Relation between exactly integrable models in resonance optics

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A unitary-transformation method that permits a Lax representation of double-resonance equations to be obtained from a given pair of Lax operators is described, with the Lax representation of the double-resonance evolution equations as the example. The integrability of the polarization models of Raman resonance is proved by the method of the inverse scattering matrix problem.

Many processes involving passage of high-power ultrashort pulses (USP) of light[1)] through a resonance medium, including those of practical importance (in spectroscopy, information transmission and processing, and others) can be successfully described by using models based on nonlinear evolution equations[1] that are integrable by the method of the inverse scattering matrix problems (ISP). Since these models correspond to unlike resonance conditions, they are studied as a rule independently of one another. This is perfectly natural from the mathematical viewpoint, since both the evolution equations and their Lax representations, which are needed to apply the ISP method, differ substantially for different models (see below). Yet various optical-resonance theories can be developed in a single manner by using a unitary transformation of the total Hamiltonian of the system.[2] This raises the question of the relation between exactly integrable models of optical resonance. It should be noted that the gauge equivalence of certain exactly integrable theories had been discussed earlier. The equivalence of the nonlinear Schrödinger equation to the Heisenberg-ferromagnet equation was established in this manner.[3] As to optical-resonance theories, it must be emphasized that they cannot be mutually gauge equivalent, since each of them is equivalent to a principal chiral field on different groups, for example $SL(3)$ and $SU(2)$ for models of propagation of two-frequency USP in double (Fig. 1a) and Raman (Fig. 1b) resonance, respectively.[2)]

It is shown in the present paper that a unitary transformation,[2] just as it relates the Hamiltonians of different optical-resonance models, makes it possible to obtain from the Lax representation of one exactly integrable theory a Lax representation of another theory. It becomes possible by the same token to construct new exactly integrable models of nonlinear optics simultaneously with their Lax representation. As an example, we consider exactly integrable models of double resonance with resonance energy levels that are nondegenerate[4,5] and degenerate[6] in different orientations of the total angular momentum. This choice is not accidental. On the one hand, one can see here particularly clearly the role of the unitary transformation that excludes adiabatically, when applied to the Hamiltonian of a three-level system, a common level of adjacent optically allowed transitions (level E_b, Fig. 1a), transforming the system to a Raman resonance with an optically forbidden transition. On the other hand, new results are obtained: The integrability of polarization models of Raman resonance by the ISP is proved for the first time ever, and the type of polarized USP of stationary shape is established, with the character of the USP polarization dependent on the degeneracy of the energy levels.

Section 1 describes in detail the use of the proposed method for double resonance with nondegenerate levels, and repeats the results previously obtained for Raman resonance by differential-geometry methods.[7,8] In the next section are obtained the Lax representations of polarization models of Raman resonance with nondegenerate and threefold degenerate energy level. The profile of polarized Raman-resonance USP of stationary type is also discussed. The Conclusion lists a number of problems whose investigation by the proposed method seems promising.

1. UNITARY-TRANSFORMATION METHOD

We obtain first the equations that describe the propagation of two-frequency USP having an electric field of the form

$$\mathbf{E}=\vec{\mathscr{E}}_1 e^{-i\Phi_1}+\vec{\mathscr{E}}_2 e^{-i\Phi_2}+\text{c.c.},\qquad \Phi_j=\omega_j t-k_j z,\quad j=1,\ 2 \tag{1}$$

in a half-space $z>0$ filled with three-level particles. We start with the classical Maxwell equations

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}=\frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{P} \tag{2}$$

and the quantum-mechanical equation for the density matrix ρ:

$$i\hbar\frac{\partial}{\partial t}\rho=[H_0-\mathbf{Ed},\rho], \tag{3}$$

where H_0 and $\mathbf{d}$ are the Hamiltonian and dipole-moment operator of the three-level particle, and $\mathbf{P}$ is the polarization of the medium

$$\mathbf{P}=\operatorname{Sp}\rho\mathbf{d}. \tag{4}$$

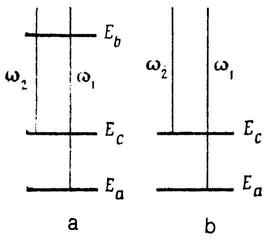

FIG. 1. Energy-level structure for double (a) and Raman (b) resonances.

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The density matrix is normalized to the density N of the three-level particles, $\mathrm{Sp}\,\rho = N$.

We have left out of (3) the relaxation terms by virtue of the short duration of the USP. We have furthermore neglected the inhomogeneous broadening of the spectral lines. We assume also that the energy levels E_a, E_b, and E_c form a Λ configuration: the transitions $E_b \rightarrow E_a$ and $E_b \rightarrow E_c$ are optically allowed, $E_c \rightarrow E_a$ is optically forbidden, and $E_a < E_c < E_b$. We neglect in the present section the energy-level degeneracy and the USP polarization.

