Nonlinear surface acoustic waves in dilute parametric dielectrics

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(Submitted 3 March 1989)
Zh. Eksp. Teor. Fiz. **97**, 235–245 (January 1990)

A theory of nonlinear surface acoustic waves in solid dielectrics containing paramagnetic impurities is constructed. Waves are considered that are formed under conditions where the effects of anharmonic lattice vibrations and dispersive and coherent interactions between the wave momentum and impurities are important. It is shown that, under the conditions in which the momentum envelope has an area $\Theta \ll 1$, nonlinear waves are formed in the shape of double solitons. The formation of a nonlinear acoustic wave as a result of anharmonism, dispersion, and attenuation is studied. The conditions for the experimental observation of the predicted nonlinear phenomena are discussed.

1. INTRODUCTION

Surface acoustic waves (SAW) can propagate along the interfaces of different media. At sufficiently low amplitudes of these waves, their properties can be described adequately within the framework of linear SAW theory, which has been developed in great detail.[1] With increase in amplitude, nonlinear effects become important. These effects are especially pronounced for high-frequency SAW with $\omega \sim 10^9$–10^{10} Hz. Among nonlinear phenomena, the greatest interest attaches to those which lead to the formation of solitons. Theoretical investigations of the properties of SAW solitons have been carried out, for example, in Refs. 2 and 3, while experimental observations are reported in Ref. 4.

The investigations of the properties of nonlinear SAW in dielectrics follow two trends. In the first the solitons are formed under conditions of nonlinear coherent interaction of SAW with paramagnetic impurities contained in the medium, i.e., under the conditions of acoustic self-induced transparency (SIT). In the second the nonlinear waves are formed as a result of effects connected with anharmonic lattice vibration and dispersion. In the case of acoustic SIT, at sufficiently large values of the coefficient of resonant acoustic absorption $\alpha_0 \sim G^2 n_0 k$, the following condition is necessary for the observation of a soliton (or double soliton): the area of the SAW pulse envelope must be $\Theta > \pi$ or $\Theta \ll 1$, where $\Theta \sim GTku$, G being the constant of spin-phonon coupling, n_0 the concentration of the parametric impurity, k the wave number, T the pulse length, and u the value of the strain vector.[2]

From the inequalities given above, it is evident that for high-frequency SAW, in the case of typical values such as $k \sim 10^4$–10^5 cm^{-1}, $T \sim 10^{-6}$–10^{-7} s, and large values of $G \sim 10^2$–10^3 cm^{-1}/unit strain (such values are realized, for example, for the impurities V^{3+}, Fe^{2+} and Ni^{2+} in the MgO crystal[5,6]), the conditions of formation of nonliner SAW are easily satisfied at rather low values of the strain vector. At small values of u, the anharmonic effects lattice vibrations are insignificant and can be neglected. In solids, however, the concentration of paramagnetic impurities and the constants of spin-phonon coupling change over quite wide limits.[6] Moreover, in certain situations (see, for example, Ref. 7), if the direction of propagation of the wave makes an angle $\vartheta \neq 0$ with the external constant magnetic field, then the coefficient of resonant acoustic absorption is

$$\alpha_0 \sim G^2 n_0 k \cos\vartheta,$$

and the area of the momentum envelope is

$$\Theta \sim GTku \cos\vartheta.$$

Therefore, in the general case, the quantity $GTk \cos\vartheta$ can turn out to be rather small, and for satisfaction of the inequalities given above it is necessary to apply pulses for which u is no longer small. In this case, the anharmonic effects which, together with the dispersion, produce nonlinear SAW, become important. Consequently, we arrive at a physically interesting situation—two completely different mechanisms of formation of nonlinear SAW can be effective simultaneously. In this case, the picture of the nonlinear wave process becomes very complicated and there is not answer yet to the question as to what sort of laws govern the formation of nonlinear SAW. It is precisely this question that we shall consider in this paper.

2. DERIVATION OF EQUATIONS

It is known that SAW propagating along a free surface of a solid or along the boundary of a solid with other media have different forms and are distinguished from one another by boundary conditions characterizing the wave process on the boundaries in the media.[1] We shall consider the following case in detail: the propagation of a Rayleigh wave along a free plane boundary of a nonmetallic diamagnetic solid containing a small concentration of paramagnetic impurities with electron spins J and nuclear spins I. For simplicity, we shall assume that $J = I = 1/2$. We shall also assume that the solid medium occupies the half-space $x_1 < 0$. We consider the case in which a Rayleigh-wave pulse with duration $T \ll T_{1,2}$, frequency $\omega_{\mathbf{k}}$, and wave vector $\mathbf{k}$ propagates with the positive x_3 direction along the surface of the half-space ($T_{1,2}$ are the longitudinal and transverse relaxation times). An external, constant magnetic field H_0 is applied to this same direction. Upon satisfaction of the condition $\omega_{\mathbf{k}} = \omega_J + \omega_I$, the Rayleigh wave can induce resonance transitions in the electron-nuclear spin system (ω_J and ω_I are the Zeeman frequencies of the electron and nuclear spins).[2] Using an expansion in the coherent states of the acoustic field, the components of the strain vector of the Rayleigh wave can be represented in the following form[2,3,8]:

$$u_j(\mathbf{r},t) = \sum_{\mathbf{k}} \mu_{\mathbf{k}} [a_{\mathbf{k}}(t) + a_{-\mathbf{k}}^{+}(t)] \beta_{\mathbf{k}}^{(j)}(\mathbf{r}),$$

