Propagation of femtosecond solitons in single-mode fibers

A. B. Grudinin, V. N. Men'shov, and T. N. Fursa

Institute of General Physics, USSR Academy of Sciences
(Submitted 14 June 1989)
Zh. Eksp. Teor. Fiz. **97**, 449–454 (February 1990)

We show that it is possible to find an analytic solution of the nonlinear Schrödinger equation that takes account of third-order dispersion and relaxation of the nonlinearity of the medium. We identify the domain of existence of nonlinear solitary waves, which take the form of "bright" and "dark" solitons.

With the current utilization of optical fibers as the transmission medium in long-distance communications systems, nonlinear (soliton) operating modes appear quite promising—they make nondispersive propagation of light pulses feasible at very high data rates, they improve noise immunity, and they make new frequency bands accessible.[1] The development of methods for generating ultrashort light pulses[2–5] (shorter than 100 fsec) and ongoing improvement in the technology required to produce optical fibers meeting prior specifications have provided a powerful impetus for research into the physics of nonlinear interactions at femtosecond time-scales.

The possibility that a high-frequency signal might develop and evolve with a soliton envelope in an optical fiber was first examined by Hasegawa and Tappert.[6] There it was shown that under certain circumstances, the process is adequately described by the nonlinear Schrödinger (NLS) equation:

$$i\left(\frac{\partial\Phi}{\partial z}+\beta_0'\frac{\partial\Phi}{\partial t}\right)-\frac{\beta_0''}{2}\frac{\partial^2\Phi}{\partial t^2}+\frac{\beta_0 n_2}{n_0}|\Phi|^2\Phi=0, \tag{1}$$

where $\Phi(z,t)$ is the slowly varying amplitude of the electric field distribution

$$E(z,t)=\Phi(z,t)\exp i(kz-\omega_0 t)$$

along the fiber, t and z are the temporal and spatial coordinates, respectively,

$$\beta_0^{(n)}=\left.\frac{\partial^n k}{\partial\omega^n}\right|_{\omega=\omega_0}$$

provides a quantitative measure of nth order dispersion at the signal carrier frequency ω_0, and $v_g\equiv(\beta_0')^{-1}$ is the group velocity of the medium. The physical meaning of the existence of soliton excitations in a single-mode optical fiber is that under certain conditions, the dispersive spreading of a wave packet can be exactly cancelled by nonlinear self-induced compression due to the (Kerr) dependence of the medium's refractive index on the field strength: $n=n_0+n_2|E|^2$.

The inverse scattering method (ISM) provides a mathematically sound basis for the description of nonlinear wave fields; it has been used to prove rigorously that the NLS equation (1) supports soliton solutions, and to investigate the dynamics of N-soliton processes.[7]

In the simplest case, if the group dispersion is negative ($\beta_0''<0$), Eq. (1) is satisfied by the function

$$\Phi(z,t)=\Phi_0\,\mathrm{sech}\left(\frac{\tau}{\tau_0}\right)\exp i(\mu z+\Delta\omega\tau)\equiv\Phi^b(z,\tau), \tag{2}$$

the so-called "bright" soliton; $\tau=t-z/v$. Its parameters are related to the coefficients in Eq. (2) by

$$\tau_0^{-2}=\frac{2\mu|\beta_0''|-\alpha^2}{|\beta_0''|^2},$$

$$\Phi_0^2\equiv\Phi^2(0,0)=\frac{n_0\beta_0''}{n_2\beta_0\tau_0^2},\qquad \Delta\omega=\frac{\alpha}{\beta_0''}, \tag{3}$$

where $\alpha=1/v_g-1/v$, v is the soliton envelope velocity, $\mu>\alpha^2/2|\beta_0''|$ is the shift [relative to $k(\omega_0)=\beta_0$] in the wave number, and $\Delta\omega$ is the frequency shift (relative to ω_0) due to pulse self-modulation. Ultrashort pulses of this kind in optical fiber were first detected experimentally in 1980.[8]

