Calculation of the field strength and coherent radiation of relativistic bunches

V. N. Melekhin

Institute for Physical Problems, Academy of Sciences of the USSR

(Submitted 14 July 1989)
Zh. Eksp. Teor. Fiz. **97**, 757–765 (March 1990)

The Lienard-Wiechert potentials are used to obtain approximate formulas for the fields $\mathbf{E}(t)$ and $\mathbf{H}(t)$ in the vicinity of a point charge as functions of $\mathbf{u}(t)$, $\dot{\mathbf{u}}(t)$, $\ddot{\mathbf{u}}(t)$, and $\mathbf{r}$, where $\mathbf{u}$ is the velocity of the charge and $\mathbf{r}$ is the radius-vector joining at the instant t the charge and the observation point. Formulas are obtained for the bunches which permit one to determine the effect of the proper field. In particular we calculate the deceleration, which determines the power of the coherent radiation of the bunch, for a thread-like filament of angular length Φ moving on a circle. For $u \ll c$ the calculation is valid for arbitrary Φ. For $u \approx c$ a more exact expression is found that is valid only for small Φ, which, however, may satisfy the inequality $\Phi \gg (1 - u^2/c^2)^{3/2}$; when this inequality is satisfied the expression found coincides with the familiar asymptotic formula. Circumstances are indicated under which the results of the calculation are applicable to bunches moving in a plane undulator.

INTRODUCTION

In recent years in connection with the employment of a forced radiation undulator and various applications of powerful relativistic electron beams certain problems of classical electrodynamics have again become of interest. These include the determination of the power of spontaneous radiation of bunches, on whose value depend the initial conditions for the production of forced radiation, as well as the calculation of the proper field of charged bunches. In both cases it is advisable to make use of the field of a point charge, which is, in effect, the Green's function for problems of classical electrodynamics connected with application to charged bunches. However the Lienard-Wiechert potentials determine the field of a point charge only indirectly, since at every observation point it is necessary to solve a transcendental equation in order to determine the retardation time $\tau = t - t'$ that enters in these potentials.

In the general case τ can be determined only by numerical calculations. In calculating the radiation power of a point charge this difficulty is immaterial since the power is determined by integration of the current of the Poynting vector in the wave zone at a fixed value of τ. However in passing to bunches, whose length is comparable to the wavelength of the radiation field, the problem becomes substantially more complicated. Even in the simplest case of motion of relativistic bunches on a circle, one is able to calculate only the asymptotics for sufficiently long bunches, that significantly exceed the wavelength at the maximum of the radiation spectrum (see, e.g., Ref. 1), or one has to fall back on numerical calculations.[2] We note that difficulties also arise for the point charge in the calculation of the field in the near zone, where all the terms in the Lienard-Wiechert potential are of comparable size, while the value of τ changes significantly with a shift in the observation point.

In this paper we find an analytic solution to the indicated problems. In the vicinity of the point charge, moving with arbitrary velocity on the specified trajectory, we find approximate expressions for the electric and magnetic fields $\mathbf{E}(t)$ and $\mathbf{H}(t)$ at the instant of time t as functions of the distance to the charge r at that same time and the velocity of the charge $\mathbf{u}(t)$ and its derivatives $\dot{\mathbf{u}}(t)$ and $\ddot{\mathbf{u}}(t)$. The calculation is carried out using the method of expansion in a series in the small parameter τ/t_0, where t_0 is a characteristic time of variation of $\mathbf{u}(t)$ in motion along the trajectory; in this expansion we keep terms inclusive up through second order. Previously such formulas have been obtained only for the turning points, where the velocity of the charge vanishes[3–5] and the problem simplifies substantially — we note that mistakes were made in a number of papers leading to incorrect results.[4,5]

The new formulas for the field of a point charge solve, in particular, the above-mentioned problem: the calculation of the power radiated by a nonrelativistic bunch of arbitrary length moving on a circle. In the case of relativistic motion we find a more exact expression, applicable to bunches whose angular length Φ lies in the interval $0 < \Phi < \Phi_{max}$, where Φ_{max} satisfies the inequalities $1/\gamma^3 \ll \Phi_{max} \ll 2\pi$. For $\Phi \to 0$ this expression describes the power radiated by a point charge, for $\Phi \gg 1/\gamma^3$ it coincides with a previously found asymptotic formula,[1,2] and for intermediate values of Φ it agrees with the results of numerical calculations.[2] The circumstances are indicated under which this expression can be applied to the calculation of undulator radiation, and the powers of incoherent and coherent spontaneous radiation are compared.