where the function

 0038-5646/90/010131-06$03.00

$$\beta_{\mathbf{k}}^{(j)}(\mathbf{r})=e^{ikx_3}\begin{cases}\dfrac{\varkappa_l(k)}{ik}\left\{\exp[\varkappa_l(k)x_1]-\dfrac{2k^2}{\varkappa_t^2(k)+k^2}\exp[\varkappa_t(k)x_1]\right\}, & j=1,2,\\ \exp[\varkappa_l(k)x_1]-\dfrac{2\varkappa_l(k)\varkappa_t(k)}{\varkappa_t^2(k)+k^2}\exp[\varkappa_t(k)x_1], & j=3\end{cases}$$

takes into account the transverse structure of the field and is determined from physical considerations, namely the condition of the stresses on the free surface of the medium and the Bose creation and annihilation operators $a_+^{\mathbf{k}}$ and $a_{\mathbf{k}}$ of the Rayleigh modes, while the quantities

$$\varkappa_{l,t}(k)=[k^2-\omega_{\mathbf{k}}^2/c_{l,t}^2]^{1/2}$$

determining the rate of damping of the wave along the x_1 axis; c_l and c_t are the velocities of the longitudinal and transverse acoustic waves, $\mu_2^{\mathbf{k}}=\hbar/2\rho\omega_{\mathbf{k}}N_0V$, ρ is the density of the medium, N_0 is the number of sites in the lattice, V is the volume of the medium. In what follows, we shall assume that $\hbar=V=n_0=1$.

The dispersion relation is given by the equation

$$\omega_{\mathbf{k}}^2=c^2k^2(1-\Delta),\quad \Delta=(hk)^2, \tag{1}$$

where c is the velocity of linear SAW and h is a quantity of the order of the lattice constant.

The Hamiltonian of the investigated model has the form

$$H=H_Z+H_A+H_f+H_a+H_{sf} \tag{2}$$

where

$$H_Z=\omega_JJ^z-\omega_II^z$$

corresponds to the Zeeman interaction with the magnetic field,

$$H_A=A\sum_i J_i^zI_i^z$$

is the Hamiltonian of the hyperfine interaction,

$$H_f=\sum_{\mathbf{k}}\omega_{\mathbf{k}}(a_{\mathbf{k}}^+a_{\mathbf{k}}+{}^1/_2)$$

is the Hamiltonian of the system of phonons,

$$H_a=\sum_{\mathbf{k}\mathbf{k}'\mathbf{k}''}\Phi_{\mathbf{k},\mathbf{k}',\mathbf{k}''}(a_{\mathbf{k}}+a_{-\mathbf{k}}^+)(a_{\mathbf{k}'}+a_{-\mathbf{k}'}^+)$$
$$\times(a_{\mathbf{k}''}+a_{-\mathbf{k}''}^+)\delta_{\mathbf{k}+\mathbf{k}'+\mathbf{k}'',0}$$

is the Hamiltonian of phonon-phonon interaction, the vertex function of three-phonon interaction is

$$\Phi_{\mathbf{k},\mathbf{k}',\mathbf{k}''}=(2\rho)^{-3/2}(\omega_{\mathbf{k}}\omega_{\mathbf{k}'}\omega_{\mathbf{k}''})^{-1/2}$$
$$\times\sum_{\substack{ijk\\lmn}}C_{ijklmn}\int\frac{\partial\beta_{\mathbf{k}}^{(i)}}{\partial x_j}\frac{\partial\beta_{\mathbf{k}'}^{(k)}}{\partial x_l}\frac{\partial\beta_{\mathbf{k}''}^{(m)}}{\partial x_n}d\mathbf{r},$$

where the coefficients c_{ijklmn} are known as the third-order elastic constant,[3] A is the constant of hyperfine interaction,

$$H_{sf}={}^1/_2L(\varepsilon^+S^-+\varepsilon^-S^+)$$

is the Hamiltonian of spin-phonon interaction in the rotating-wave approximation, where

$$L=A\mu_BH_0F_{zzzz}/4\omega_{\mathbf{k}},$$

$$\varepsilon^\pm=\mp2i\sum_{\mathbf{k}}k\mu_{\mathbf{k}}a_{\mathbf{k}}^\pm e^{\mp ikx_3}\gamma_{\mathbf{k}}(x_1),$$

$$\gamma_{\mathbf{k}}(x_1)=\beta_{\mathbf{k}}^{(3)}e^{-ikx_3},$$

F_{zzzz} is the component of the tensor of the SAW spin-phonon interaction, μ_B is the Bohr magneton, $\varepsilon_{zz}=\frac{1}{2}(\varepsilon^++\varepsilon^-)$ is the component of the SAW strain tensor, and $S^+=J^\pm+I^\mp$.

The equation of motion for the operator quantities $a_{-\mathbf{k}}^+$ and $a_{\mathbf{k}}$ are easily obtained by using the Hamiltonian (2):

$$i\dot a_{-\mathbf{k}}^+=-\omega_{\mathbf{k}}a_{-\mathbf{k}}^+-3\sum_{\mathbf{k}'}\Phi_{-\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'}A_{\mathbf{k}'}A_{\mathbf{k}-\mathbf{k}'}+i\frac{L\mu_{\mathbf{k}}k}{2}e^{-ikx_3}S^+,$$
$$i\dot a_{\mathbf{k}}=\omega_{\mathbf{k}}a_{\mathbf{k}}+3\sum_{\mathbf{k}'}\Phi_{-\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'}A_{\mathbf{k}'}A_{\mathbf{k}-\mathbf{k}'}-i\frac{L\mu_{\mathbf{k}}k}{2}e^{-ikx_3}S^-, \tag{3}$$

where the dot denotes differentiation with respect to time and $A_{\mathbf{k}}=a_{\mathbf{k}}+a_{\mathbf{k}}^+$.

