Partition function of a chromoelectric hadron string

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The partition function of an open hadron string is calculated and a procedure is determined for the renormalization of the string tension coefficient and of the mass of the quark located at the end of the string. It is shown that the Gell-Mann–Low function for the nonperturbative phase of the $SU(N)$ gauge theory coincides with the first term of the expansion of the β-function in the strong coupling approximation in field theory on a lattice. The string partition function depends only on the Euler characteristic of the string world sheet.

1. INTRODUCTION

In the limit of a large number of colors $N\to\infty$ the nonabelian $SU(N)$ gauge theory of interacting quarks and gluons goes over into the theory of a chromoelectric string, binding quarks in a hadron.[1,2] The partition function Z of this string is in the form of a functional integral over the contributions of the world sheet Σ of the string $x_\mu = x_\mu(z)$ in R^4 and an integral over the inner one-dimensional metrics $\lambda(\gamma)$, defined on the sheet boundary $\partial\Sigma$ (see Appendix 1):

$$Z=\int Dx_\mu(z)D\lambda(\gamma)\exp\{-S[x(z),\lambda(\gamma)]\},\tag{1}$$

$$S[x(z),\lambda(\gamma)]=k_0\int_\Sigma d^2z\{\det h_{ab}[x(z)]\}^{1/2}+{}^1/_2\oint_{\Gamma=\partial\Sigma}d\gamma\left\{\frac{\dot{x}^2[z(\gamma)]}{\lambda(\gamma)}+\lambda(\gamma)m_0^2\right\},\tag{2}$$

$a,b = 1,2$; $\mu = 1,2,3,4$.

Here λ is a parametrization of the contour Γ, m_0 is the bare mass of the quark,

$$\dot{x}_\mu=dx_\mu/d\gamma=(\partial x_\mu/\partial z^a)(dz^a/d\gamma);\quad h_{ab}=(\partial x_\mu/\partial z^a)(\partial x_\mu/\partial z^b)$$

is the induced metric on the sheet Σ. The expression for the bare tension coefficient k_0 of the string has the form

$$k_0=\frac{e^2}{2\delta^2}\frac{N^2-1}{2N},\qquad e^2=\frac{e_0^2}{N},\tag{3}$$

where δ is a regularization parameter with dimensions of length. In the process of subsequent renormalization of the quantity k the parameter δ should be taken to zero. We note that Eq. (3) exactly coincides with the expression obtained for this quantity to lowest order in the strong coupling approximation[3] in the Hamiltonian formulation of lattice gauge theory.[4]

Let us recall that the hadron string in question arises only in the presence of quarks and describes the topologically nontrivial configuration of the gauge field, which realizes the extremum of the two-dimensional effective action $S_{\rm eff}$, defined on the surfaces Σ. The action $S_{\rm eff}$ was obtained in Refs. 1, 2, and 5 in evaluating the four-dimensional hadron field correlators (A1) in the framework of the $1/N$ expansion. The contribution of the quantum fluctuations of the gauge field is small of order $\propto 1/N$ (Ref. 2). It is important to note that the stable configuration of the string disappears in the abelian limit $N\to 1$.[1] The hadronic string, which arises naturally for $N\gg 1$, differs from the customary models of strings in that it possesses a narrower space of quantum states. This restriction is caused by the following specific properties of the hadron string.[1,2,5]

1. Its action is quantized:

$$k_0\int_\Sigma d^2zh^{1/2}=\pi|Q|,\tag{4}$$

$$h=\det h_{ab},\quad Q=\pm1,\pm2,\dots$$

As a consequence the area of the world sheet $A(\Sigma)$ does not change continuously, as in conventional string theory, but discretely. This circumstance should be taken into account in evaluating the integral over surfaces in Eq. (1).

2. In the integral (1) it is necessary to sum only over those surfaces, whose curvature scalar is constant and equal to

$$|R|=2e^2/\delta^2=\text{const}.\tag{5}$$

We take the restrictions (4) and (5) into account by imposing constraints and introducing the corresponding Lagrange multipliers into the integrand in (1).

As a consequence of the condition (4) the full partition function Z breaks up into a sum of contributions from different topological sectors[1,5]:

$$Z=\sum_{|Q|=1}^{\infty}Z_{|Q|}=\sum_{|Q|}(Z_{Q=+|Q|}+Z_{Q=-|Q|})\equiv\sum_{|Q|}(Z_{Q^+}+Z_{Q^-}).\tag{6}$$

Since the action (2) is not quadratic in the variable $x_\mu(z)$, the evaluation of the integral (1) is a difficult problem. In order to solve this problem we go over to the first-order formalism by means of the replacement[6,7]

$$\int Dx_\mu(z)\exp(-S[x])\to\int Dx_\mu(z)Dg_{ab}(z)\exp(-S[x,g]),\tag{7}$$

where $S[x]=k_0\int_\Sigma d^2z\{h[x(z)]\}^{1/2}$,

$$S[x,g]=\frac{k_0}{2}\int_\Sigma d^2zg^{1/2}g^{ab}\,\partial_ax_\mu\,\partial_bx_\mu.\tag{8}$$

One may put the equality sign in the expression (7) if it is agreed that the evaluation of the integral over the metrics g_{ab} is taken only in the leading order of the saddle-point approximation.