If the group dispersion is positive ($\beta_0''>0$), the solution of Eq. (1) is a "dark" soliton,[9]

$$\Phi(z,t)=\Phi_0\left[1-i\frac{\tau_1}{\tau_0}\tanh\left(\frac{\tau}{\tau_0}\right)\right]\exp i(\mu z+\Delta\omega\tau), \tag{4}$$

with a complex temporal envelope. The parameters of this solution are

$$\frac{1}{\tau_0^2}+\frac{1}{\tau_1^2}=\frac{\alpha\Delta\omega+\mu}{\beta_0''},$$

$$\Phi_0^2\equiv\Phi^2(0,0)=\frac{n_0\beta_0''}{n_2\beta_0\tau_1^2},\qquad \Delta\omega=\frac{\alpha}{\beta_0''}-\frac{1}{\tau_1}. \tag{5}$$

A practical embodiment of dark-soliton excitations in an optical waveguide was recently reported by Krokel *et al.*[10]

The development of optical communications technology has stimulated theoretical studies of pulse dynamics in fiber, and these have refined and supplemented the model implicit in Eq. (1). There exist operating regimes in which the NLS equation (1) is incapable of describing the pulse dynamics, which must be modeled by introducing new terms. For example, at frequencies for which $\beta_0''\approx 0$, a description of the dispersive deformation of a signal requires that the next order of dispersion (terms $\sim\beta_0'''$) be taken into account. The fact that the nonlinearity is not instantaneous (due to the term $\sim\beta_0 n_2/\omega_0 n_0$) can lead to asymmetrically shaped signals and the formation of shock waves.[11] It becomes crucially important to take these terms into consideration [essentially terms of higher order in an expansion like (1)] when one attempts to describe the propagation of high-power femtosecond light pulses,[1] as effects due to third-order dispersion become comparable to those due to second-order dispersion (as is clear from a comparison of the second- and third-order dispersion lengths, i.e., $z_g^{(2)}=\tau_0^2/\beta_0''$ and $z_g^{(3)}=\tau_0^3/\beta_0'''$), and the nonlinear response time of the medium is of the order of several femtoseconds.[12] These effects result in the following modified version of the NLS equation (1):

 0038-5646/90/020249-03$03.00

$$i\left(\frac{\partial\Phi}{\partial z}+\beta_0'\frac{\partial\Phi}{\partial t}\right)-\frac{\beta_0''}{2}\frac{\partial^2\Phi}{\partial t^2}-i\frac{\beta_0'''}{6}\frac{\partial^3\Phi}{\partial t^3}+\frac{\beta_0 n_2}{n_0}|\Phi|^2\Phi+\frac{2i\beta_0 n_2}{\omega_0 n_0}\frac{\partial}{\partial t}(|\Phi|^2\Phi)=0. \qquad (6)$$

The role played by the terms $\sim\beta_0'''$ and $\sim\omega_0^{-1}$ in this equation has been analyzed by Hasegawa and Kodama[13] using soliton perturbation theory based on the ISM. The variations in pulse velocity and occurrence of frequency modulation obtained in Ref. 13 are consistent with numerical results[14] obtained for $\beta_0'''=0$. It has been pointed out[15] that with $\beta_0'''=0$, Eq. (6) is completely integrable, and the first three integrals of the motion for this equation have indeed been found[16]; furthermore, it has been shown[16] that soliton states like (2) can exist (in contrast to the NLS equation) even for $\beta_0''>0$. Another instance in which the last term in (6) can be neglected (formally, $\omega_0\to\infty$ for $\beta_0'''\neq 0$) has been the subject principally of numerical studies.[17,18] Finally, we wish to draw attention to Ref. 19, in which it was first demonstrated (apparently for the first time) that exact soliton-like solutions exist for Eq. (6).