Similarly, for the spin operators S_i^+ and $S_i^z=\frac{1}{2}(J_i^z-I_i^z)$ we have the equation

$$\dot S_i^+=i\omega_0S_i^+-iL\varepsilon^+S_i^z,\quad \dot S_i^z={}^1/_2iL(\varepsilon^+S_i^--\varepsilon S_i^+), \tag{4}$$

where $\omega_0=\omega_J+\omega_I$. It is assumed here that $T_{1,2}\to\infty$.

It should be noted that the obtained Eqs. (3) and (4) remain valid, apart from notation, even in the case in which the SAW excites only electron spins of the paramagnetic impurities with $J=1$. Generally speaking, Eqs. (3) and (4) are quantum-mechanical and, as a consequence, the dynamical laws following from them are very complicated. However, the investigation of the corresponding classical equations is sufficient for our purposes. The transition to them is carried out under the assumption that the quantum correlations between spins and phonons are negligible. In this approximation, the mean of the products of operators is equal to the product of the means, i.e., $\langle S^{\pm,z},\varepsilon^\mp\rangle\approx\langle S^{\pm,z}\rangle\langle\varepsilon^\mp\rangle$, (the semiclassical approximation). Moreover, we take it into account the soliton (double soliton) SAW is a nonlinear wave, corresponding to the state in which a macroscopically large number of surface phonons is excited and, consequently, the quantum fluctuations can be neglected. Using the well-known properties of coherent states[3,8,9]:

$$a_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle=\alpha_{\mathbf{k}}|\alpha_{\mathbf{k}}\rangle,\quad \frac{1}{\pi}\int|\alpha_{\mathbf{k}}\rangle\langle\alpha_{\mathbf{k}}|d^2\alpha_{\mathbf{k}}=1,$$

$$|\{\alpha_q\}\rangle=\prod_q|\alpha_q\rangle,$$

$$|\alpha_{\mathbf{k}}\rangle=\exp\left(-\frac{|\alpha_{\mathbf{k}}|^2}{2}\right)\sum_{n_{\mathbf{k}}=0}^{\infty}\frac{(\alpha_{\mathbf{k}})^{n_{\mathbf{k}}}}{(n_{\mathbf{k}}!)^{1/2}}|n_{\mathbf{k}}\rangle,$$

$$\langle\alpha_{\mathbf{k}}|\beta_{\mathbf{k}}\rangle=\exp\left(\alpha_{\mathbf{k}}^{*}\beta_{\mathbf{k}}-\frac{|\alpha_{\mathbf{k}}|^{2}}{2}-\frac{|\beta_{\mathbf{k}}|^{2}}{2}\right),\quad N_{k}=a_{\mathbf{k}}^{+}a_{\mathbf{k}},$$

$$N_{\mathbf{k}}|n_{\mathbf{k}}\rangle=n_{\mathbf{k}}|n_{\mathbf{k}}\rangle,$$

$$\frac{1}{\pi}\int d^{2}\alpha_{\mathbf{k}}f(\alpha_{\mathbf{k}}^{*})\exp(\alpha_{\mathbf{k}}\beta_{\mathbf{k}}^{*}-|\alpha_{\mathbf{k}}|^{2})=f(\beta_{\mathbf{k}}^{*}),$$

$$d^{2}\alpha_{\mathbf{k}}=d(\operatorname{Re}\alpha_{\mathbf{k}})\,d(\operatorname{Im}\alpha_{\mathbf{k}}),$$

and considering the remarks that have been made, we obtain from the set of equations (3) and (4) the following set of classical equations of motion for the acoustic field:

$$\begin{aligned}i\dot{\alpha}_{-\mathbf{k}}^{*}&=-\omega_{\mathbf{k}}\alpha_{-\mathbf{k}}^{*}-3\sum_{\mathbf{k}'}\Phi_{-\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'}\tilde{A}_{\mathbf{k}'}\tilde{A}_{\mathbf{k}-\mathbf{k}'}\\&\quad+i\frac{L\mu_{\mathbf{k}}k}{2}e^{-ikx_{3}}\langle S^{+}\rangle,\\ i\dot{\alpha}_{\mathbf{k}}&=\omega_{\mathbf{k}}\alpha_{\mathbf{k}}+3\sum_{\mathbf{k}'}\Phi_{-\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'}\tilde{A}_{\mathbf{k}'}\tilde{A}_{\mathbf{k}-\mathbf{k}'}-i\frac{L\mu_{\mathbf{k}}k}{2}e^{-ikx_{3}}\langle S^{-}\rangle,\end{aligned}\tag{5}$$

and for the variables $N=\langle S_{l}^{z}\rangle$ and $B^{\pm}=\langle S_{l}^{\pm}\rangle$:

$$\begin{aligned}\dot{B}^{+}&=i\omega_{0}B^{+}-iLu^{+}N,\\ \dot{N}&={}^{1}/_{2}iL(u^{+}B^{-}-u^{-}B^{+}),\end{aligned}\tag{6}$$

where $u^{\pm}=\langle\varepsilon^{\pm}\rangle$, $\tilde{A}_{\mathbf{k}}=a_{\mathbf{k}}+\alpha^{*}_{-\mathbf{k}}$, $|\alpha_{\mathbf{k}}\rangle$ is the vector of the coherent state of the kth mode of the surface phonons, the symbol $\{\alpha_{\mathbf{k}}\}$ denotes the set of all amplitudes $\alpha_{\mathbf{k}}$. In what follows, we shall for simplicity use the notation $x=x_{1}$ and $z=x_{3}$.