The variation of the action (8) equals

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$$\delta S[x,g]=-k_0\int_\Sigma d^2z g^{1/2}\,\delta x_\mu\,\Delta_g x_\mu - \frac{k_0}{2}\int_\Sigma d^2zg^{1/2}\,\delta g_{ab}T^{ab}+k_0\oint_{\partial\Sigma} ds\,\delta x_\mu n^a\partial_a x_\mu, \qquad (9)$$

where

$$\Delta_g=\frac{1}{g^{1/2}}\frac{\partial}{\partial z^a}\left(g^{1/2}g^{ab}\frac{\partial}{\partial z^b}\right), \qquad (10)$$

$$T_{ab}=\partial_a x_\mu\partial_b x_\mu-{}^1/_2 g_{ab}g^{cd}\partial_c x_\mu\partial_d x_\mu, \qquad (11)$$

n^a is the outer normal to the boundary $\partial\Sigma$, $ds^2=g_{ab}\,dz^a\,dz^b$. The solution of the equation $T_{ab}=0$ for the energy-momentum tensor (11) has the general form

$$\tilde g_{ab}=f(x,\ g)\,h_{ab}, \qquad (12)$$

where f is an arbitrary function. Substitution of this solution into the expression (8) for the action $S[x,g]$ gives

$$S[x,\tilde g]=k_0\int_\Sigma d^2zh^{1/2}=S[x]. \qquad (13)$$

In what follows we agree to systematically ignore loop corrections to g_{ab} so that the following equality is valid:

$$\int Dx(z)\exp\{-S[x]\}\overset{\circ}{=}\int Dx(z)Dg(z)\exp\{-S[x,g]\}, \qquad (14)$$

which permits one to evaluate the integral (1) by using the action $S[x,g]$, which is quadratic in $x_\mu(z)$. The symbol $\overset{\circ}{=}$ indicates that the integral over the metric is calculated only to leading order in the saddle-point approximation. The summation in Eq. (1) is carried out over surfaces of genus zero, since the contribution of manifolds with genera$\geqslant 1$ is suppressed[8] by the factor $1/N$.

For this method of evaluation of the functional integral (14) with condition (4) it is sufficient to implement the requirement (5) of constant curvature as follows. According to the Gauss-Bonnet theorem

$$\frac{1}{2}\int_\Sigma d^2zg^{1/2}R+\oint_{\partial\Sigma} ds\varkappa_g=2\pi\chi, \qquad (15)$$

the integral over the boundary $\partial\Sigma$ of the geodesic curvature $\varkappa_g$ is constant as a result of the restrictions (4) and (5). Therefore the condition (5) gives rise to the additional constraint

$$q\equiv\int_{\partial\Sigma} ds\varkappa_g=\text{const.} \qquad (16)$$

Taking into account the indicated contribution of the topological sector to $|Q|$ we may write the partition function Z in the form

$$Z_{|Q|}=Z_{Q^+}+Z_{Q^-},$$

$$Z_{Q^\pm}\overset{\circ}{=}\int Dx_\mu(z)Dg_{ab}(z)D\lambda(\gamma)\exp\int-\frac{k_0}{2}\int_\Sigma d^2zg^{1/2}g^{ab}\,\partial_a x_\mu\,\partial_b x_\mu$$
$$+\alpha\Big(\pm\pi Q-k_0\int_\Sigma d^2zg^{1/2}\Big)+\beta\Big(\oint_{\partial\Sigma} ds\varkappa_g-q\Big)$$
$$-\frac{1}{2}\oint_{\partial\Sigma} d\gamma\Big(\frac{\dot x^2}{\lambda}+\lambda m_0^2\Big)\Big\}. \qquad (17)$$

where α,β are Lagrange multipliers.

We note that the quantization condition, connected with the multiplier α, fixes the factor $f=1$ in expression (12), since only then is it equivalent to Eq. (4).

If at this stage one evaluates the variation of the action and sets $\delta S_Q/\delta g_{ab}=0$ one obtains, with Eq. (17) taken into account, the equation

$$\partial_a x_\mu\partial_b x_\mu-{}^1/_2 g_{ab}g^{cd}\partial_c x_\mu\partial_d x_\mu=\alpha g_{ab}, \qquad (18)$$

with solution

$$\alpha=0,\quad \tilde g_{ab}=\partial_a x_\mu\partial_b x_\mu=h_{ab}.$$

However our strategy is to first perform the integration over the contributions $x_\mu(z)$. Elimination of the resulting divergences permits the renormalization of the quark mass and the string tension coefficient k. It is only afterwards that we carry out variation of the resultant expression with respect to the metric g_{ab}.

2. EVALUATION OF THE PARTITION FUNCTION $Z_{|Q|}$

Since the quark trajectory Γ is the boundary of the surface Σ (i.e., $\Gamma=\partial\Sigma$) the metrics on these two objects should agree, i.e.

$$ds\equiv(g_{ab}dz^adz^b)^{1/2}=m_0\lambda(\gamma)\,d\gamma \qquad (19)$$

[the mass m_0 is included here to ensure correct dimensions for the action (17)]. As was noted in Ref. 9a, this identification of the metric is most natural from the point of view of unification of the dynamics of strings and particles (quarks). After introduction into the integral over $\lambda(\gamma)$ of the δ-function, which takes into account the constraint (19), we obtain after integration the following factor in (17):

$$\int D\lambda(\gamma)\exp\Big\{-\frac{1}{2}\oint_{\partial\Sigma} d\gamma\Big(\frac{\dot x^2}{\lambda}+\lambda m_0^2\Big)$$
$$\Rightarrow\exp\Big\{-\frac{m_0}{2}\oint_{\partial\Sigma} d\gamma\dot x^2(g_{ab}\dot z^a\dot z^b)^{-1/2}-\frac{m_0}{2}\oint_{\partial\Sigma} ds\Big\}. \qquad (20)$$