1. THE NEAR FIELD OF A POINT CHARGE

We shall write the field of a point charge, determined by the Lienard-Wiechert potentials, in the form

$$\mathbf{E}(\mathbf{r}, t) = s^{-3}\{(c\mathbf{R} - R\mathbf{v})[c^2 - v^2 + (\mathbf{R}\dot{\mathbf{v}})] - sR\dot{\mathbf{v}}\}, \quad \mathbf{H} = \frac{1}{R}[\mathbf{R}\mathbf{E}]. \tag{1}$$

Here $\mathbf{r}$ is the radius-vector to the observation point from the point where the charge finds itself at the time t; $\mathbf{R}$ is the radius-vector to the same observation point from the point where the charge was at the time t' with $R = c(t - t')$; $\mathbf{v}(t')$ and $\dot{\mathbf{v}}(t')$ are the velocity and acceleration of the charge; $s = Rc - (\mathbf{R}\mathbf{v})$, where c is the velocity of light; the magnitude of the charge e is taken to be unity.

Using the Taylor series we express $\mathbf{R}$, $\mathbf{v}(t')$ and $\dot{\mathbf{v}}(t')$ in terms of $\mathbf{r}$, $\mathbf{u}$, $\dot{\mathbf{u}}$, and $\ddot{\mathbf{u}}$, where the velocity of the charge $\mathbf{u}$ and its derivatives refer to the time t. Substituting these expressions into Eq. (1) we obtain formulas for $\mathbf{E}$ and $\mathbf{H}$ in the

0038-5646/90/030423-05$03.00

form of a power series, in which we shall keep the main terms (order $1/r^2$), first order of smallness terms ($\propto \dot{\mathbf{u}}/r$) and second order of smallness terms ($\propto \ddot{\mathbf{u}}$ and $\propto \dot{\mathbf{u}}^2$). The terms of the expansion that are ignored are proportional to $r|\dot{\mathbf{u}}|^3$, $r(\dot{\mathbf{u}}\ddot{\mathbf{u}})$, $r\ddot{\mathbf{u}}$, $r^2\dot{\mathbf{u}}^4$, ... Whence follows that this approximation is legal for arbitrary dependence $\mathbf{u}(t)$ provided r is sufficiently small. The condition of smallness of r is given below. We shall find the retardation time $\tau = t - t'$ by representing τ in the form of the expansion $\tau = \tau_0 + \tau_1 + \tau_2$ ($\tau_0 \propto r$, $\tau_1 \propto r^2\dot{\mathbf{u}}$, $\tau_2 \propto r^3\dot{\mathbf{u}}^2$, $r^3\ddot{\mathbf{u}}$) and solving the equation $\mathbf{R}^2(\tau) = c^2\tau^2$, where the radius-vector $\mathbf{R}$ is represented by its Taylor series.

We omit the rather unwieldy intermediate steps and present the final formulas valid for arbitrary velocity $\mathbf{u}$:

$$\mathbf{E}=A\mathbf{r}+B\dot{\mathbf{u}}+D\ddot{\mathbf{u}},$$

$$\mathbf{H}=-\frac{A}{c}[\mathbf{ru}]+\frac{B}{c}[\mathbf{u}\dot{\mathbf{u}}]+\frac{[\mathbf{r}\dot{\mathbf{u}}](\mathbf{ru})}{\rho^3c^3}\left(1-\frac{3r^2(\mathbf{r}\dot{\mathbf{u}})}{2\rho^2c^2}\right)$$

$$+\frac{3r^4}{2\rho^5c^5}[\mathbf{r}\dot{\mathbf{u}}](\mathbf{u}\dot{\mathbf{u}})+\frac{D}{c}[\mathbf{u}\ddot{\mathbf{u}}]+\frac{r^2[\mathbf{r}\ddot{\mathbf{u}}]}{2\rho^3c^3}. \qquad (2)$$