3. DOUBLE SOLITON SAW

Using the method of slowly changing profile, we can materially simplify the set of equations (5) and (6). For this purpose, we represent the function $U=\frac{1}{2}(u^{+}+u^{-})$ in the form

$$U=\sum_{l}Z_{l}E_{l},\tag{7}$$

where $Z_{l}=\exp[il(kz-\omega_{\mathbf{k}}t)]$, E_{l} are the slowly changing complex amplitudes of the acoustic wave, and l runs through the values ±1, $\pm2\ldots$. To guarantee the reality of the quantity U, we set $E_{l}=E^{*}_{-l}$. We note that such a representation of the solution of a nonlinear wave equation has been widely used in the theory of nonlinear waves.[10]

Taking it into account that the carrier frequency of the acoustic wave $\omega_{\mathbf{k}}$ is at resonance with paramagnetic impurities that have two energy levels in the considered simplest case, we can represent the mean value of the magnetization of the paramagnetic impurities in the form

$$B^{\pm}=\sum_{l}\tilde{B}_{l}Z_{-l}\delta_{l,1}.$$

This expression, together with (7), allows us to write down the set of equations (6) in terms of the slow variables

$$\dot{\tilde{B}}_{+1}=(\omega_{\mathbf{k}}-\omega_{0})\tilde{B}_{+1}-2iN\dot{\Theta}_{-1},\quad\dot{N}=i(\dot{\Theta}_{-1}\tilde{B}_{-1}-\dot{\Theta}_{+1}\tilde{B}_{+1}),\tag{8}$$

where

$$\Theta_{l}=\frac{L}{2}\int_{-\infty}^{t}E_{l}\,dt'$$

is the area of the momentum envelope of the SAW. Here we have used in approximation of a rotating wave, which consists in the discarding of terms that oscillate with the frequencies $2\Omega_{\mathbf{k}}$, $3\varepsilon_{\mathbf{k}},\ldots$.

In the interaction of an acoustic wave with a resonantly absorbing medium, the most significant effects are usually observed at exact resonance. Therefore, for simplicity, we consider Eq. (8) at exact resonance $\omega_{\mathbf{k}}=\omega_{0}$ and the simplest initial condition, when all the paramagnetic impurities are initially in the ground states, i.e., $N_{\text{init}}=-\frac{1}{2}$.

For the determination of the explicit form of the quantity $B^{+}+B^{-}$, we expand the quantities Θ_{l}, B_{l}, and N a perturbation-theory series in the small nonlinearity parameter ε:

$$\Theta_{l}=\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}\Theta_{l}^{(\alpha)},\quad\tilde{B}_{l}=\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}B_{l}^{(\alpha)},\quad N=\sum_{\alpha=0}^{\infty}\varepsilon^{\alpha}N^{(\alpha)}.$$

Substituting these expansions in the set of equations (8), we obtain

$$B^{+}+B^{-}=in_{0}\sum_{l}\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}\rho_{l}^{(\alpha)}Z_{l}\Delta_{l},$$

where

$$\rho_{\pm1}^{(1,2)}=\Theta_{\pm1}^{(1,2)},\quad\rho_{\pm1}^{(3)}=\Theta_{\pm1}^{(3)}\mp2\int_{-\infty}^{t}\dot{\Theta}_{\pm1}^{(1)}\Theta_{+1}^{(1)}\Theta_{-1}^{(1)}\,dt',$$

$$\Delta_{l}=\delta_{l,-1}-\delta_{l,+1}.$$

Taking this expression into account, it is easy to transform Eq. (5) into

$$\begin{aligned}\ddot{\tilde{A}}_{\mathbf{k}}&=-\omega_{\mathbf{k}}^{2}\tilde{A}_{\mathbf{k}}-6\omega_{\mathbf{k}}\sum_{\mathbf{k}'}\Phi_{-\mathbf{k},\mathbf{k}',\mathbf{k}-\mathbf{k}'}\tilde{A}_{\mathbf{k}'}\tilde{A}_{\mathbf{k}-\mathbf{k}'}\\&\quad-\frac{1}{2}L\mu_{\mathbf{k}}k\omega_{\mathbf{k}}n_{0}e^{-ikz}\\&\quad\times\sum_{l}\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}\rho_{l}^{(\alpha)}Z_{l}\Delta_{l}.\end{aligned}\tag{9}$$

We multiply this equation by the quantity $ik\mu_{\mathbf{k}}\exp\gamma_{\mathbf{k}}(0)$ and sum over $\mathbf{k}$. As a result, we obtain the following expression on the interface of the media at $x=0$:

$$\begin{aligned}\ddot{U}&+\frac{1}{2\pi}\int W(z_{1})U(z-z_{1},t)\,dz_{1}+\frac{1}{4\pi^{2}}\int\int P(z_{1},z_{2})U(z-z_{1},t)\\&\times U(z-z_{1}-z_{2},t)\,dz_{1}\,dz_{2}\\&+\frac{i}{2}\sum_{\mathbf{k}}L\mu_{\mathbf{k}}^{2}k^{2}\omega_{\mathbf{k}}n_{0}\gamma_{\mathbf{k}}(0)\sum_{l}\sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}\rho_{l}^{(\alpha)}Z_{l}\Delta_{l}=0,\end{aligned}\tag{10}$$