To perform the integration over the variable x_μ in Eqs. (17), (20) we represent $x_\mu(z)$ in the form

$$x_\mu(z)=x_\mu^{cl}(z)+y_\mu(z), \qquad (21)$$

where x^{cl} is the solution of the classical equation

$$\Delta_g x_\mu^{cl}=0 \qquad (22)$$

with the boundary condition

$$k_0\frac{ds}{d\gamma}n^a\,\partial_a x_\mu^{cl}=m_0\frac{d}{d\gamma}[\dot x^{cl}(g_{ab}\dot z^a\dot z^b)^{-1/2}], \qquad (23)$$

which is obtained from variation of the action in (17), (20). In view of Eq. (22) for x^{cl} it is necessary that the integral

$$\int_\Sigma d^2zg^{1/2}\,\Delta_g x_\mu^{cl}=\oint_{\partial\Sigma} d\gamma\frac{ds}{d\gamma}n^a\,\partial_a x_\mu^{cl}$$
$$=\frac{m_0}{k_0}\int_0^{\gamma_{max}} d\gamma\frac{d}{d\gamma}[\dot x_\mu^{cl}(g_{ab}\dot z^a\dot z^b)^{-1/2}] \qquad (24)$$

should vanish, which gives rise to the periodicity condition

$$x^{cl}(0)=\dot x^{cl}(\gamma_{max}). \qquad (25)$$

In this manner the classical solution (22) is connected with the closed periodic quark orbits $\Gamma = \partial\Sigma$. In particular, these orbits have no kinks, we was implicit in formula (15). Summation over Q in Eq. (4) is in fact equivalent to summing the contributions of all periodic orbits.

Note that

$$S[x,g] = \frac{k_0}{2}\int_\Sigma d^2z\, g^{1/2} g^{ab}\,\partial_a x_\mu\,\partial_b x_\mu + \frac{m_0}{2}\Big[\oint_{\partial\Sigma} d\gamma\,\dot{x}^2 (g_{ab}\dot{z}^a\dot{z}^b)^{-1/2} + \oint_{\partial\Sigma} ds\Big] \tag{26}$$

is equal at the classical level to

$$S[x^{cl},g] = \frac{m_0}{2}\oint_{\partial\Sigma} ds,$$

since

$$\int_\Sigma d^2z g^{1/2} g^{ab}\,\partial_a x_\mu\,\partial_b x_\mu = -\int_\Sigma d^2z g^{1/2} x_\mu \Delta_g x_\mu + \oint_{\partial\Sigma} ds x_\mu n^a\,\partial_a x_\mu,$$

where

$$n^a = g^{ab} g^{1/2} e_{bc}\frac{dz^c}{ds}, \qquad n_a n^a = 1, \tag{27}$$

and according to (23) we have

$$\frac{k_0}{2}\oint_{\partial\Sigma} d\gamma\,\frac{ds}{d\gamma}\,x_\mu n^a\,\partial_a x_\mu$$

$$= \frac{m_0}{2}\oint_{\partial\Sigma} d\gamma\, x_\mu \frac{d}{d\gamma}[\dot{x}_\mu (g_{ab}\dot{z}^a\dot{z}^b)^{-1/2}] = -\frac{m_0}{2}\oint_{\partial\Sigma} d\gamma\,\dot{x}^2 (g_{ab}\dot{z}^a\dot{z}^b)^{-1/2},$$

which ensures the cancellation of the corresponding term in the right side of Eq. (26). Substitution of the expansion (21) into the action (26) with Eqs. (22) and (23) taken into account leads to the following result:

$$S[x,g] = -\frac{k_0}{2}\int_\Sigma d^2z g^{1/2} y_\mu \Delta_g y_\mu + \frac{k_0}{2}\oint_{\partial\Sigma} ds y_\mu n^a\,\partial_a y_\mu + \frac{m_0}{2}\oint_{\partial\Sigma} ds\Big(\frac{dy_\mu}{ds}\frac{dy_\mu}{ds} + 1\Big). \tag{28}$$

The last term in (28) was obtained using the relations

$$\frac{ds}{d\gamma} = (g_{ab}\dot{z}^a\dot{z}^b)^{1/2}, \qquad \dot{y}_\mu \dot{y}_\mu = \Big(\frac{dy_\mu}{ds}\frac{dy_\mu}{ds}\Big)\Big(\frac{ds}{d\gamma}\Big)^2.$$

We perform the integration over y_μ (over fluctuations of the embedding of the surface Σ) using the Neumann boundary condition

$$\partial_n y_\mu(z) \equiv n^a \partial_a y_\mu(z) = 0, \quad z \in \partial\Sigma. \tag{29}$$

Further, assume that the periodic classical quark orbit has invariant length

$$L = \oint_{\partial\Sigma} ds, \tag{30}$$

where $x^{cl}_\mu(s=0) = x^{cl}_\mu(s=L)$. It is natural to impose periodicity here on the fluctuation of the quark trajectory $y_\mu[z(s)] \equiv y_\mu(s)$:

$$y_\mu(0) = y_\mu(L), \quad dy_\mu(0)/ds = dy_\mu(L)/ds. \tag{31}$$

We note that we have included the interaction of the quark with the field of the string by imposing the boundary conditions (19), (23). With this taken into account expression (17) for Z_Q takes on the following form

$$\begin{aligned} Z_{Q^\pm} \stackrel{\circ}{=} &\int Dg_{ab}(z) Dy_\mu(z) Dy_\mu(s) \\ &\times \exp\Big\{\frac{k_0}{2}\int_\Sigma d^2z g^{1/2} y_\mu(z)\Delta_g y_\mu(z) + \frac{m_0}{2}\oint_{\partial\Sigma} ds \\ &\times\Big[y_\mu(s)\frac{d^2}{dy^2} y_\mu(s) - 1\Big] + \alpha\Big(\pm\pi Q - k_0\int_\Sigma d^2z\, g^{1/2}\Big) \\ &+ \beta\Big(\oint_{\partial\Sigma} ds\,\varkappa_g - q\Big)\Big\} \\ \doteq &\text{ const}\int Dg_{ab}(z)\Big(\oint_{\partial\Sigma}\frac{ds}{2\pi}\Big)^{D/2}\Big[\det{}'\Big(-\frac{d^2}{ds^2}\Big)\Big]^{-D/2} \\ &\times\Big(\int_\Sigma d^2z\frac{g^{1/2}}{2\pi}\Big)^{D/2}[\det{}'(-\Delta_g)]^{-D/2} \\ &\times\exp\Big\{-\frac{m_0}{2}\oint_{\partial\Sigma} ds + \alpha\Big(\pm\pi Q - k_0\int_\Sigma d^2z g^{1/2}\Big) \\ &+\beta\Big(\oint_{\partial\Sigma} ds\,\varkappa_g - q\Big)\Big\}. \end{aligned} \tag{32}$$