We transform the density matrix using the unitary operator e^{iS}:

$$\tilde\rho = e^{-iS}\rho e^{iS}.$$

The equation for the transformed density matrix $\tilde\rho$,

$$i\hbar\frac{\partial}{\partial t}\tilde\rho = [\tilde H, \tilde\rho], \tag{3'}$$

is determined by the Hamiltonian

$$\tilde H = e^{-iS}H_0e^{iS} - e^{-iS}E\,de^{iS} - i\hbar e^{-iS}\frac{\partial}{\partial t}e^{iS},$$

which we expand in usual fashion:

$$\tilde H = H_0 - i[S, H_0] - {}^1/_2[S, [S, H_0]]$$
$$\ldots - Ed + i[S, Ed] + {}^1/_2[S, [S, Ed]] + \ldots - i\hbar e^{-iS}\frac{\partial}{\partial t}e^{iS}.$$

The polarization (4) of the medium also takes a similar form

$$P = \mathrm{Sp}\{\tilde\rho(d - i[S, d] - {}^1/_2[S, [S, d]] - \ldots)\}. \tag{4'}$$

We represent S and $\tilde H$ by series in powers of the electric field intensity:

$$S = S^{(1)} + S^{(2)} + \ldots, \qquad \tilde H = \tilde H^{(0)} + \tilde H^{(1)} + \tilde H^{(2)} + \ldots. \tag{5}$$

Here

$$\tilde H^{(0)} = H_0,$$
$$\tilde H^{(1)} = -Ed - i[S^{(1)}, H_0] + \hbar\frac{\partial}{\partial t}S^{(1)}, \tag{6}$$
$$\tilde H^{(2)} = \frac{i}{2}[S^{(1)}, Ed] - \frac{i}{2}[S^{(1)}, \tilde H^{(1)}] - i[S^{(2)}, H_0] + \hbar\frac{\partial}{\partial t}S^{(2)}.$$

The succeeding simplifications differ here and depend on the USP propagation conditions.

For the double resonance

$$\omega_1 \approx \omega_{ba}, \quad \omega_2 \approx \omega_{bc}, \quad \omega_{ba} = (E_b - E_a)\hbar^{-1}, \quad \omega_{bc} = (E_b - E_c)\hbar^{-1},$$

we stipulate that the only nonzero matrix element of the operator $\tilde H^{(1)}$ be the following:

$$\tilde H^{(1)}_{ba} = -\mathscr{E}_1 d_{ba}e^{-i\Phi_1} = \tilde H^{(1)*}_{ab}, \qquad \tilde H^{(1)}_{bc} = -\mathscr{E}_2 d_{bc}e^{-i\Phi_2} = \tilde H^{(1)*}_{cd},$$

and that the operator $\tilde H^{(2)}$ be diagonal and contain no oscillating exponentials $\exp(\pm i\Phi_j)$. We obtain then from (6)

$$S^{(1)}_{ba} = -\frac{id_{ba}}{\hbar}\left(\frac{\mathscr{E}_2 e^{-i\Phi_2}}{\omega_{ba}-\omega_2} + \frac{\mathscr{E}_2^* e^{i\Phi_2}}{\omega_{ba}+\omega_2} + \frac{\mathscr{E}_1^* e^{i\Phi_1}}{\omega_{ba}+\omega_1}\right),$$
$$S^{(1)}_{bc} = -\frac{id_{bc}}{\hbar}\left(\frac{\mathscr{E}_1 e^{-i\Phi_1}}{\omega_{bc}-\omega_1} + \frac{\mathscr{E}_1^* e^{i\Phi_1}}{\omega_{bc}+\omega_1} + \frac{\mathscr{E}_2^* e^{i\Phi_2}}{\omega_{bc}+\omega_2}\right).$$

We do not present here expressions for $S^{(2)}$, which will not be needed. It is important that the $S^{(1)}$ and $S^{(2)}$ contain no resonant denominators and confirm that the assumptions made concerning $\tilde H$ are not contradictory.

The effective double-resonance operator defined in this manner

$$H^D = \tilde H^{(0)} + \tilde H^{(1)} + \tilde H^{(2)}$$

has the matrix elements

$$H_{ba}{}^D = -\mathscr{E}_1 d_{ba}e^{-i\Phi_1} = H_{ab}{}^{D*}, \qquad H_{bc}{}^D = -\mathscr{E}_2 d_{bc}e^{-i\Phi_2} = H_{cb}{}^{D*},$$

$$H_{\alpha\alpha}{}^D = E_\alpha + E_\alpha{}^{St},$$
$$E_a{}^{St} = |\mathscr{E}_2|^2\Pi_a(\omega_2) - |d_{ba}\mathscr{E}_1|^2/\hbar(\omega_{ba}+\omega_1),$$
$$E_c{}^{St} = |\mathscr{E}_1|^2\Pi_c(\omega_1) - |d_{bc}\mathscr{E}_2|^2/\hbar(\omega_{bc}+\omega_2),$$
$$E_b{}^{St} = -|\mathscr{E}_1|^2\Pi_c(\omega_1) - |\mathscr{E}_2|^2\Pi_a(\omega_2) + |d_{ba}\mathscr{E}_1|^2/\hbar(\omega_{ba}+\omega_1) + |d_{bc}\mathscr{E}_2|^2/\hbar(\omega_{bc}+\omega_2),$$

