1<\delta N_0$) the soliton slows down and the wave vector of the wave increases. It is qualitatively understandable that on the average this process is more likely than the reverse process: as waves accumulate in the region of high values of k their phase volume increases and the entropy of the system increases accordingly (since the number of soliton degrees of freedom is much less than the number of weakly turbulent degrees of freedom it may be assumed in practice that the value of the entropy is determined by the contribution of the weakly turbulent spectrum).

The kinematic calculation of the interaction of solitons with the participation of free waves is more complicated. Using (4) we can show that when two solitons $(N_1,\mathbf{v}_1)$ and $(N_2,\mathbf{v}_2)$ interact the following relations hold (in the coordinate system in which the total momentum of the waves is equal to zero):

$$\Delta N_1=\frac{(\varkappa N_2^q+v_2^2/4)\overline{\delta N}+\overline{\delta H}+N_1(\mathbf{v}_1-\mathbf{v}_2)\Delta\mathbf{v}_1/2}{\varkappa(N_1^q-N_2^q)-|\mathbf{v}_1-\mathbf{v}_2|^2/4}$$
$$\Delta N_2=-\frac{(\varkappa N_1^q+v_1^2/4)\overline{\delta N}+\overline{\delta H}-N_2(\mathbf{v}_1-\mathbf{v}_2)\Delta\mathbf{v}_2/2}{\varkappa(N_1^q-N_2^q)+|\mathbf{v}_1-\mathbf{v}_2|^2/4}\tag{6}$$

where $\overline{\delta N}$ and $\overline{\delta H}$ are carried away by the free waves. It is obvious from (6) that for low velocities the stronger soliton is strengthened and the weaker soliton is attenuated. This process corresponds to a pair "collision" and is obviously more likely than the reverse process (triple "collision"). These arguments confirm the qualitative conclusions of a thermodynamic analysis: as a result of the evolution of a given starting distribution the soliton is a statistical attractor—with time the state decays asymptotically into a soliton and a collection of weakly nonlinear waves.

3. RESULTS OF NUMERICAL MODELING

To prove this assertion directly we integrated numerically the one- and two-dimensional equations (1) to long evolution times with different nonlinearities $f(u)$. The problem was solved in the bounded region $0\leqslant r\leqslant L$ with periodic boundary and perturbed uniform initial conditions:

$$\psi(\mathbf{r},0)=\psi_0[1+\varphi(\mathbf{r})],\tag{7}$$

where $\varphi(\mathbf{r})\ll 1$ is a small perturbation.

Within the framework of Eq. (1) the growth rate of the modulation instability of the condensate $\psi_0\exp[if(|\psi_0|^2)t]$ is determined by the expression

$$\gamma(k)=k(2A-k^2)^{1/2},\quad A=\left(u\frac{\partial f}{\partial u}\right)_{u=|\psi_0|^2}.\tag{8}$$

Its maximum is reached at $k=A^{1/2}$ and is equal to $\gamma_{\max}=A$. The corresponding modulation length $\lambda_{\rm mod}=2\pi/A^{1/2}$ determines the number $t>\gamma_{\max}^{-1}$ of solitons formed for times $n\approx(L/\lambda_{\rm mod})^d$. The parameters of the model were chosen so that the competing requirements that the statistics of the solitons be convincing $n\gg1$ and the asymptotic soliton be adequately resolved $(\lambda\Delta)^d\ll1$ (Δ is the linear size of the cell and λ is the inverse characteristic size of the soliton; for a power-law nonlinearity $\lambda\sim N^{s/(4-sd)}$) always holds in the calculations. The calculations were performed on the ES-1037-ES-2706 multiprocessor complex of the Space Research Institute of the Academy of the Sciences of the USSR. Equation (1) was integrated using an FFT algorithm following a procedure analogous to that employed in Ref. 8. The integrals of the motion (3) were used to monitor the calculations. Aside from the power-law nonlinearity, systems with saturation of the type

$$f(u)=u(1-au),\ f(u)=u(1+b_1u)/(1+b_2u)$$

were also studied.