where

$$U=\sum_{\mathbf{k}}ik\mu_{\mathbf{k}}\tilde{A}_{\mathbf{k}}e^{ikz}\gamma_{\mathbf{k}}(0),\quad W(z_{1})=\sum_{\mathbf{k}}\omega_{\mathbf{k}}^{2}e^{ikz_{1}},$$

$$P(z_{1},z_{2})=\sum_{\mathbf{k}_{1}\mathbf{k}_{2}}P_{\mathbf{k}_{1},\mathbf{k}_{2}}\exp[i(k_{1}z_{1}+k_{2}z_{2})],$$

$$P_{\mathbf{k}_{1},\mathbf{k}_{2}}=-\frac{6ik_{1}\omega_{\mathbf{k}_{1}}\mu_{\mathbf{k}_{1}}}{k_{2}(k_{1}-k_{2})\mu_{\mathbf{k}_{2}}\mu_{\mathbf{k}_{1}-\mathbf{k}_{2}}}\Phi_{-\mathbf{k}_{1},\mathbf{k}_{2},\mathbf{k}_{1}-\mathbf{k}_{2}}.$$

The nonlinear equation (10) describes a nonlinear wave process in a bounded elastic medium containing paramagnetic impurities. Substituting (7) in (10), we obtain the following equation for the quantities $\Psi_l^{(\alpha)}(z,t)=\Theta_l^{(\alpha)}(x=0,z,t)$:

$$\sum_l \sum_{\alpha=1}^{\infty} \varepsilon^\alpha Z_l \Big\{ W_l \dot\Psi_l^{(\alpha)} - 2il\omega_{\mathbf{k}} \ddot\Psi_l^{(\alpha)} - iB_l \frac{\partial}{\partial z}\dot\Psi_l^{(\alpha)} + \dddot\Psi_l^{(\alpha)} - C_l \frac{\partial^2}{\partial z^2}\dot\Psi_l^{(\alpha)} + \sum_{l'}\sum_{\alpha'=1}^{\infty} \varepsilon^{\alpha'}\Big[F_{l,l'}\dot\Psi_{l-l'}^{(\alpha)}\dot\Psi_{l'}^{(\alpha')} - \frac{i}{k}F^{(1)}_{l,l'}\frac{\partial\dot\Psi^{(\alpha)}_{l-l'}}{\partial z}\dot\Psi^{(\alpha')}_{l'} - \frac{i}{k}(F^{(1)}_{l,l'}+F^{(2)}_{l,l'})\dot\Psi^{(\alpha)}_{l-l'}\frac{\partial \dot\Psi^{(\alpha')}_{l'}}{\partial z} + O\Big(\frac{Q^2}{k^2}\Big)\Big] + iR^2\rho_l^{(\alpha)}\Delta_l\Big\} = 0, \tag{11}$$

where

$$W_l=\omega_{\mathbf{k}l}^2-l^2\omega_{\mathbf{k}}^2,\quad B_l=\frac{1}{k}\frac{\partial\omega_{\mathbf{k}l}^2}{\partial l},\quad C_l=\frac{1}{2k^2}\frac{\partial^2\omega_{\mathbf{k}l}^2}{\partial l^2},$$
$$F_{l,l'}=\frac{2}{L}P_{l\mathbf{k},l'\mathbf{k}},\quad F^{(1)}_{l,l'}=\frac{2}{L}\frac{\partial}{\partial l}P_{l\mathbf{k},l'\mathbf{k}},$$
$$F^{(2)}_{l,l'}=\frac{2}{L}\frac{\partial}{\partial l'}P_{l\mathbf{k},l'\mathbf{k}},\quad R^2=\sum_{\mathbf{k}}\frac{L^2n_0k^2}{8\rho}.$$

For further analysis of this equation, we make use of the perturbative reduction method, developed in Ref. 11, according to which the quantity $\Psi_l^{(\alpha)}(z,t)$ can be represented in the form

$$\Psi_l^{(\alpha)}(z,t)=\sum_{n=-\infty}^{+\infty}Y_n(z,t)\varphi_{l,n}^{(\alpha)}(\zeta,\tau), \tag{12}$$

where

$$Y_n\propto\exp[in(Qz-\Omega t)],\quad \zeta=\varepsilon^q(Qz-\lambda t),$$
$$\tau=\varepsilon^{q+1}t,\quad \lambda=Qv_g,\quad v_g=\partial\Omega/\partial Q,$$

the quantity q is subject to determination. Such a representation allows us to separate from $\Psi_l^{(\alpha)}$ the still more slowly changing quantity $\varphi_{l,n}^{(\alpha)}(\zeta,\tau)$. Consequently, it is assumed that the quantities Ω, Q and $\varphi_{l,n}^{(\alpha)}$ satisfy the inequalities

$$\omega_{\mathbf{k}}\gg\Omega,\ k\gg Q,\ |\dot\varphi_{l,n}^{(\alpha)}|\ll\Omega|\varphi_{l,n}^{(\alpha)}|,\ |\partial\varphi_{l,n}^{(\alpha)}/\partial z|\ll Q|\varphi_{ln}^{(\alpha)}|.$$