where

$$\Pi_\alpha(\omega) = \sum_{\alpha'}\frac{|d_{\alpha\alpha'}|^2}{\hbar}\left(\frac{1}{\omega_{\alpha\alpha'}+\omega} + \frac{1}{\omega_{\alpha\alpha'}-\omega}\right),$$

$$\omega_{\alpha\alpha'} = (E_\alpha - E_{\alpha'})\hbar^{-1},$$

$\alpha, \alpha' = a,b,c$; the quantities E_α^{St} are the Stark shifts of the levels, and their terms that do not contain the factor Π_α are the so-called Bloch–Siegert shifts.[9]

The main contribution to the polarization (4') of the medium is made by the first term, namely $P = \mathrm{Sp}\,\tilde\rho d$. Transforming in (2) and (3') to slowly changing variables[3)] and reducing them to dimensionless form, we obtain evolution equations that can be written here for convenience in the form

$$i\frac{\partial\hat E}{\partial\zeta} = -\frac{1}{2}[\hat R, \hat J], \qquad i\frac{\partial\hat R}{\partial\tau} = -\frac{\delta}{2}[\hat R, \hat J] + [\hat R, \hat E]. \tag{7}$$

This equation contains new independent variables $\zeta = z/ct_0$ and $\tau = (t - z/c)/t_0$ and also

$$\hat J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat R = \begin{pmatrix} r^b & r^{bc} & r^{ba} \\ r^{bc*} & r^c & r^{ca} \\ r^{ba*} & r^{ca*} & r^a \end{pmatrix},$$

$$\hat E = \begin{pmatrix} 0 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_2^* & 0 & 0 \\ \varepsilon_1^* & 0 & 0 \end{pmatrix},$$

$$r^\alpha = \tilde\rho_{\alpha\alpha}N^{-1}, \qquad r^{ba} = \tilde\rho_{ba}N^{-1}e^{i\Phi_1}, \qquad r^{bc} = \tilde\rho_{bc}N^{-1}e^{i\Phi_2},$$

$$r^{ca} = \tilde\rho_{ca}N^{-1}e^{i(\Phi_1-\Phi_2)},$$

$$\varepsilon_j = t_0 d_{ab}\mathscr{E}_j\hbar^{-1}, \qquad t_0 = (\hbar/2\pi\omega_1|d_{ba}|^2N)^{1/2},$$

$$\delta = (\omega_1 - \omega_{ba})t_0 = (\omega_2 - \omega_{bc})t_0.$$

With an aim at further analysis of the exactly integrable case, we neglect in (7) the Stark level shifts and assume that

$$\omega_1|d_{ba}|^2 = \omega_2|d_{bc}|^2. \tag{8}$$

In the case of Raman resonance

$$\omega_1 - \omega_2 \approx \omega_{ca}, \qquad \omega_{ca} = (E_c - E_a)\hbar^{-1},$$

we must put $H^{(1)}=0$ and $\widetilde{H}^{(2)}_{ba}=\widetilde{H}^{(2)}_{bc}=0$, retain in the expressions for $\widetilde{H}^{(2)}_{aa}$ only the terms that do not contain rapidly varying exponentials, and retain in $\widetilde{H}^{(2)}_{ca}$ only the term proportional to $\exp[-i(\Phi_1-\Phi_2)]$. The unitary transformation is then determined by the matrix

$$S^{(1)}_{aa'}=-\frac{id_{aa'}}{\hbar}\sum_{j=1,2}\left(\frac{\mathscr{E}_j e^{-i\Phi_j}}{\omega_{aa'}-\omega_j}+\frac{\mathscr{E}_j^* e^{i\Phi_j}}{\omega_{aa'}+\omega_j}\right),$$

and the matrix elements of the effective Raman-resonance Hamiltonian

$$H^R=\tilde{H}^{(0)}+\tilde{H}^{(2)},$$

take the form

$$H_{ca}{}^R=-\mathscr{E}_1\mathscr{E}_2{}^*\Pi_{ca}(\omega_1)e^{-i(\Phi_1-\Phi_2)}=H_{ac}{}^{R*},\quad H_{aa}{}^R=E_a+E_a{}^{St},$$
$$E_a{}^{St}=|\mathscr{E}_1|^2\Pi_a(\omega_1)+|\mathscr{E}_2|^2\Pi_a(\omega_2),$$

where

$$\Pi_{ca}(\omega)=\sum_\alpha\frac{d_{c\alpha}d_{\alpha a}}{\hbar}\left(\frac{1}{\omega_{\alpha c}+\omega}+\frac{1}{\omega_{\alpha a}-\omega}\right),$$

with $\Pi_{ca}(\omega_1)=\Pi_{ca}(-\omega_2)$, and $\Pi_\alpha(\omega)$ the same as in the case of double resonance.