The coefficients A, B, and D in these formulas are given by the following

$$A=\frac{1}{\gamma^2\rho^3}-\frac{1}{2\rho^5c^4}\{[\rho^2c^2-3(\mathbf{ru})^2](\mathbf{r}\dot{\mathbf{u}})+3r^2(\mathbf{ru})(\mathbf{u}\dot{\mathbf{u}})$$

$$+\frac{3}{4}r^4\dot{\mathbf{u}}^2-r^2(\mathbf{ru})(\mathbf{r}\ddot{\mathbf{u}})+r^4(\mathbf{u}\ddot{\mathbf{u}})\}$$

$$+\frac{3r^2}{8\rho^7c^6}\{[\rho^2c^2-5(\mathbf{ru})^2](\mathbf{r}\dot{\mathbf{u}})^2+10r^2(\mathbf{ru})(\mathbf{r}\dot{\mathbf{u}})(\mathbf{u}\dot{\mathbf{u}})-5r^4(\mathbf{u}\dot{\mathbf{u}})^2\},$$

$$B=-\frac{1}{4\rho^5c^{10}}\{2\rho^2r^2c^8-3r^4c^6(\mathbf{r}\dot{\mathbf{u}})-\gamma^6(\mathbf{u}\dot{\mathbf{u}})[8\rho^5c^5$$

$$+(\mathbf{ru})(15\rho^4c^4-10\rho^2c^2(\mathbf{ru})^2+3(\mathbf{ru})^4)]\},$$

$$D=\frac{2\gamma^4}{3c^3}+\frac{\gamma^4(\mathbf{ru})[3\rho^2c^2-(\mathbf{ru})^2]}{3\rho^3c^6}.$$

Here $\gamma = (1 - u^2/c^2)^{-1/2}$ is the relativistic factor and

$$\rho=r\left(1-\frac{u^2}{c^2}\sin^2\theta_0\right)^{1/2}$$

is the effective value of r, which depends on the angle θ_0 between the vectors $\mathbf{r}$ and $\mathbf{u}$.

For $\mathbf{u}(t)\equiv$const only the leading terms of order $1/r^2$ survive in Eq. (2) and describe in that case the field for arbitrary r. The results in that case, naturally, coincide with familiar formulas. Below we shall refer to these terms conditionally as the Coulomb field (for $\mathbf{E}$) and the Biot-Savart field (for $\mathbf{H}$) not only for uniform but also for accelerated motion. Together they form what we shall call the accompanying field, since for uniform motion it shifts together with the particle. We shall call the remaining terms the radiation field [in Eq.(1) this field depends on $\dot{\mathbf{v}}$ and in Eq. (2) it depends on $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$]. It is seen from Eq. (2) that both the accompanying field and the radiation field exhibit the "flattening" effect in the direction of motion with increasing velocity of the charge, since for $\theta_0 = 0$ we have $\rho = r$ while for $\theta_0 = \pi/2$ the effective distance is reduced ($\rho = r/\gamma$).

Let us compare the formulas (2) with the familiar expressions. This method of calculation was first used by Lorentz[3] for $\mathbf{u} = 0$ and $\dot{\mathbf{u}}\|\ddot{\mathbf{u}}$, i.e., at the turning point of the charge in rectilinear motion. Formula (2) for $\mathbf{E}(\mathbf{r},t)$ coincides with the expression obtained in Ref. 3. However thereafter in calculating the field $\mathbf{E}$ in the case of $\mathbf{u} = 0$, but for arbitrary $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ (see, e.g., the first edition of the monograph, Ref. 4, or Ref. 5) an error was made: in the expansion in the Taylor series the retardation time was taken equal to $\tau = \tau_0 = r/c$, whereas even for $\mathbf{u} = 0$ one must consider τ_1 and τ_2. In application to a uniformly charged sphere the obtained incorrect formulas by chance give the correct expression for the radiation reaction, which can be independently obtained from energy considerations. For this reason the error remained unnoticed for some time. In the third edition of Heitler's monograph (see Ref. 4) the error is corrected. We also note that if the field is calculated with the help of the potentials used in Ref. 6 in the derivation of the Darwin Lagrangian, then it will coincide with the first order terms in formula (2) if one expands them in u/c accurate to terms $\sim u^2/c^2$.