Here det′ means that the zero modes of the corresponding operator were omitted. These modes give rise to the appearance of factors

$$\Big(\oint_{\partial\Sigma} ds/2\pi\Big)^{D/2}, \Big(\int_\Sigma d^2z g^{1/2}/2\pi\Big)^{D/2}$$

on the right side of relation (32) (Ref. 10). The quark determinant, regularized by the ζ-function method, equals[11]

$$\det{}'(-d^2/ds^2) = \text{const}\oint_{\partial\Sigma} ds \tag{33}$$

and cancels the analogous factor in (32), connected with the zero mode.

Evaluation of the determinant

$$[\det{}'(-\Delta_g)]^{-D/2} \equiv \exp\{-\Phi[g]\}, \tag{34}$$

where D is the dimension of the space R^D (in our work $D = 4$), is carried out by making use of the conformal anomaly, following the method described in Refs. 7, 9b, and 12. In the gauge

$$g_{ab}(z) = \rho(z)\delta_{ab} = \exp[\varphi(z)]\delta_{ab} \tag{35}$$

we have

$$\delta\Phi/\delta\rho = -{}^1/_2 D[Y(z,z;t=0^+) - \Pi(z)], \tag{36}$$

where $Y(z,z';t)$ is the kernel of the heat equation (see Ref. 9b) and $\Pi(z)$ is the kernel of the projector on the zero mode of the operator Δ_g with boundary condition (29). In our case

$$\Pi(z) \equiv \sum_i \langle z|i\rangle\langle i|z\rangle = 1\Big/\int_\Sigma d^2z g^{1/2}, \tag{37}$$

since the operator Δ_g has one normalizable scalar zero mode for each of the D dimensions:

$$u_0=A^{-1/2}(\Sigma), \quad A(\Sigma)=\int_\Sigma d^2z g^{1/2}. \tag{38}$$

By making use of the asymptotic expression for $Y(z,z;t\to 0^+)$,[12,13] we obtain analogously to Ref. 9b

$$\frac{\delta\Phi}{\delta\rho}=-D\left\{\frac{1}{8\pi t}+\frac{R(z)}{48\pi}+\frac{\delta_B(z)}{16(\pi\rho t)^{1/2}}+\frac{1}{16\pi\rho^{1/2}}\partial_n\delta_B(z)+\frac{\delta_B(z)\varkappa_g}{24\pi\rho^{1/2}}-\frac{1}{2A(\Sigma)}\right\}. \tag{39}$$

Here $R(z)$ is the scalar curvature equal to

$$R(z)=-\partial^2\varphi(z)/\rho(z), \tag{40}$$

and $\delta_B(z)$ is a planar one-dimensional δ-function in the variable orthogonal to the boundary of the integration region in the space of conformal parameters z^a.

Integrating (39) we obtain in the gauge (35)

$$\Phi(\varphi)=-D\left\{\frac{1}{48\pi}\int_\Sigma d^2z\frac{1}{2}(\partial_a\varphi)^2+\frac{1}{8\pi t}\int_\Sigma d^2z\exp\varphi+\frac{1}{8(\pi t)^{1/2}}\oint_{\partial\Sigma}ds+\frac{1}{24\pi}\oint_{\partial\Sigma}dz\varphi\varkappa_f-\frac{1}{8\pi}\oint_{\partial\Sigma}ds\varkappa_g-\frac{1}{2}\ln A(\Sigma)\right\}+\text{const}, \tag{41}$$

where the constant does not depend on φ. Here $ds=dz\rho^{1/2}=dz\exp(\varphi/2)$, $\varkappa_f$ is the plane curvature of the boundary in the z^a parameter space,

$$\varkappa_g=n_a(t^b\nabla_b)t^a=(^1/_2\partial_N\varphi+\varkappa_f)/\rho^{1/2} \tag{42}$$

is the geodesic curvature; t^a is the tangent vector to the boundary ($t^a=dz^a/ds$, $t^at^a=1$); ∂_N is the plane derivative in the direction of the outer normal to the boundary

$$\partial_N=\rho^{1/2}\partial_n. \tag{43}$$

Let us note that the term in (41) containing $\ln A(\Sigma)$ cancels, upon substitution in (34), the factor $A(\Sigma)^{D/2}$ in Eq. (32). For that reason we omit it in what follows.

We turn now to the renormalization of the bare parameters k_0, m_0. The second and third terms in (41) produce divergences in the limit $t\to 0^+$. The appropriate counterterms needed to eliminate these divergences were introduced in Refs. 7 and 9 "by hand." It is remarkable that in a consistent derivation of the chromoelectric string from a nonabelian gauge theory the counterterms k_0, m_0 appear automatically [Eqs. (2), (3), (32)].

We define the renormalized quantities k, m with the help of the relations

$$\lim_{\substack{t\to 0\\ \delta\to 0}}\left(\alpha k_0-\frac{D}{8\pi t}\right)=\frac{D}{48\pi}\alpha k, \tag{44}$$

$$\lim_{\substack{t\to 0\\ m_0\to\infty}}\left(\frac{m_0}{2}-\frac{D}{8(\pi t)^{1/2}}\right)=\frac{D}{48\pi}m. \tag{45}$$

(The factor $D/48\pi$ is extracted here for convenience). After carrying out the renormalization we obtain

$$Z_{Q^\pm}\overset{\circ}{=}\int D\varphi\exp[-F(\varphi)]\exp[\pm\alpha\pi Q-\beta q],$$

where

$$F(\varphi)=-\frac{D}{48\pi}\int_\Sigma d^2z\left[\frac{1}{2}(\partial_a\varphi)^2-\alpha k\exp\varphi\right]+\frac{D}{48\pi}m\oint_{\partial\Sigma}dz\exp\frac{\varphi}{2}-\frac{D}{24\pi}\int_{\partial\Sigma}dz\varphi\varkappa_f+\left(\frac{D}{8\pi}-\beta\right)\oint_{\partial\Sigma}dz\left(\exp\frac{\varphi}{2}\right)\varkappa_g. \tag{46}$$

As was indicated in Sec. 1, the integral over φ in Eq. (46) should be evaluated to lowest order in the saddle-point approximation only, so that

$$Z_{Q^\pm}=\exp[\pm\alpha\pi Q-\beta q-F(\hat\varphi)], \tag{47}$$

where $\hat\varphi$ is the extremum point of $F(\varphi)$.