The polarization of the medium in Raman resonance is determined by the second term of (4′):

$$P=\tilde{\rho}_{ca}D_{ac}+\text{c.c.}+\tilde{\rho}_{aa}D_{aa}+\tilde{\rho}_{cc}D_{cc},$$

where $D=i[d,S^{(1)}]$ denotes the effective dipole moment,

$$D_{ac}=\sum_{j=1,2}[\mathscr{E}_j\Pi_{ca}{}^*(-\omega_j)e^{-i\Phi_j}+\mathscr{E}_j{}^*\Pi_{ca}{}^*(\omega_j)e^{i\Phi_j}]=D_{ca}{}^*,$$

$$D_{aa}=-\sum_{j=1,2}\mathscr{E}_j\Pi_a(\omega_j)e^{-i\Phi_j}+\text{c.c.}$$

The sought evolution equations[10] are obtained from (2) and (3′) by going to slowly varying amplitudes and neglecting the reaction of the waves generated at the combined $2\omega_1-\omega_2$ and $|2\omega_2-\omega_1|$ frequencies. It is important to emphasize that in contrast to Ref. 10 the matrix elements H^R have turned out to be interrelated by virtue of the restriction of the initial particle model to three levels.

The Hamiltonian H^R can also be obtained by a unitary transformation and from H^D:

$$H^R=e^{-i\tilde{S}}H^De^{i\tilde{S}}-i\hbar e^{-i\tilde{S}}\frac{\partial}{\partial t}e^{i\tilde{S}}. \tag{9}$$

As a rule,[4,5] however, Stark levels are not taken into account when double resonance is considered. This leads to an effective Hamiltonian $\widetilde{H}^R$ and an effective dipole moment $\widetilde{D}$ with somewhat different parameters:

$$\tilde{H}_{ca}{}^R=-\mathscr{E}_1\mathscr{E}_2{}^*d_{cb}d_{ba}[2\hbar(\omega_{bc}-\omega_2)]^{-1}\exp[-i(\Phi_1-\Phi_2)]=\tilde{H}_{ac}{}^{R*},$$

$$\tilde{H}_{aa}{}^R=E_a-|\mathscr{E}_1d_{ba}|^2/\hbar(\omega_{bc}-\omega_2),$$

$$\tilde{H}_{cc}{}^R=E_c-|\mathscr{E}_2d_{bc}|^2/\hbar(\omega_{bc}-\omega_2),$$

$$\tilde{D}_{ac}=d_{ab}d_{bc}(\mathscr{E}_2e^{-i\Phi_2}+\mathscr{E}_1{}^*e^{i\Phi_1})/\hbar(\omega_{bc}-\omega_2),$$
$$\tilde{D}_{aa}=|d_{ab}|^2\mathscr{E}_1e^{-i\Phi_1}/\hbar(\omega_{bc}-\omega_2)+\text{c.c.},$$
$$\tilde{D}_{cc}=|d_{cb}|^2\mathscr{E}_2e^{-i\Phi_2}/\hbar(\omega_{bc}-\omega_2)+\text{c.c.}$$

It is recognized here that by virtue of the Raman-resonance condition we have $\omega_{ba}-\omega_1\approx\omega_{bc}-\omega_2$.

We do not need the evolution equations for Raman resonance, since we obtain their Lax representation from the Lax representation of the double-resonance equations(7). The ensuing restrictions are determined entirely by the relation between the parameters of the Hamiltonian $\widetilde{H}^R$ and by condition (8).

We proceed now to analyze the Lax representation of the double-resonance equations. According to Refs. 4 and 5, Eqs. 7 are the condition for the compatibility of the solutions of a system of third-order linear equations

$$\frac{\partial}{\partial\tau}q=\hat{L}q,\quad \frac{\partial}{\partial\zeta}q=\hat{A}q. \tag{10}$$

Here, in contrast to Refs. 4 and 5, we write the Lax operators $\hat{L}$ and $\hat{A}$ in the form

$$\hat{L}=i(\lambda-\delta/2)\hat{J}+i\hat{E},\quad \hat{A}=i(2\lambda)^{-1}\hat{R}. \tag{11}$$

Recall that the evolution equations (7) are obtained from the zero-curvature representation

$$\frac{\partial\hat{L}}{\partial\zeta}=\frac{\partial\hat{A}}{\partial\tau}+[\hat{A},\hat{L}]$$

[the conditions for the compatibility of the solutions of (10)] after substituting in them (11) and equating the expressions for equal powers of the spectral parameter λ.