The "stretchable" variables ζ and τ, and also the quantity q, are chosen such that between the nonlinear effects and the dispersion and absorption effects there can arise an interaction of exactly the same order and unambiguously determined by the solvable set of equations (11). In this case, with the help of the standard procedure of Refs. 11 and 12, we show that this condition is satisfied at $q=1$. Substituting Eqs. (12) in Eq. (11), and calculating the derivatives $\dot\Psi_l^{(\alpha)}$, $\ddot\Psi_l^{(\alpha)}$, $\dddot\Psi_l^{(\alpha)}$, $\partial\dot\Psi_l^{(\alpha)}/\partial z$ and $\partial^2\dot\Psi_l^{(\alpha)}/\partial z^2$ we obtain the nonlinear wave equation

$$\sum_l Z_l\Big\{\sum_{\alpha n}\varepsilon^\alpha Y_n\Big[(\mathscr{W}_{l,n}+\varepsilon J_{l,n}\partial_\zeta+i\varepsilon^2\mathscr{H}_{l,n}\partial_{\zeta\zeta}^2+\varepsilon^2h_{l,n}\partial_\tau)\varphi_{l,n}^{(\alpha)} - \sum_{\alpha' l' n'}\varepsilon^{\alpha'}\chi_{l,l'}^{n,n'}\varphi_{l-l',n-n'}^{(\alpha)}\varphi_{l',n'}^{(\alpha')}\Big]+iR^2\Delta_l\rho_l'+O(\varepsilon^4)\Big\}=0, \tag{13}$$

where

$$\mathscr{W}_{l,n}=-in\Omega[W_l-n(2l\omega_{\mathbf{k}}\Omega-B_lQ)-n^2\Omega^2+C_ln^2Q^2]+iR^2\Delta_l,$$
$$J_{l,n}=-Qv_gW_l+4n\Omega Qv_gl\omega_{\mathbf{k}}-B_lnQ(\Omega+Qv_g)+3n^2\Omega^2Qv_g-C_ln^2Q^2(2\Omega+Qv_g), \tag{14}$$
$$\mathscr{H}_{l,n}=[-2l\omega_{\mathbf{k}}v_g^2+B_lv_g-3n\Omega v_g^2-C_ln(\Omega+2Qv_g)]Q^2,$$
$$h_{l,n}=W_l-4nl\omega_{\mathbf{k}}\Omega+nQB_l-3n^2\Omega^2+C_ln^2Q^2,$$

$$\chi_{l,l'}^{n,n'}=F_{l,l'}\Omega^2n'(n-n')+O(Q/k),$$

$$\rho_l'=2i\varepsilon^3\Omega\sum_{nn'n''}(n-n'-n'')\int_{-\infty}^{t}Y_n\varphi^{(1)}_{l,n-n'-n''}\varphi^{(1)}_{l,n'}\varphi^{(1)}_{-l,n''}dt'.$$

To determine the values of $\varphi_{l,n}^{(\alpha)}$, we set equal to zero the terms corresponding to like powers of ε. As a result, we obtain a chain of equations: in first order in ε

$$\mathscr{W}_{l,n}\varphi_{l,n}^{(1)}=0, \tag{15}$$

in second order in ε

$$\mathscr{W}_{l,n}\varphi_{l,n}^{(2)}+J_{l,n}\partial_\zeta\varphi_{l,n}^{(1)}-\sum_{l'n'}\chi_{l,l'}^{n,n'}\varphi^{(1)}_{l-l',n-n'}\varphi^{(1)}_{l',n'}=0 \tag{16}$$

and in third order in ε

$$\mathscr{W}_{l,n}\varphi_{l,n}^{(3)}+J_{l,n}\partial_\zeta\varphi_{l,n}^{(2)}+(i\mathscr{H}_{l,n}\partial_{\zeta\zeta}^2+h_{l,n}\partial_\tau)\varphi_{l,n}^{(1)}-\chi_{l,l'}^{n,n'}(\varphi^{(1)}_{l-l',n-n'}\varphi^{(2)}_{l,n'}+\varphi^{(2)}_{l-l',n-n'}\varphi^{(1)}_{l,n'})$$
$$=2R^2i\frac{\Delta_l}{n}\sum_{n'n''}(n-n'-n'')\varphi^{(1)}_{l,n-n'-n''}\varphi^{(1)}_{l,n'}\varphi^{(1)}_{-l,n''}. \tag{17}$$

It follows from Eq. (1) that in dispersive media $W_0=W_{\pm1}=0$ and $W_{|l|>1}=0$. Consequently, according to Eq. (15), only the following terms of all the quantities $\varphi_{l,n}^{(1)}$ differ from zero: $\varphi_{0,n}^{(1)}\varphi_{\pm1,\mp1}^{(1)}$ and $\varphi_{\pm1,\pm1}^{(1)}$. Here, the connection between the quantities Ω and Q, at fixed values of l and n, is determined from the relations

$$n(2l\omega_{\mathbf{k}}\Omega-B_lQ)+\Omega^2-C_lQ^2+nR^2\Delta_l/\Omega=0.$$

Taking into account that $F_{0,n}=F_{0,n}^{(1)}=F_{0,n}^{(2)}=0$, we get at $l=0$ from (17) $d^2_{\zeta\zeta}\varphi_{0,n}^{(1)}=0$. In what follows, we shall also be interested in solutions of (17) that vanishes at $\zeta\to\pm\infty$. Consequently, under the stated conditions, we obtain $\varphi_{0,n}^{(1)}=0$. We consider below in detail the situation in which $\ln=l_0n_0=-1$. In this case, we have

$$\varphi^{(1)}_{\pm1,\mp1}\neq0, \tag{18}$$

$$2\omega_{\mathbf{k}}\Omega+n_0B_{l_0}Q-\Omega^2+C_{l_0}Q^2-R^2/\Omega=0. \tag{19}$$

Substituting the expressions (14) and (19), we easily see that the following relations hold:

$$J_{\pm1,\mp1}=0,\quad \mathscr{H}_{\pm1,\mp1}=\pm\Omega Q^2[v_g^2(1+R^2/\Omega^3)-C_{l_0}]\equiv\pm\Omega\beta,$$
$$h_{\pm1,\mp1}=2\Omega(\omega_{\mathbf{k}}-\Omega+R^2/2\Omega^2)\equiv2\Omega\alpha, \tag{20}$$
$$v_g=-(n_0B_{l_0}+2QC_{l_0})/2\alpha.$$