The variation $\delta F/\delta\varphi=0$ results in the Liouville equation:

$$\partial^2\varphi+\alpha k\exp\varphi=0, \quad z\in\Sigma \tag{48}$$

with a definite type of boundary conditions, which will be discussed in Sec. 3. It follows from (48) that the curvature scalar (40) equals

$$R=\alpha k=\text{const}. \tag{49}$$

The sign of the curvature is determined by the sign of the renormalized quantity k (the quantity α is considered positive by definition). Formally the quantity k may change sign under renormalization. It will be seen below that this implies no contradictions. Accordingly we consider in Sec. 3 both versions: $R>0$ and $R<0$. The final choice of the sign of the curvature should be made on the basis of physical considerations after calculating hadron scattering amplitudes in this framework.

3. SOLUTIONS OF THE EQUATION $\delta F/\delta\varphi=0$

1. $R>0$. In this case k is positive and the curvature equals

$$R=\alpha k=2/a^2. \tag{50}$$

The solution to Eq. (48) is a constant curvature metric

$$\hat\rho(z)=\exp[\hat\varphi(z)]=4(1+|z|^2/a^2)^{-2}=4(1+r^2/a^2)^{-2}, \tag{51}$$

where $r^2=|z|^2$, $z=x+iy$.

From the variation $\delta F/\delta\varphi$ in addition to Eq. (48) we also obtain the boundary condition on φ:

$$\frac{D}{48\pi}\delta_B(z)\left[\frac{m}{2}\exp\frac{\varphi}{2}-\partial_N\varphi-2\varkappa_f\right]-\frac{1}{2}\left(\frac{D}{8\pi}-\beta\right)\partial_N\delta_B(z)=0, \quad z\in\partial\Sigma. \tag{52}$$

The last term in (52) is obtained by taking into account (42) and the relations[9b]

$$\oint_{\partial\Sigma}ds\varkappa_g=-\frac{1}{2}\int_\Sigma d^2z\varphi\,\partial_N\delta_B(z)$$

+ terms independent of φ. Taking into consideration that the metric (51) is not singular on $\partial\Sigma$ we conclude that (52) is equivalent to the two equations

$$\tfrac{1}{2}m \exp(\varphi/2) - \partial_N\varphi - 2\varkappa_f = 0, \quad z \in \partial\Sigma, \tag{53}$$

$$\beta = D/8\pi. \tag{54}$$

The first of these has a solution when the boundary $\partial\Sigma$ is a circle in the parameter space (x,y). Then

$$\varkappa_f = 1/r, \quad \partial_N = \partial_r, \quad \exp(\hat\varphi/2) = 2(1 + r^2/a^2)^{-1}. \tag{55}$$

Upon substitution of (55) into (53) we obtain the equation

$$r^2 + \tfrac{1}{2}ma^2r - a^2 = 0, \tag{56}$$

which allows one to find the radius of the circle $r_0^2 = x^2 + y^2$, $x, y \in \partial\Sigma$,

$$r_0 = -ma^2/4 + [(ma^2/4)^2 + a^2]^{1/2} > 0. \tag{57}$$

We shall also evaluate the geodesic curvature of the boundary $\partial\Sigma$. According to (42), (55) we have

$$\varkappa_g = (a^2 - r^2)/2r_0a^2.$$

Using Eq. (56) we obtain

$$\varkappa_g = m/4. \tag{58}$$

In this way, the renormalized mass of the quark located at the end of the string determines the geodesic curvature of the boundary $\partial\Sigma$. Upon substitution of the solutions (51), (54), (58) into the expression (46) for $F(\varphi)$ and evaluation of the simple integrals

$$m\oint_{\partial\Sigma} dz \exp\frac{\hat\varphi}{2} = m\int_0^{2\pi} r_0\, d\vartheta \frac{2}{(1+r_0^2/a^2)} = \frac{4\pi m r_0}{(1+r_0^2/a^2)} = 4\oint_{\partial\Sigma} ds\varkappa_g,$$

$$\oint_{\partial\Sigma} dz\varkappa_f\hat\varphi = \int_0^{2\pi} r_0\, d\vartheta \frac{2}{r_0}\ln\frac{2}{(1+r_0^2/a^2)} = 2\pi\ln\frac{4}{(1+r_0^2/a^2)^2},$$

$$\frac{D}{48\pi}\int_\Sigma d^2z\left[\frac{1}{2}(\partial_a\hat\varphi)^2 - \alpha k\exp\hat\varphi\varphi\right] = \frac{D}{6\pi a^4}\left[\int_\Sigma d^2z\frac{(r^2-a^2)}{(1+r^2/a^2)^2}\right]$$

$$= \frac{D}{6}\left[\ln(1+r_0^2/a^2) - \frac{2r_0^2}{(a^2+r_0^2)}\right], \quad d^2z = r\,dr\,d\vartheta,$$

$$\frac{1}{2}\int_\Sigma d^2z\,\hat g^{1/2}R(\hat g) = \frac{1}{a^2}\int_\Sigma d^2z\hat\rho = \frac{4\pi r_0^2}{(a^2+r_0^2)}, \tag{59}$$

we find that

$$F(\hat\varphi) = -\frac{D}{6}\left[\ln\left(1+\frac{r_0^2}{a^2}\right) - \frac{2r_0^2}{(a^2+r_0^2)}\right] - \frac{D}{6}\ln\frac{2}{(1+r_0^2/a^2)} + \frac{D}{12\pi}\oint_{\partial\Sigma} ds\varkappa_g$$

$$= \frac{D}{12\pi}\left[\frac{1}{2}\int_\Sigma d^2z\hat g^{1/2}R(\hat g) + \oint_{\partial\Sigma} ds\varkappa_{\hat g}\right] + \frac{D}{6}\ln 2 = \frac{D}{6}\chi + \text{const}. \tag{60}$$

Here χ is the Euler characteristic (15).