We transform the auxiliary equations (10) in a manner similar to the one used above for Eq. (3):

$$\tilde{q}=e^{-iQ}q,\quad \frac{\partial}{\partial\tau}\tilde{q}=\hat{\tilde{L}}\tilde{q},\quad \frac{\partial}{\partial\zeta}\tilde{q}=\hat{\tilde{A}}\tilde{q},$$
$$\hat{\tilde{L}}=e^{-iQ}\hat{L}e^{iQ}-e^{-iQ}\frac{\partial}{\partial\tau}e^{iQ}, \tag{12}$$
$$\hat{\tilde{A}}=e^{-iQ}\hat{A}e^{iQ}-e^{-iQ}\frac{\partial}{\partial\zeta}e^{iQ}.$$

We represent the new Lax operators and the matrix Q by series in powers of the field $\hat{E}$ [just as in Eq. (5)]:

$$\hat{\tilde{L}}=\hat{\tilde{L}}^{(0)}+\hat{\tilde{L}}^{(1)}+\hat{\tilde{L}}^{(2)}+\ldots,\quad \hat{\tilde{A}}=\hat{\tilde{A}}^{(0)}+\hat{\tilde{A}}^{(1)}+\ldots,$$
$$Q=Q^{(1)}+Q^{(2)}+\ldots$$
$$\hat{\tilde{L}}^{(0)}=i(\lambda-\delta/2)\hat{J},$$
$$\hat{\tilde{L}}^{(1)}=-i[Q^{(1)},\hat{L}^{(0)}]+i\hat{E}-i\frac{\partial}{\partial\tau}Q^{(1)},$$
$$\hat{\tilde{L}}^{(2)}=\frac{1}{2}[Q^{(1)},\hat{E}]-\frac{i}{2}[Q^{(1)},\hat{\tilde{L}}^{(1)}]-i[Q^{(2)},\hat{\tilde{L}}^{(0)}]-i\frac{\partial}{\partial\tau}Q^{(2)}. \tag{13}$$

Assume that the common level E_b of the adjacent optically allowed transitions $E_b\to E_a$ and $E_b\to E_c$ is not at resonance with the two-frequency USP, or more accurately, that the detuning δ from the resonance is much larger than the spectral width of the USP. This allows us to require furthermore that the effective operator

$$\hat{L}^e=\hat{\tilde{L}}^{(0)}+\hat{\tilde{L}}^{(1)}+\hat{\tilde{L}}^{(2)}$$

have the block form

$$\hat{L}^e=\begin{pmatrix} \cdot & 0 & 0 \\ 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{pmatrix}. \quad (14)$$

To this end we must have

$$Q_{11}^{(1)}=Q_{22}^{(1)}=Q_{33}^{(1)}=Q_{23}^{(1)}=Q_{32}^{(1)}=0,$$

$$Q_{12}^{(1)}=\int_{-\infty}^{\tau}\varepsilon_2 e^{i(2\lambda-\delta)(\tau'-\tau)}d\tau', \quad Q_{21}^{(1)}=\int_{-\infty}^{\tau}\varepsilon_2^* e^{-i(2\lambda-\delta)(\tau'-\tau)}d\tau',$$

$$Q_{13}^{(1)}=\int_{-\infty}^{\tau}\varepsilon_1 e^{i(2\lambda-\delta)(\tau'-\tau)}d\tau', \quad Q_{31}^{(1)}=\int_{-\infty}^{\tau}\varepsilon_1^* e^{-i(2\lambda-\delta)(\tau'-\tau)}d\tau',$$

and the principal role is played here, by virtue of the assumption made, by terms resulting from integration by parts:

$$\tilde{Q}_{12}^{(1)}=\frac{\varepsilon_2}{i(2\lambda-\delta)}, \quad \tilde{Q}_{21}^{(1)}=-\frac{\varepsilon_2^*}{i(2\lambda-\delta)}, \quad \tilde{Q}_{13}^{(1)}=\frac{\varepsilon_1}{i(2\lambda-\delta)},$$

$$\tilde{Q}_{31}^{(1)}=-\frac{\varepsilon_1^*}{i(2\lambda-\delta)}.$$

Defining $Q^{(2)}$ as the solution of the equation

$$\frac{1}{2}[Q^{(1)}-\tilde{Q}^{(1)},\hat{E}]-i[Q^{(2)},\hat{L}^{(0)}]-i\frac{\partial}{\partial\tau}Q^{(2)}=0,$$

we obtain

$$\hat{L}^e=i(\lambda-\delta/2)\hat{J}+{}^1/_2[\tilde{Q}^{(1)},\hat{E}],$$

$$\hat{L}^e=i\begin{pmatrix} -(\lambda-\delta/2)-(|\varepsilon_1|^2+|\varepsilon_2|^2)/(2\lambda-\delta) & 0 & 0 \\ 0 & \lambda-\delta/2+|\varepsilon_2|^2/(2\lambda-\delta) & \varepsilon_1\varepsilon_2^*/(2\lambda-\delta) \\ 0 & \varepsilon_1^*\varepsilon_2/(2\lambda-\delta) & \lambda-\delta/2+|\varepsilon_1|^2/(2\lambda-\delta) \end{pmatrix}. \quad (15)$$

As to the operator $\tilde{A}$ we find that, accurate to first order in the field inclusive and with allowance for the evolution equations (7), the effective terms

$$\hat{A}^e=\hat{A}-i[\tilde{Q}^{(1)},\hat{A}]-i\frac{\partial}{\partial\tau}\tilde{Q}^{(1)} \quad (16)$$

also take the block form (14).