From Eq. (16) we obtain the connection between $\varphi^{(1)}_{\pm1,\mp1}$ and $\varphi^{(2)}_{\pm2,\mp2}$:

$$\mathscr{W}_{\pm2,\mp2}\varphi^{(2)}_{\pm2,\mp2}=\chi^{\mp2,\mp1}_{\pm2,\pm1}(\varphi^{(1)}_{\pm1,\mp1})^2. \tag{21}$$

Substituting Eqs. (18)–(21) in Eq. (17) we obtain after simple transformations an equation for the quantities

$$\xi_{\pm}=\varepsilon\varphi^{(1)}_{\pm1,\mp1}:$$

$$\mp i(\partial_t+v_g\partial_z)\xi_{\pm}+p\partial_{zz}{}^2\xi_{\pm}+q\xi_{\pm}|\xi_{\pm}|^2=0, \tag{22}$$

where

$$p=\frac{\beta}{2\alpha Q^2},\quad q=\frac{1}{\alpha}\left(\frac{R^2}{\Omega}+\frac{2}{\eta}\mathscr{L}^2\right),$$
$$\mathscr{L}=\frac{\Omega\tilde{P}_{\pm2,\pm1}}{Lk},\quad \tilde{P}_{l,m}=\frac{i|mk|\,|lk-mk|}{|lk|}P_{lk,mk}, \tag{23}$$
$$\eta=3c^2k^2\Delta-\frac{R^2}{\Omega}.$$

Thus, we have reduced the set of equations (15)–(17) to a single equation for the unknown function $\xi_{\pm}(z,t)$. If we now go over in the latter equation to the variables $y=z-v_g t$ and t, the equation takes on the form

$$\mp i\partial_t\xi_{\pm}+p\partial_{yy}{}^2\xi_{\pm}+q\xi_{\pm}|\xi_{\pm}|^2=0. \tag{24}$$

This is the well known nonlinear Schrödinger equation (NSE)[13,14]. Under conditions when the quantities p and q are positive in the nonlinear equation (22) it is convenient to carry out the following change of variables: $v=q^{1/2}\xi_-$, $z=p^{1/2}X+v_g t$. Then we obtain the NSE in the following form

$$i\partial_t v+\partial_{XX}{}^2v+v|v|^2=0.$$

This equation has the soliton solution

$$v(X,t)=2^{1/2}K\,\mathrm{sech}[K(X-v_lt)]\exp\left\{i\left[\frac{v_l}{2}X-\left(\frac{v_l^2}{4}-K^2\right)t\right]\right\}, \tag{25}$$

where K and v_l are the amplitude and velocity of the soliton[3,13].

Substituting the solution (25) in (12), we obtain for the quantity Ψ_{-1}

$$\Psi_{-1}=\mathrm{Re}\left\{Y_{+1}\left(\frac{2}{q}\right)^{1/2}K\,\mathrm{sech}\left\{K\left[\frac{z}{p^{1/2}}-\left(\frac{v_g}{p^{1/2}}+v_l\right)t\right]\right\}\right.$$
$$\left.\times\exp\left(i\left[\frac{v_l}{2p^{1/2}}z-\left(\frac{v_gv_l}{2p^{1/2}}+\frac{v_l^2}{4}-K^2\right)t\right]\right)\right\}. \tag{26}$$

The latter solution is a double soliton of the breather type. The appearance in Eq. (26) of the factor $\exp[i(Qz-\Omega t)]$ indicates the formation of periodic (slow in comparison with $\exp[i(kz-\omega_{\mathbf{k}}t)]$) beats in coordinates and time, with characteristic parameters Q and Ω, as a result of which the soliton solution (25) is transformed into the breather solution (26). It should be noted that for plane waves a solution of type (26) has been obtained in Ref. 12 for the sine-Gordon equation, and interpreted as a bound state of a soliton and an antisoliton. The solution (26) describes a quasimonochromatic wave with wave number $\mathbf{k}$, frequency $\omega_{\mathbf{k}}$, and slowly varying amplitude having the form of a breather.

4. CONCLUSION

A quasimonochromatic pulse is usually characterized by a central frequency $\omega_{\mathbf{k}}$ (or a corresponding wave number $\mathbf{k}$) and a width $\omega_{\mathbf{k}}-\omega$ (or a corresponding width in wave-number space $\mathbf{k}-\mathbf{k}'$). In the propagation of the pulse in a dispersive medium, its shape will not remain unchanged—the width of the pulse will increase during propagation if $\partial^2\omega_{\mathbf{k}}/\partial k^2\neq0$. This is due to the fact that waves of different length $2\pi k^{-1}$ propagate in a dispersive wave with different velocities. Along with the dispersion, the absorption processes also lead to broadening of the profile. In the NSE, both these effects are taken into account through the term $p\partial^2_{yy}\xi_{\pm}$, with terms proportional to the quantity Δ responsible for the dispersion, while terms containing R^2 are responsible for the absorption. It should be noted that the resonance absorption of sound by the spin system leads to dispersion of the acoustic wave.[6]