In our present work we have attempted to solve the modified NLS equation (6) analytically. We have proposed a special procedure, whereby it becomes possible to write out the simplest solutions of (6) explicitly, be they crests or troughs in the amplitude of the temporal envelope (one-soliton solutions), and to determine the region in which a given solution exists. In addition, we have related our results to those obtained by other authors.[6,11,19]

We therefore seek traveling-wave solutions, to the original equation (6),

$$\Phi(z,t)=f(\tau)\exp i(\mu z+\nu\tau), \qquad (7)$$

where "running" time is $\tau=t-z/v$. Plugging Eq. (7) into (6), we obtain an ordinary differential equation for the function $f(\tau)$,

$$f'''+iaf''+bf'+icf+d(|f|^2f)'+i\gamma|f|^2f=0, \qquad (8)$$

with coefficients

$$a=3\left(\nu-\frac{\beta_0''}{\beta_0'''}\right),$$
$$b=3\nu\left(2\frac{\beta_0''}{\beta_0'''}-\nu\right)-\frac{6\alpha}{\beta_0'''},$$
$$c=\nu^2\left(3\frac{\beta_0''}{\beta_0'''}-\nu\right)-\frac{6}{\beta_0'''}(\mu+\alpha\nu), \qquad (9)$$
$$d=-\frac{12\beta_0 n_2}{\omega_0 n_0\beta_0'''},$$
$$\gamma=\frac{6\beta_0 n_2}{n_0\beta_0'''}\left(1-\frac{2\nu}{\omega_0}\right).$$

We can now show that when certain relationships hold among the coefficients (9), some (although of course not all) solutions of the third-order equation (8) are identical to the solutions of a somewhat simpler second-order equation. To this end, we rewrite (6) in the form

$$\frac{d}{d\tau}\{f''+imf'+[b+m(a-m)]f+d|f|^2f\}+i[(a-m)(f''+imf)+cf+\gamma|f|^2f]=0, \qquad (10)$$

where m is real. It is then clear that when

$$\frac{c}{a-m}=b+m(a-m),$$
$$d=\frac{\gamma}{a-m} \qquad (11)$$

the problem reduces to an investigation of the equation

$$f''+imf'+[b+m(a-m)]f+d|f|^2f=0. \qquad (12)$$

Substitution of the explicit form of the coefficients (9) into (11) yields a relation between ν and m,

$$m=2\nu-\frac{\omega_0}{2}(\eta-1), \qquad (13)$$

as well as one between μ and α,

$$\alpha=-\frac{2\mu}{\omega_0}+\frac{\omega_0\beta_0''}{4}\left(1-\frac{1}{\eta}\right). \qquad (14)$$

Here we have introduced the convenient dimensionless quantity $\eta=6\beta_0''/\omega_0\beta_0'''$, which at a given carrier frequency ω_0 is a parameter of the fiber. Equation (12) is the time-independent form of the NLS equation corresponding to (1), so the desired solutions of Eq. (6) will take the same form as (2) and (4).

Next, we determine the conditions under which self-localized states of the temporal envelope resembling (2) or (4) can arise in a system described by Eq. (6). We can make use of (9) and (14) to calculate the parameters of the "bright" soliton $\Phi^b(z,\tau)$ given by (2):

$$\frac{1}{\tau_0^2}=-\left[\frac{2\mu\eta}{\beta_0''}+(\Delta\omega)^2\right],$$
$$\Phi_0^2=-\frac{n_0\beta_0''}{n_2\beta_0\tau_0^2\eta}, \qquad (15)$$
$$\Delta\omega=\frac{\omega_0(\eta-1)}{4}.$$

Clearly, the solitary wave $\Phi^b(z,\tau)$ appears when $\beta_0'''<0$. It is also apparent from (15) that $\Delta\omega$ and μ determine the parameters τ_0 and Φ_0 of the soliton pulse, as well as its velocity:

$$\alpha=-2\frac{\mu}{\omega_0}-\frac{\beta_0''}{\eta}\Delta\omega$$

[see (14)]. If with the aid of (15) we now rewrite Eq. (13) in the form $m=2(\nu-\Delta\omega)$, the physical meaning of m becomes clear: it governs the frequency modulation of the nonlinear pulse. Using (14) and (15), we can determine the range of propagation velocities accessible to a bright soliton in a fiber as a function of the parameter η characterizing the material:

$$(1-\eta)(3+\eta)>\frac{96\alpha}{\omega_0^2|\beta_0'''|}.$$

In the limit as $\beta_0'''\to 0$ and $\omega_0\to\infty$ simultaneously, Eq. (6) goes into Eq. (1) with $\eta\to 1$, and the expressions (15) becomes identical to (3). Departures of the parameter η from unity attest to the need to take higher-order terms [by comparison with (1)] into account in the pulse-dynamics equation (6).