We now reduce $\hat{L}^e$ and $\hat{A}^e$ to 2×2 matrices consisting of $\hat{L}^e$ and $\hat{A}^e$ elements with indices 2 and 3. We leave out of $\hat{L}^e$ the inessential identical diagonal elements $i(\lambda-\delta/2)$, and confine ourselves in $\hat{A}^e$ to terms of zeroth and first (which equals zero) orders in the field. This results in the following Lax operators:

$$\hat{L}^R=\frac{i}{2\lambda-\delta}\begin{pmatrix} |\varepsilon_2|^2 & \varepsilon_1\varepsilon_2^* \\ \varepsilon_1^*\varepsilon_2 & |\varepsilon_1|^2 \end{pmatrix}, \quad \hat{A}^R=\frac{i}{2\lambda}\begin{pmatrix} r^c & r^{ca} \\ r^{ca*} & r^a \end{pmatrix}, \quad (17)$$

which realize the Lax representations of the Raman-resonance equations with Hamiltonian $\tilde{H}^R$, with account taken of the condition (8). This can be easily verified directly. In addition, the Lax representation (17) coincides with Steudel's result[7] in which we put $f=0$ and redesignate the spectral parameter. It must also be noted that after determining (17) it is easy to dispense with the restrictions of the model and obtain the results of Ref. 7 with $f\neq0$.

The above derivation of the Lax representation of the Raman-resonance equations seems quite natural if it is noted that the role of the operator $\hat{L}$ in the double-resonance problem is played by H^D (after discarding the Stark shifts of the levels and separating the slowly changing variables), while the first equation of (10) is an abbreviated Schrödinger equation.[4)] The requirement (14) has therefore turned out to be perfectly analogous to the assumptions that reduce H^D to $\tilde{H}^R$ under the transformation (9). What remains surprising is only the corresponding transformation of the operator $\hat{A}$. It must also be emphasized that direct substitution of the abbreviated combination-resonance Hamiltonian $\tilde{H}^R$ for the operator $\hat{L}$ does not make it possible, according to Maĭmistov's calculations,[5)] to find a Lax representation in analogy with Refs. 11 and 12.

2. LAX REPRESENTATION OF POLARIZATION MODELS OF RAMAN RESONANCE

Following the proposed method, we discuss now the theory of propagation of arbitrarily polarized USP [Eq. (1)] under Raman-resonance conditions on the basis of the exactly integrable double-resonance polarization models. It is shown in Ref. 6 that in double resonance of two-frequency USP with energy levels E_a, E_b, and E_c characterized by angular momenta $j_a=j_c=j_b-1=0$ and $j_a=j_c=j_b+1=1$, the evolution equations can be integrated by the ISP method and are the conditions for the compatibility of the solutions of systems of auxiliary linear equations (10) of fourth and fifth order, respectively, with Lax operators of form (11) but with matrices $\hat{J}$, $\hat{E}$, and $\hat{R}$ of their own.

In the case $j_a=j_c=j_b-1=0$ we have

$$\hat{J}=\mathrm{diag}(-1,-1,1,1),$$

$$\hat{E}=-\begin{pmatrix} 0 & 0 & \varepsilon_1^{-1} & \varepsilon_2^{-1} \\ 0 & 0 & \varepsilon_1^{1} & \varepsilon_2^{1} \\ \varepsilon_1^{-1*} & \varepsilon_1^{1*} & 0 & 0 \\ \varepsilon_2^{-1*} & \varepsilon_2^{1*} & 0 & 0 \end{pmatrix},$$

$$\hat{R}=\begin{pmatrix} \mu_{-1-1} & \mu_{1-1} & p_1^{-1} & p_2^{-1} \\ \mu_{-11} & \mu_{11} & p_1^{1} & p_2^{1} \\ p_1^{-1*} & p_1^{1*} & m & r^* \\ p_2^{-1*} & p_2^{1*} & r & \nu \end{pmatrix},$$

$$\mu_{qq'}=\tilde{\rho}^{bb}_{-q'-q}N^{-1}, \quad m=\tilde{\rho}^{aa}_{00}N^{-1}, \quad \nu=\tilde{\rho}^{cc}_{00}N^{-1},$$

$$r=\tilde{\rho}^{ca}_{00}N^{-1}e^{i(\Phi_1-\Phi_2)}, \quad p_1^q=\tilde{\rho}^{ba}_{-q0}N^{-1}e^{i\Phi_1}, \quad p_2^q=\tilde{\rho}^{bc}_{-q0}N^{-1}e^{i\Phi_2}. \quad (18)$$