The computational results demonstrated that the observed space-time dynamics of the system is in complete agreement with the qualitative picture of soliton turbulence predicted above. Figure 1 shows the results of integrating the one-dimensional equation with $f(u)=u^{1/2}$. A lattice of solitons with a period of the order of $\lambda_{\rm mod}$ forms as a result of the development of the modulation instability (Fig. 1a). In the course of further evolution the system is increasingly

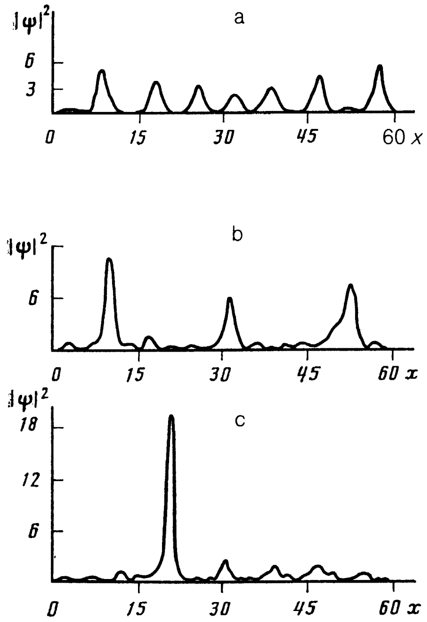

FIG. 1. Fragments of the evolution of the solution of the equation $i\psi_t+\psi_{xx}+|\psi|\psi=0$ with the parameters $\psi_0=1$ and $L=60$; the time $t=17.4$ (a), 365.4 (b), and 730.8 (c).

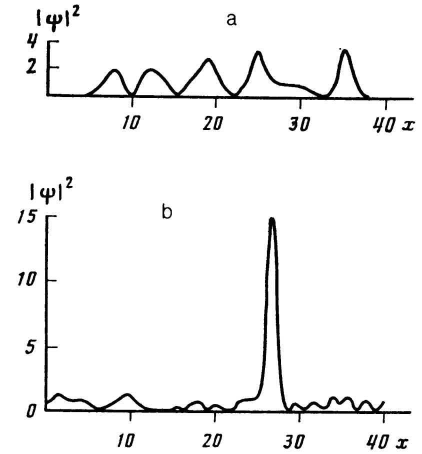

FIG. 2. Fragments of the evolution of the solution of the equation $i\psi_t + \psi_{xx} + |\psi|^2[(1+0.1|\psi|^2)/(1+0.5|\psi|^2)] = 0$ with the parameters $\psi_0 = 1$ and $L = 40$; the time $t = 52.2$ (a) and 765.4 (b).

separated into solitons and weakly nonlinear free waves. The interaction of solitons with one another and with free waves leads to gradual transfer of waves from weak solitons into stronger solitons, and the amplitudes of the solitons increases as the number of solitons decreases (Fig. 1b). Over long times the system is reduced to a single soliton of small size and large amplitude (Fig. 1c). The measured velocity v of the soliton is much less than the group velocity $(\partial\omega/\partial k)_{k=\lambda}$, this is completely obvious: a motionless soliton minimizes the energy. The form of the asymptotic soliton is adequately described by the exact solution of Eq. (2), in which the quantity λ is calculated from the measured amplitude. Free waves account for about 15–20% of the starting integral N.

Analogous results were also obtained for other types of nonlinearities (Fig. 2 corresponds to the nonlinearity $f(u) = u[(1+0.1u)/(1+0.5u)]$), as well as in the solution of the two-dimensional problem. In the two-dimensional case [the results of the integration of Eq. (1) with $f(u) = u^{1/4}$ are presented in Fig. 3], however, the evolution time increases to such an extent (as the number of solitons decreases the probability that they interact with one another decreases) that the problem of achieving a single soliton becomes very complicated for numerical modeling. However, the qualitative picture of the evolution of turbulence, as we can see, is identical to the picture described above. It should be noted that in studying the evolution of turbulence we did not observe any cases in which the number of solitons increased owing to fragmentation of a soliton under the action of weakly nonlinear waves; this indicates that the process is statistically irreversible. This behavior agrees with the calculations of Ref. 9, where it was found that the fragmentation of a soliton by a sound wave requires a special sound packet with large amplitude.