The variation $\delta\ln Z_Q/\delta\alpha = 0$ in (46), (47) gives rise to the quantization condition, which in contrast to expression (4) contains the renormalized tension coefficient (44):

$$\pi|Q| = \frac{Dk}{48\pi}\int_\Sigma d^2z g^{1/2}. \tag{61}$$

For $|Q| > 1$ the region $\Sigma_{Q=1}$ covers Q once.

The condition (61) (for $|Q| = 1$) makes it possible to express the Lagrange multiplier α in terms of the renormalized parameters k and m.

Indeed, from (61) and the last integral in (59) we have

$$1 = \frac{Dk}{48\pi^2}\int_\Sigma d^2z\hat g^{1/2} = \frac{Dka^2r_0^2}{12\pi(a^2+r_0^2)}. \tag{62}$$

Taking into account the relation (50), which implies $ka^2 = 2/\alpha$, we obtain from here an equation for the quantity α:

$$6\pi\alpha = Dr_0^2/(r_0^2+a^2).$$

Its positive solution may be found with expressions (56), (57) taken into account. It has the form

$$\alpha = \frac{m^2}{16k}\left[\frac{4Dk}{3\pi m^2} - 1 + \left(1 + \frac{8kD}{3\pi m^2}\right)^{1/2}\right]. \tag{63}$$

For $kD/m^2 \ll 1$ we obtain from here

$$\alpha = \frac{D}{6\pi}\left(1 - \frac{Dk}{3\pi m^2} + \ldots\right);$$

for $kD/m^2 \gg 1$ we have $\alpha \approx D/12\pi$; for $4kD/3\pi m^2 = 1$ we have $\alpha = D/4\sqrt{3}\pi$.

2. $R < 0$. In this case we have $k < 0$ and the curvature scalar equals

$$R = -\alpha|k| = -2/a^2. \tag{64}$$

Equation (48) takes on the form

$$\partial^2\varphi = (2/a^2)e^\varphi, \quad z \in \Sigma. \tag{65}$$

For the solution of this equation it is convenient to make use of the metric of the Lobachevski plane in the Klein model

$$\hat\varphi = \ln\frac{a^2}{y^2} = \ln\left|\frac{2a}{z - z^*}\right|^2, \quad \operatorname{Im} z > 0. \tag{66}$$

In this model the boundary of the region Σ is a straight line parallel to the real axis: $y = \text{const} \equiv y_0$. Then $\varkappa_f = 0$ and $\partial_N = -\partial_y$. Equation (52) takes the form

$$\frac{D}{48\pi}\delta(y-y_0)\left[\frac{m}{2}\exp\frac{\varphi}{2} + \partial_y\varphi\right] + \frac{1}{2}\left(\frac{D}{8\pi} - \beta\right)\partial_y\delta(y-y_0) = 0. \tag{67}$$

We require that the metric (66) have no singularities. Then one must set $y_0 \neq 0$. In that case Eq. (67) reduces to the equations

$$\tfrac{1}{2}m\exp(\varphi/2) + \partial_y\varphi = 0, \tag{68}$$

$$\beta = D/8\pi. \tag{69}$$

Substituting $\hat\varphi$ in the form (66) into the equation (68) we find

$$1/a = m/4, \quad R = -\alpha|k| = -m^2/8. \tag{70}$$

The geodesic curvature of the boundary equals

$$\varkappa_g = -\frac{1}{2}\frac{\partial\varphi}{\partial y}\exp\left(-\frac{\varphi}{2}\right) = \frac{1}{a} = \frac{m}{4}. \tag{71}$$

We have obtained the same expression for $\varkappa_g$ as in the case $R > 0$ [see (58)]. It is most interesting that for $R < 0$ the renormalized mass of the quark determines not only the geodesic curvature of the boundary $\partial\Sigma$, but also the curvature scalar R. Comparison of the expressions (71) and (64) leads to the relation

$$\varkappa_g^2 = -R/2, \tag{72}$$

which is characteristic of the Beltrami pseudo-sphere. (Regarding this surface see, e.g., Ref. 14). This circumstance permits making the region of definition of the parameters $z = x + iy$ in (66) more precise. For winding number $Q = 1$ the properties of the pseudo-sphere imply $0 \leqslant x \leqslant 2\pi a$, $a \leqslant y < \infty$, $y_a = a$. For arbitrary Q we obtain $0 \leqslant x \leqslant 2\pi a Q$, while the range of variation of the coordinate y remains the same as before. Substitution of the constant negative curvature metric (66) into the effective action (46), with Eqs. (70) and (71) taken into account, leads to a result coinciding in form to that obtained in Sec. 3, §1 for the case of positive curvature:

$$\begin{aligned} F(\hat\varphi) = &-\frac{D}{48\pi}\int_\Sigma d^2z\left[\frac{1}{2}(\partial_a\hat\varphi)^2 + \alpha|k|\exp\hat\varphi\right] \\ &+\frac{Dm}{48\pi}\oint_{\partial\Sigma} ds \\ &= \frac{D}{48\pi}\left[\int_\Sigma d^2z\left(-\frac{4}{y^2}\right) + 4\oint_{\partial\Sigma} ds\varkappa_{\hat g}\right] \\ &= \frac{D}{12\pi}\left[\frac{1}{2}\int_\Sigma d^2z g^{1/2}R(\hat g) + \oint_{\partial\Sigma} ds\varkappa_{\hat g}\right] = \frac{D}{6}\chi. \end{aligned} \tag{73}$$

However, we must note that for a manifold with the topology of a disc the Euler characteristic satisfies $\chi = 1$, while for the Beltrami pseudo-sphere it satisfies $\chi = 0$.