In the case $j_a = j_c = j_b - 1 = 0$ we have

$$\hat{E} = -\begin{pmatrix} 0 & \varepsilon_1^{-1} & \varepsilon_1^{1} & \varepsilon_2^{-1} & \varepsilon_2^{1} \\ \varepsilon_1^{-1*} & 0 & 0 & 0 & 0 \\ \varepsilon_1^{1*} & 0 & 0 & 0 & 0 \\ \varepsilon_2^{-1*} & 0 & 0 & 0 & 0 \\ \varepsilon_2^{1*} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{R} = \begin{pmatrix} \mu & p_1^{-1} & p_1^{1} & p_2^{-1} & p_2^{1} \\ p_1^{-1*} & m_{-1-1} & m_{-11} & r_{-1-1} & r_{-11} \\ p_1^{1*} & m_{1-1} & m_{11} & r_{1-1} & r_{11} \\ p_2^{-1*} & \overset{*}{r}_{-1-1} & \overset{*}{r}_{1-1} & \nu_{-1-1} & \nu_{-11} \\ p_2^{1*} & \overset{*}{r}_{-11} & \overset{*}{r}_{11} & \nu_{1-1} & \nu_{11} \end{pmatrix},$$

$$\mu = \tilde{\rho}_{00}^{bb} N^{-1}, \qquad m_{qq'} = \tilde{\rho}_{qq'}^{aa} N^{-1}, \qquad \nu_{qq'} = \tilde{\rho}_{qq'}^{cc} N^{-1},$$

$$r_{qq'} = \tilde{\rho}_{q'q}^{ca} N^{-1} e^{i(\Phi_2 - \Phi_1)}, \qquad p_1{}^{q} = \tilde{\rho}_{0q}^{ba} N^{-1} e^{i\Phi_1}, \qquad p_2{}^{q} = \tilde{\rho}_{0q}^{bc} N^{-1} e^{i\Phi_2}. \tag{19}$$

Here $\varepsilon_j^{\pm 1}$ are dimensionless spherical components of the vector $\vec{\mathscr{E}}_j$, the superscripts in the density matrix $\tilde{\rho}$ label the matrix elements for transitions between energy levels whose lower indices indicate Zeeman sublevels with different q, $q' = 0, \pm 1$ components along the quantization axis of the corresponding total angular momentum. For a unified description of various cases $j_a = j_c = j_b - 1 = 0$and $j_a = j_c = j_b + 1 = 1$, the notation differs somewhat from that in Ref. 6.

We transform Eqs. (10), (11) and (18), and (19) in accordance with (12) and (13) assuming, just as in the preceding section, that the detuning from the resonance is large enough. We stipulate here that the block form of the effective operator $\hat{L}^e$ be the following:

$$\hat{L}^e = \begin{pmatrix} \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{pmatrix}, \quad j_a = j_c = j_b - 1 = 0$$

$$\hat{L}^e = \begin{pmatrix} \cdot & 0 & 0 & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad j_a = j_c = j_b + 1 = 1.$$

Such a transformation is effected by the matrices at $j_a = j_c = j_b - 1 = 0$

$$\tilde{Q}_{13}^{(1)} = -\frac{\varepsilon_1^{-1}}{i(2\lambda-\delta)}, \quad \tilde{Q}_{31}^{(1)} = \frac{\varepsilon_1^{-1*}}{i(2\lambda-\delta)},$$

$$\tilde{Q}_{23}^{(1)} = -\frac{\varepsilon_1^{1}}{i(2\lambda-\delta)}, \quad \tilde{Q}_{32}^{(1)} = \frac{\varepsilon_1^{1*}}{i(2\lambda-\delta)},$$

$$\tilde{Q}_{14}^{(1)} = -\frac{\varepsilon_2^{-1}}{i(2\lambda-\delta)}, \quad \tilde{Q}_{41}^{(1)} = \frac{\varepsilon_2^{-1*}}{i(2\lambda-\delta)},$$

$$\tilde{Q}_{24}^{(1)} = -\frac{\varepsilon_2^{1}}{i(2\lambda-\delta)}, \quad \tilde{Q}_{42}^{(1)} = \frac{\varepsilon_2^{1*}}{i(2\lambda-\delta)};$$

and for $j_a = j_c = j_b + 1 = 1$ we have

$$\tilde{Q}_{12}^{(1)} = -\frac{\varepsilon_1^{-1}}{i(2\lambda-\delta)}, \quad \tilde{Q}_{21}^{(1)} = \frac{\varepsilon_1^{-1*}}{i(2\lambda-\delta)},$$

$$\tilde{Q}_{13}^{(1)} = -\frac{\varepsilon_1^{1}}{i(2\lambda-\delta)}, \quad \tilde{Q}_{31}^{(1)} = \frac{\varepsilon_1^{1*}}{i(2\lambda-\delta)},$$

$$\tilde{Q}_{14}^{(1)} = -\frac{\varepsilon_2^{-1}}{i(2\lambda-\delta)}, \quad \tilde{Q}_{41}^{(1)} = \frac{\varepsilon_2^{-1*}}{i(2\lambda-\delta)},$$

$$\tilde{Q}_{15}^{(1)} = -\frac{\varepsilon_2^{-1}}{i(2\lambda-\delta)}, \quad \tilde{Q}_{51}^{(1)} = \frac{\varepsilon_2^{1*}}{i(2\lambda-\delta)},$$

the remaining matrix elements are zero.