There is one aspect of this treatment to note here that we believe to be important. The condition $\beta_0'''<0$ is necessary for the existence of femtosecond soliton-like pulses. In conventional single-mode fibers, $\beta_0'''>0$ holds and only in overcompensated light guides with a flat dispersion curve can one have $\beta_0''' < 0$. Nonlinear femtosecond pulses are therefore only stable in structures of the latter variety, and their attenuation in conventional light guides,[12] all else aside, can be accounted for by the violation of the necessary condition $\beta_0''' < 0$.

Likewise, we can use (9), (13), and (14) to find typical parameters of a "dark" soliton $\Phi^d(z,\tau)$ as given by the model (6):

$$\frac{1}{\tau_0^2}+\frac{1}{\tau_1^2}=\frac{\mu\eta}{\beta_0''}+\frac{\Delta\omega}{\tau_1}+\frac{(\Delta\omega)^2}{2},$$
$$\Phi_0^2=\frac{n_0\beta_0''}{n_2\eta\tau_1^2\beta_0},\tag{16}$$
$$\nu=\Delta\omega,\quad \tau_1=m/2,$$
$$\Delta\omega=\frac{\omega_0(\eta-1)}{4}-\frac{1}{\tau_1}.$$

Soliton-like troughs of this kind exist for all cubic dispersions with $\beta_0''' > 0$, and they can take on velocities that correspond to the inequality

$$(\eta-1)(\eta+3)+\left(\frac{2m}{\omega_0}\right)^2>\frac{96\alpha}{\omega_0^2\beta_0'''}.$$

As m increases, the domain of existence of a dark soliton shrinks. The quantity $\tau_1 = m/2$ specifies the phase shift of the complex temporal envelope of a dark soliton relative to the rapid oscillations of the carrier at a particular point z, and it governs the depth of the trough in $\Phi^d(z,\tau)$ [see (16)]. In the limit as $\beta_0''' \to 0$ and $\omega_0 \to \infty$ $(\eta \to 1)$, Eqs. (16) go over to (5).

It is important to note that the procedure proposed above for finding soliton (or more accurately, soliton-like) solutions of the modified NLS equation, as well as the results derived therefrom, are valid only if one makes equivalent allowance for cubic dispersion and relaxation of nonlinear polarization in the medium. We know that the envelope of a femtosecond light pulse in a fiber can be stabilized in bright or dark soliton form because of the mutual cancellation of dispersive (second- or third-order) and nonlinear (due to stationary and non-stationary polarization of the medium) deformations of the pulse. Equation (10) formally reflects this cancellation process. Therefore, in particular, if the pertinent conditions of a problem make it possible to neglect higher-order terms in the expansion of (6), then a proper transition from (15) to (2) or (16) to (4) requires $\beta_0''' \to 0$ and $\omega_0 \to \infty$ simultaneously, with $\eta \to 1$; if the opposite is true, then the proposed computational method is inapplicable. For example, if we put $\beta_0''' = 0$ in Eq. (6) but $\omega = \text{const} \neq \infty$, then for $\beta_0'' < 0$ the solution of this equation is the function

$$\tilde{\Phi}^s(z,\tau)=\exp i[\mu z+\nu(\tau)]f(\tau),$$
$$f^2(\tau)=f^2(0)\frac{1+\chi}{\mathrm{ch}^2(\tau/\tau_0)+\chi},\tag{17}$$
$$\nu'(z)=\xi+\zeta f^2(\tau).$$

This result was obtained in Ref. 11, which also contains explicit expressions for the parameters ξ, $\zeta f(0)$, and χ. Even structurally, the functions $\Phi^s(z,\tau)$ and $\tilde{\Phi}^s(z,\tau)$ [compare (2) and (17)] differ are markedly different. Likewise, when $\omega_0 \to \infty$ but $\beta_0''' = \text{const} \neq 0$, it is all but impossible, in general, to solve (6) exactly.