The quantization rule (61) holds as before; it should be emphasized here that for $R < 0$ the terms in the first line in expression (17) change places, i.e.,

$$Z_{Q^+} \propto \exp(-\alpha\pi Q) = \exp(-\alpha\pi|Q|);$$
$$Z_{Q^-} \propto \exp(\alpha\pi Q) = \exp(-\alpha\pi|Q|).$$

Moreover the variant with $R < 0$ is distinguished by the fact that in this case a relation arises between the parameters m^2 and k (this was already noted in Ref. 9b). Indeed, from the condition (61) we obtain

$$1 = \frac{D|k|}{48\pi^2}\int_\Sigma d^2z\exp\hat\varphi = \frac{D|k|}{48\pi^2}\int_0^{2\pi a} dx\int_a^\infty dy\frac{a^2}{y^2} = \frac{D|k|}{24\pi}a^2. \tag{74}$$

With (64) taken into account we have $\alpha|k| = 2/a^2$, which gives

$$\alpha = D/12\pi. \tag{75}$$

On the other hand, expression (70) means that

$$\alpha = m^2/8|k|. \tag{76}$$

As a result we obtain

$$|k| = 3\pi m^2/2D, \quad k \approx 1{,}2m^2 \text{ for } D = 4. \tag{77}$$

Thus in this case there is only one renormalized parameter, whose numerical value should be determined by comparison with experimental data.

The constant q entering expressions (32) and (47) is easily evaluated by taking into account that $\hat\rho^{1/2} = a/y$ [see (66)]:

$$\begin{aligned} q &= \oint_{\partial\Sigma} ds\varkappa_{\hat g} = \frac{1}{a}\oint_{\partial\Sigma} ds = \frac{1}{a}\int_0^{2\pi a|Q|} dx\hat\rho^{1/2} \\ &= \frac{1}{a}\int_0^{2\pi a|Q|} dx\frac{a}{y}\bigg|_{y=a} = 2\pi|Q|. \end{aligned} \tag{78}$$

Collecting together all the results (75), (69), (78), (73) obtained above we arrive at the final expression for (17), (47):

$$\begin{aligned} Z_{|Q|} &= Z_{Q^+} + Z_{Q^-} = 2\exp[-\alpha\pi|Q| - \beta q - F(\hat\varphi)] \\ &= 2\exp[-D|Q|(1/12 + 1/4 + 1/6\chi)] = 2\exp[-D|Q|(1/6\chi + 1/3)]. \end{aligned} \tag{79}$$

Hence

$$Z = \sum_{|Q|=1}^{\infty} Z_{|Q|} = \frac{2b}{(1-b)}, \quad b = \exp\left[-D\left(\frac{\chi}{6} + \frac{1}{3}\right)\right]. \tag{80}$$

Note that for $R > 0$ the quantity q has a more complicated form than (78). In that case we obtain, according to (59)

$$q = \oint_{\partial\Sigma} ds\varkappa_{\hat g} = \frac{\pi m r_0 a^2}{(a^2 + r_0^2)}|Q|.$$

It follows from here that $q = |Q|f(k,m)$, where f is a function of k and m whose explicit form should be found by making use of the relations (57), (50), (63). It should be noted that for $R < 0$ the results obtained here look much simpler than for $R > 0$.

4. THE GELL-MANN-LOW FUNCTION FOR THE NONPERTURBATIVE PHASE OF THE GAUGE FIELD THEORY

In Sec. 1 it was noted that the bare string tension coefficient (3) has the form[1,5]

$$k_0 = (e^2/2\delta^2)C_2(F). \tag{81}$$

Here $C_2(F) = (N^2 - 1)/2N$ is the quadratic Casimir operator in the fundamental representation of the $SU(N)$ group. In the renormalization process (44) we make the substitution

$$k_0 \to \tilde k \equiv (D/48\pi)k \tag{82}$$

(the factor $D/48\pi$ was extracted for convenience in further calculations). Here $|\tilde k|$ is the physical string tension coefficient, whose numerical value should be fixed by comparison with experimental data that are relevant to distances of the order of the radius of confinement ($1 \sim R_c$). Since $|\tilde k|$ is a dimensional quantity it is natural to write it in the form, analogous to (81)

$$|\tilde k| = [e^2(l)/2l^2]C_2(F), \tag{83}$$

where $e(l)$ is the renormalized charge. In a nonperturbative analysis of a field theory with confined quarks it is natural to consider a renormalization scheme in which the string tension is fixed[3]:

$$d|\tilde k|/dl = 0. \tag{84}$$

From this condition and from Eq. (83) we easily find the following expression for the Gell-Mann–Low function

$$\beta(e) \equiv -l\frac{de}{dl} = -e(l). \quad (85)$$

It coincides with the first term in the expansion of the β-function in powers of the charge, obtained previously in the strong coupling approximation for the Hamiltonian formulation of the gauge field theory on a lattice.[3,4] The Hamilton formulation is distinguished by being able to separate the contributions from the chromoelectric and chromomagnetic fields. In the strong coupling approximation the contribution of the electric field dominates. At distances $l \ll R_c$, where the charge is small, it is also necessary to take into account the contribution of the magnetic field. In that case, as was shown in Ref. 3, a smooth transition takes place from expression (85) to the standard formulas for $\beta(e)$ in the weak coupling approximation:

$$\beta(e) = -b_0e^3 - b_1e^5 \ldots,$$
$$b_0 = \frac{11}{48\pi}N, \quad b_1 = \frac{34}{3}\left(\frac{N}{16\pi^2}\right)^2.$$

The agreement between our results and those of Refs. 3 and 4 is a natural consequence of the fact that the chromoelectric string discussed above was consistently derived[1,2,5] from a nonabelian gauge theory (to leading order in $1/N$).