4. CONCLUSIONS

Thus in the process of evolution of long-lived soliton turbulence a nonintegrable system approaches the state of a soliton gas; this makes it possible to regard this state as a statistical attractor. This result was obtained by three methods: analytically near the equilibrium state[4,5] and far from equilibrium (see Sec. 2 of this work) as well as by direct

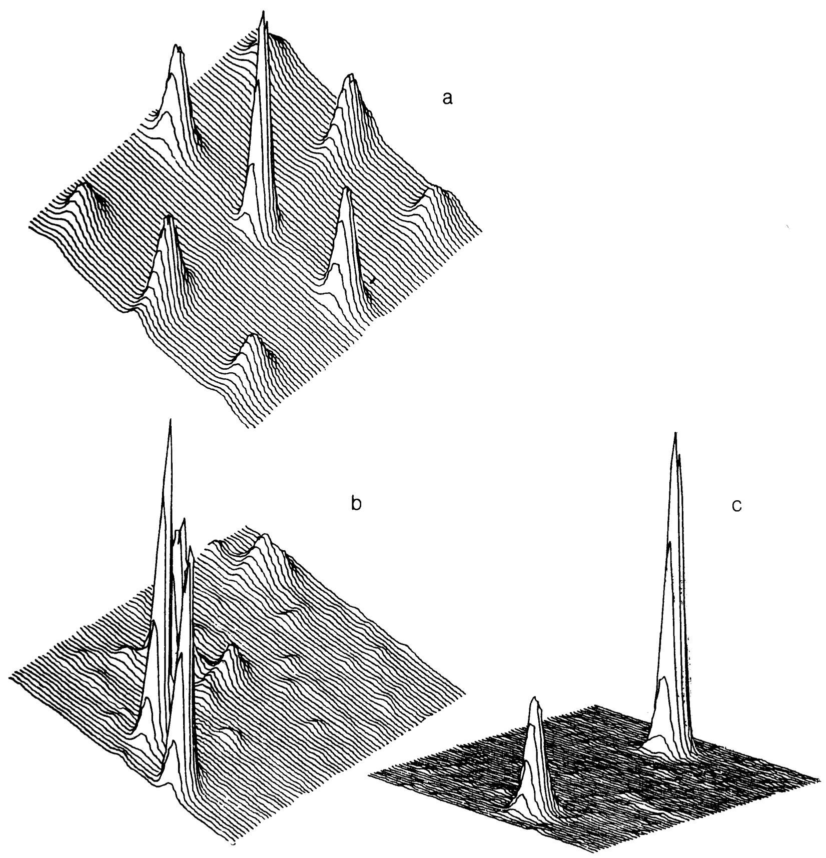

FIG. 3. Fragments of the evolution of the solution of the equation $i\psi_t + \psi_{xx} + \psi_{yy} + |\psi|^{1/2}\psi = 0$ with the parameters $\psi_0 = 1$ and $L = 37.7$: a—$t = 13$ and $|\psi|^2_{max} = 33.2$; b—$t = 277$ and $|\psi|^2_{max} = 63.9$; and, c—$t = 2681$ and $|\psi|^2_{max} = 137.3$.

numerical modeling (Sec. 3). The result does not depend on the dimension of the space and on the details of the interaction (if, of course, the solitons are stable).

The separation of the soliton state was confirmed for the one-dimensional case and a power-law nonlinearity in a detailed paper by Lebowitz *et al.*,[10] where, unlike direct modeling, analytical assumptions about the properties of the thermodynamically equilibrium state are made first and then this state is studied numerically. The well-known difficulties of introducing a measure on function spaces[5] were not mentioned in Ref. 10.

The analytical work of Protogenov and Fraĭman[6] is based on the Langevin force method. They introduce a phenomenological equation, which is chosen so as to obtain the correct answer for linear waves, to describe the soliton. This equation, which differs from the equation derived analytically in Ref. 11, does not take into account direct soliton-soliton interactions and the result obtained—that there are no strong solitons in the asymptotic limit—contradicts the results of the numerical experiment performed in this work.

We note that for infinite-dimensional systems the Hamiltonian character of the system does not prevent the existence of statistical attractors of the soliton type. Strictly speaking, dissipation, which ensures the possibility of the existence of true attractors, also appears as a result of averaging over the many degrees of freedom of the starting Hamiltonian system.

We also note that damping of waves with large wave numbers as well as different nonlinear mechanisms of damping are always present in real physical systems. These effects lead to the fact that solitons of quite small size and high intensity will rapidly dissipate. The concentration of energy in solitons of small size thus turns out to be a strongly nonlinear mechanism for the absorption of energy; this mechanism can be compared with the "collapse" mechanism of dissipation owing to catastrophic development of singularities of the wave field. Thus it turns out that the "soliton" and "collapse" variants of wave turbulence are qualitatively not too different from one another.

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