The exact solutions of Eq. (6) obtained above are finite-energy solitary waves which retain their shape during propagation. Solutions with these properties have come to be known as solitons in the broad sense that this term is used, although they are scarcely true solitons in the sense of the ISM.[7] Nevertheless, the patently obvious analogy between the modified NLS equation (6) and the Hirota equation,[20] which is integrable within the scope of the ISM, must be pointed out; the distinction lies in the structure of the last term of Eq. (6).

Methodologically speaking, the very fact that Eq. (6) possesses exact solutions is important. In the femtosecond regime, the description of signal propagation in a fiber can start with the presence of stable soliton-like pulses like (2), (15) or (4), (16), making it possible to derive new analytical results, perhaps without even reverting to the use of a computer.

It is to be hoped that the results that have been obtained may serve in large measure to help prepare for an experimental search (especially at femtosecond pulse widths) for stable, localized pulses, which are ideal instruments for data transmission over fiber-optic communications lines. With regard to recently initiated work (see, e.g., Ref. 10) on the detection and investigation of dark solitons, our present analysis of pulse behavior when $\beta_0''' > 0$ is most timely.

[1] A. S. Belanov, E. A. Golovchenko, E. M. Dianov, *et al.*, Trudy Inst. Obshch. Fiz. AN SSSR **5**, 35 (1987).
[2] E. M. Dianov, A. Ya. Karasik, P. V. Manyshev, *et al.* Pis'ma Zh. Eksp. Teor. Fiz. **41**, 242 (1985) [JETP Lett. **41**, 294 (1985)].
[3] A. B. Grudinin, E. M. Dianov, D. V. Korotkin, *et al.*, Pis'ma Zh. Eksp. Teor.Fiz. **45**, 211 (1987) [JETP Lett. **45**, 260 (1987)].
[4] A. S. Gluveia-Neto, A. S. L. Gomes, and J. R. Taylor, Electron. Lett. **23**, 1034 (1987).
[5] B. Zysset, P. Beaud, and W. Hodel, Appl. Phys. Lett. **50**, 1027 (1987).
[6] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973).
[7] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov, *Theory of Solitons: the Inverse Scattering Method*, Consultants, New York (1984).
[8] L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, Phys. Rev. Lett. **45**, 1085 (1980).
[9] A. M. Kosevich, B. A. Ivanov, and A. S. Kovalev, *Nonlinear Magnetization Waves. Dynamical and Topological Solitons* [in Russian], Naukova Dumka, Kiev (1983).
[10] D. Krokel, N. J. Halas, G. Giuliani, and D. Grischkowsky, Phys. Rev. Lett. **60**, 29 (1988).
[11] D. Anderson and M. Lisak, Phys. Rev. **A27**, 1393 (1983).
[12] A. B. Grudinin, E. M. Dianov, D. V. Korotkin, *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. **46**, 175 (1987) [JETP Lett. **46**, 221 (1987)].
[13] A. Hasegawa and Y. Kodama, Proc. IEEE **69**, 1145 (1981).
[14] V. A. Vysloukh and V. N. Serkin, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 170 (1983) [JETP Lett. **38**, 199 (1983)].
[15] D. J. Kauo and A. C. Newell, J. Math. Phys. **19**, 798 (1978).
[16] E. A. Golovchenko, E. M. Dianov, A. M. Prokhorov, and V. N. Serkin, Dokl. Akad. Nauk SSSR **228**, 851 (1986) [Sov. Phys. Doklady **31**, 494 (1986)].
[17] P. K. A. Wai, C. R. Menyuk, H. H. Chen, and Y. C. Lee, Opt. Lett. **12**, 628 (1987).
[18] V. A. Vysloukh, Kvant. Elektron. (Moscow) **10**, 1688 (1983) [Sov. J. Quantum Electron. **13**, 1113 (1983)].
[19] D. N. Christodoulides and R. I. Joseph, Electron. Lett. **20**, 659 (1984).
[20] R. Hirota, J. Math. Phys. **14**, 805 (1973).

Translated by Marc Damashek