5. CONCLUSION

Let us indicate the differences between the hadron string and the traditional (mathematical) string models, in addition to properties (4) and (5). In those models cancellation of divergences, arising from summing over the insertions of the string world sheet, is achieved by introducing counterterms proportional to the area[7] and perimeter[9] of the sheet. These counterterms are introduced, in fact, "by hand." In contrast to this, in our approach the counterterms k_0, m_0 appear automatically in the process of deriving the string Lagrangian. The second difference has to do with the use of the unphysical limit $D \to \infty$, which is needed to justify the saddle-point approximation in Ref. 9b. In application to the hadron string this requirement is superfluous, since according to the property (5) the summation over surfaces should be restricted just to the contributions from the constant curvature metric. This metric is an extremum of the effective action (46), therefore no further corrections need be considered.

According to the result (80) the partition function of the hadron string Z depends only on the Euler characteristic χ of the string world sheet. The quantity χ is a topological invariant. For this reason one should conclude that in the neighborhood $\varphi \sim \tilde{\varphi}$ the theory of the chromoelectric string (in a space of dimension $D = 4$) should belong to the class of conformally invariant theories with central charge $c = 0$. Nor can one exclude a connection with the topological theories of Witten.[15]

It was noted in Ref. 16 that in conformal theories the exponential growth of the degree of degeneracy $g(n)$ of the energy levels E_n for $n \to \infty$ is determined by the central charge c of the Virasoro algebra

$$g(n) \propto \exp(2\pi^2 nc/3)^{1/2}.$$

For $c = 0$ it is to be expected that this law should go over to a power-law dependence on n. In addition, the Virasoro algebra is closed (for $D = 4$). All in all, this may give rise to dramatic consequences (in a positive sense) with regard to the properties of the hadron scattering amplitudes, which should be studied next.

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APPENDIX 1

Here we clarify the connection between the partition function (1) and the expression for the Euclidean hadron field correlator

$$K(x_1, \ldots, x_n) = i^n \langle \Psi^+(x_1)\Psi(x_1) \ldots \Psi^+(x_n)\Psi(x_n)\rangle, \quad (A1)$$

which describes the interaction of color singlet states of quark-antiquark pairs

$$\Psi^+(x)\Psi(x) \equiv \Psi_c^+(x)\Psi_c(x), \quad c = 1, \ldots, N. \quad (A2)$$

It was shown in Ref. 2 that to lowest order in the quasiclassical expansion in the parameter $1/N$ the connected part of the correlator (A1) is represented by the functional integral

$$K(x_1, \ldots, x_n) = BN \int Dx_\mu(\gamma) D\lambda(\gamma) Dx_\mu(z) \times \exp\{-S[x(z), x(\gamma), \lambda(\gamma)]\}, \quad (A3)$$

$$S[x(z), x(\gamma), \lambda(\gamma)] = k_0 \int_\Sigma d^2z h^{1/2}[x(z)] + \frac{1}{2}\oint_{\partial\Sigma} d\gamma \left[\frac{\dot{x}^2(\gamma)}{\lambda(\gamma)} + \lambda(\gamma) m_0^2\right]. \quad (A4)$$

Here $x_\mu(\gamma)$ are the coordinates of the quark trajectory, forming a closed contour Γ, passing through the fixed points x_k $(k = 1,\ldots,n)$, with $x_k = x(\gamma k)$;$\lambda(\gamma)$ is the metric on the contour Γ with parametrization γ. The quantity $h[x(z)]$ is defined by Eqs. (2) and (4); B is a normalization constant. The action (A4) should be viewed together with the supplementary conditions (4), (5). According to the character of the saddle configuration of the gauge field giving rise to the action (a4), the contour Γ should be viewed as the boundary of the surface Σ, i.e., $\Gamma = \partial\Sigma$. To take this circumstance into account one should introduce into the integral (A3) the function $\delta[x_\mu(\gamma) - x_\mu[z(\gamma)]]$ and with its help integrate over $x_\mu(\gamma)$. As a result we obtain

$$K(x_1, \ldots, x_n) = BN \int Dx_\mu(z) D\lambda(\gamma) \exp\{-S[x(z), \lambda(\gamma)]\}, \quad (A5)$$

where the integral over $x_\mu(z)$ is extended also over the last factor in Eq. (A4), with the points x_k on $\partial\Sigma$ held fixed. As a consequence Eq. (A5) may be written in the form

$$K(x_1, \ldots, x_k, \ldots, x_n) = \left\langle \int_\Gamma \prod_{k=1}^{n} d\gamma_k \lambda(\gamma_k) \delta(x_k - x(z(\gamma_k))) \right\rangle. \quad (A6)$$

Going over to the momentum representation we obtain

$$K(p_1,\ldots,p_k,\ldots,p_n)$$

$$=\left\langle \int_\Gamma \prod_{k=1}^{n} d\gamma_k \lambda(\gamma_k) \frac{\exp[-ip_k x(z(\gamma_k))]}{(2\pi)^2} \right\rangle, \qquad \text{(A7)}$$

where the averaging operation is defined by an integral of the type (A5), but with points no longer held fixed.

In order to evaluate the correlator (A7) it is necessary to learn first how to calculate the simpler expression for the partition function

$$Z=\langle 1\rangle \equiv \int Dx_\mu(z) D\lambda(\gamma) \exp\{-S[x(z),\lambda(\gamma)]\}, \qquad \text{(A8)}$$

which is the main object of study of the present work.

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