As a result of the calculations (15) and (16) of the effective operators $\hat{L}^e$ and $\hat{A}^e$ and their reduction we obtain the following Lax operators that realize a zero-curvature representation of the polarization models of Raman resonance: for $j_a = j_c = 0$

$$\hat{L}^R = \frac{i}{2\lambda-\delta}\begin{pmatrix} |\varepsilon_1|^2 & \varepsilon_1^* \varepsilon_2 \\ \varepsilon_1 \varepsilon_2^* & |\varepsilon_2|^2 \end{pmatrix}, \quad \hat{A}^R = \frac{i}{2\lambda}\begin{pmatrix} m & r^* \\ r & \nu \end{pmatrix}, \tag{20}$$

for $j_a = j_c = 1$

$$\hat{L}^R = \frac{i}{2\lambda-\delta}\begin{pmatrix} |\varepsilon_1^{-1}|^2 & \varepsilon_1^{-1*}\varepsilon_1^{1} & \varepsilon_1^{-1*}\varepsilon_2^{-1} & \varepsilon_1^{-1*}\varepsilon_2^{1} \\ \varepsilon_1^{1*}\varepsilon_1^{-1} & |\varepsilon_1^{1}|^2 & \varepsilon_1^{1*}\varepsilon_2^{-1} & \varepsilon_1^{1*}\varepsilon_2^{1} \\ \varepsilon_2^{-1*}\varepsilon_1^{-1} & \varepsilon_2^{-1*}\varepsilon_1^{1} & |\varepsilon_2^{-1}|^2 & \varepsilon_2^{-1*}\varepsilon_2^{1} \\ \varepsilon_2^{1*}\varepsilon_1^{-1} & \varepsilon_2^{1*}\varepsilon_1^{1} & \varepsilon_2^{1*}\varepsilon_2^{-1} & |\varepsilon_2^{1}|^2 \end{pmatrix},$$

$$\hat{A}^R = \frac{i}{2\lambda}\begin{pmatrix} m_{-1-1} & m_{-11} & r_{-1-1} & r_{-11} \\ m_{1-1} & m_{11} & r_{1-1} & r_{11} \\ \overset{*}{r}_{-1-1} & \overset{*}{r}_{1-1} & \nu_{-1-1} & \nu_{-11} \\ \overset{*}{r}_{-11} & \overset{*}{r}_{11} & \nu_{1-1} & \nu_{11} \end{pmatrix}. \tag{21}$$

The Lax representations (20) and (21) serve as the basis for the investigation, by the ISP method, of the polarization singularities of the propagation of USP in Raman resonance (the corresponding evolution equations are given in general form in Ref. 13). Following Refs. 14 and 15, it is easy to find the Darbou transformation and the N-soliton equations, but these results are quite unwieldy and call for a separate investigation. We indicate here a simple generalization of the expressions for USP of Lorentzian form $F_j(\tau,\zeta)$ (Refs. 16, 17) to allow for the field polarization and for the level degeneracy within the context the models considered above. It is easily found that $\boldsymbol{\varepsilon}_j = \mathbf{l}_j F_j(\tau,\zeta)$, where the unit polarization vectors $\mathbf{l}_j$ are equal, $\mathbf{l}_1 = \mathbf{l}_2 = \mathbf{l}$ for $j_a = j_c = 0$, and can be arbitrary in media with $j_a = j_c = 1$. Thus, despite the difference in profile, the polarization states of Raman-resonant USP of stationary form depend on the degree of degeneracy of the energy levels just as in the case of double resonance.

3. CONCLUSION

The examples considered do not cover all the problems in which the method proposed can be used to obtain new exactly solvable models for resonance optics. Obvious cases that can be similarly analyzed are double resonance in a three-level system with $j_a = j_b = j_c = 1/2$, and also the schemes considered in Ref. 18. It appears that new results should be expected by adiabatically eliminating from the aforementioned problem not the level E_b but another level (e.g., E_c), and/or by taking into account the second term in the expansion $Q^{(1)} = \widetilde{Q}^{(1)} + \dots$. It would be of great interest to track the analogous connection between exactly integrable quantum models, for example of Refs. 19 and 20.

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[1)] The UPS duration is much shorter than the relaxation time of the medium.

[2)] A. I. Maĭmistov, Paper at Working Conf. on Optical Solitons, Tashkent, May, 1989.

[3)] The necessary further simplifications are made in this case only in Eq. (2); in (3′) all the simplifications are already taken into account in H^D.

[4)] This circumstance was pointed out in Refs. 11 and 12 for the traditional theory of self-induced transparency in one-photon resonance.

[5)] A. I. Maĭmistov, private communication.

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