On the other hand, the nonlinearity effects produced by the anharmonic lattice vibrations, and the nonlinear coherent interaction of the wave with the paramagnetic impurities, lead to a progressive initial pulse profile deformation that increases with increase of t. In the NSE, the nonlinear effects are taken into account by the term $q\xi_{\pm}|\xi_{\pm}|^2$, with the contribution of the anharmonic oscillations taken into account by the term $2\mathscr{L}^2/\eta$ while the term R^2/Ω is responsible for the interaction of the pulse with the impurities. As a result of the competition between the nonlinearity, which increases the curvature of the profile of the pulse, and the dispersion and absorption that cause the profile to spread out, the shape of the nonlinear wave is stabilized—a breather state is formed. The conditions of balance between these effects is the simplest case, when a pulse of rectangular shape of amplitude H and width Λ is introduced into the medium, i.e.,

$$\xi(z,0)=\begin{cases}H, & 0<z<\Lambda,\\ 0, & z<0,\ z>\Lambda,\end{cases}$$

can be represented by the following relations[3]:

$$H\Lambda\sim(2p/q)^{1/2},\quad 4HT(pq/2)^{1/2}\sim\Lambda.$$

In other words, these equations reflect the connection between the effects of dispersion, anharmonism, and coherent interaction of the field with a nonlinear absorbing medium.

It should be noted that our results and their interpretation are applicable to pulses with sufficiently smooth envelopes under the condition that the size of the pulse is large in comparison with the wavelength, i.e., $\Lambda k\gg1$. Moreover, the length of the breather should be significantly greater than the characteristic length of change of the periodic "beats:" $\Lambda Q>1$. As already mentioned, the quantity Δ determines the contribution from the dispersion effects. It should be noted in this connection that the scheme considered in the preceding sections has a rather general character and the results obtained there for Rayleigh waves propagating along the free surface of a medium are easily transferred to the case in which a thin layer of another substance is placed on the surface of the medium. In this system, SAW can also propagate as, for example, Sezawa waves and Love waves.[1] In this case, the dispersion relation can be formally represented in the same form as for the Rayleigh waves with the only difference being that now, in the relation (1), the meaning of h becomes that of a quantity which is determined by the thickness of the film.[3,15] Thus, we can distinguish between two situations, "internal dispersion," i.e., governed by the discrete structure of the medium, and "external" i.e., determined by the presence of a film on the surface of the substrate—"geometric" dispersion.

We consider the quantity $2p/q$ under the conditions

$R^2 \gg \Omega^3\Delta$ and $R^2 \sim \Omega c^2 k^2 \Delta$. Then we get from (23)

$$\frac{2p}{q} = \frac{c^2}{\Omega^2}\left(1 + \frac{2\Omega\mathscr{L}^2}{R^2\eta}\right)^{-1}.$$

It is seen from this relation that the quantity $2p/q$ is always positive if $\eta > 0$. In the limiting case when the inequality $2\Omega\mathscr{L}^2 \ll R^2\eta$ is satisfied, effects associated with the anharmonic lattice vibrations become unimportant and the problem reduces to the case considered in Ref. 2 of an acoustic SIT for SAW. In the other limiting case, under the inequalities $2\Omega\mathscr{L}^2 \gg R^2\eta$ and $\Omega^3\Delta \gg R^2$, effects associated with the interaction of the pulse with the paramagnetic impurities become small. In this case, the problem reduces to that considered in Ref. 3. In the intermediate case $2\Omega\mathscr{L}^2 \sim R^2\eta$, the breather SAW is formed under conditions in which the effects of anharmonism, dispersion, and nonlinear coherent interaction of the wave with the paramagnetic impurities exist simultaneously.

The solution of Eq. (22) in the case of "external" dispersion is of interest when the inequality $2\Omega\mathscr{L}^2 \gg R^2\eta$ holds. This condition is satisfied in the case of "resonance" between the dispersion and linear absorption: $3c^2k^2\Omega\Delta \gtrsim R^2$, i.e., when η is a relatively small positive quantity. In this case, the wave is formed from effects of anharmonism, dispersion, and linear absorption, due to the interaction of the acoustic wave with paramagnetic impurities. It should be noted that this result differs in principle from acoustic SIT (both in the case $\Theta > \pi$ and for $\Theta \ll 1$), under the conditions of which the interaction of the wave with the paramagnetic impurities is essentially nonlinear. The difference between our present result and the formation of a nonlinear wave via anharmonism and dispersion is quite evident here.

For numerical estimates we choose typical values of the parameters of the acoustic field and the medium. If it is assumed that $\omega_{\mathbf{k}} \sim 10^{10}$ Hz, $\Omega \sim 10^8$ Hz, $\Lambda \lesssim 1$ cm, $T \sim 10^{-6}$–10^{-7} s, $H \sim 10^1$–10^2, $n_0 \sim 10^{18}$ cm^{-3}, $C_{ijklmn} \sim 10^{11}$–10^{12} dyn/cm^2, and $h < 10^{-5}$ cm, all the conditions given above are satisfied. This circumstance gives grounds for hoping that the considered phenomena can be observed experimentally.

Thus, it has been shown that nonlinear waves can develop in the form of double solitons in solid dielectrics containing paramagnetic impurities when a wave propagates under conditions in which the effects of anharmonism, dispersion, and coherent interaction of the wave with the paramagnetic impurities are significant. The explicit form of these waves at $x = 0$ is given by the relation (26), while the quantity $\beta_{\mathbf{k}}^{(i)}(x)$ expresses the transverse structure. The dispersion law and the connection between the quantities Ω and Q are determined by the expressions (11) and (19), respectively.

In conclusion, we note that the results for SAW at $x = 0$ given in the present paper can easily be transferred to the case of plane volume waves. For this purpose, it suffices to introduce corresponding changes of notation in the formulas given above.

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Translated by R. T. Beyer