Let us estimate the near zone in which the approximate formulas (2) are valid. An analysis of the omitted terms shows that they are small compared to the terms that were kept provided

$$\rho \ll \frac{c^2}{\gamma^4\dot{u}} = \frac{mc^2}{eE_0\gamma},$$
$$\rho \ll \frac{c^2}{\gamma^3|\dot{\mathbf{u}}|} = \frac{mc^2}{eH_0\gamma^2}. \qquad (3)$$

Here ρ is the introduced above effective distance from the charge to the observation point; $\dot{u}$ is the longitudinal acceleration of the charge (it can be represented as the result of the motion of a particle of rest mass m in the longitudinal electric field E_0); $|\dot{\mathbf{u}}|$ is the transverse acceleration (represented as the result of motion in the transverse magnetic field H_0). The first of the inequalities (3) is valid for arbitrary velocity. The second inequality was obtained for $u \approx c$, and for $u \ll c$ it depends additionally on u/c, and in that case the near zone includes the whole circle on which the charge is rotating in the magnetic field. It should be emphasized that the wave zone, in which the radiation field exceeds significantly the accompanying field, nowhere intersects with the near zone.

The formulas (2) are convenient as they readily permit the determination of the radiation reaction for an arbitrary charge distribution. This is done simplest by staying with the following procedure. One calculates first the total force acting on some pair of elements of the charge. Here one combines the forces with which the elements of the charge act on each other, and most of the terms in the expression (2) mutually cancel. This is followed by a double integration over the volume occupied by the charge, which allows taking into account all such pairs. For a sufficiently small bunch or for the procedure of mass renormalization of the charged particle the velocity of all the elements of the charge may be taken to be the same, and for each pair only those terms in (2) contribute to the total force which do not change sign upon the replacement of $\mathbf{r}$ by $-\mathbf{r}$. Here the first order of smallness terms (corresponding to the force $\mathbf{f}_1$) depend both on the size and the shape of the region occupied by the charge, and this force determines the size of the electromagnetic mass of the charged particle that enters into the total observable mass, while for the bunch the force $\mathbf{f}_1$ gives the constituent of the radiation reaction which is not connected with energy losses to radiation.

The terms of second order of smallness determine the radiative friction force, which depends neither on the size nor the shape of the charge, and the total action of this force compensates the energy loss to radiation. This force $\mathbf{f}_2$ equals

$$\mathbf{f}_2=\mathbf{f}_e+\mathbf{f}_m,$$
$$\mathbf{f}_e=-\frac{2e^2\gamma^4}{3c^3}\left[\ddot{\mathbf{u}}+\frac{3\gamma^2}{c^2}\dot{\mathbf{u}}(\mathbf{u}\dot{\mathbf{u}})\right], \tag{4}$$
$$\mathbf{f}_m=\frac{1}{c^2}[\mathbf{u}[\mathbf{u}\mathbf{f}_e]],$$

where e is the charge of the particle or the total charge of the bunch, and $\mathbf{f}_e$ and $\mathbf{f}_m$ are due to the action of the electric and magnetic fields respectively. The quantity $-(\mathbf{f}_e\mathbf{u})$ gives the power of the energy loss to radiation (averaged over the trajectory), i.e., the irreversible energy loss by the particle or the bunch, while the work of the force $\mathbf{f}_1$ determines the reversible energy change in acceleration or braking. Expanding the double cross product in Eq. (4) and combining the forces $\mathbf{f}_e$ and $\mathbf{f}_m$ we arrive at the familiar expression for the radiative friction force,[7] however the previous derivation did not allow the separation of the electric and magnetic constituents.

2. RADIATION BY THE BUNCH IN CIRCULAR AND UNDULATORY MOTION

The power radiated by a point charge is readily found by making use of Eq. (1) and integrating in the wave zone the flow of the Poynting vector through the surface of a sphere of radius $R=$ const. However for an extended bunch such calculations becomes drastically more complicated, and only asymptotic formulas have been obtained for sufficiently long bunches,[1] rotating on a circle. We will find the radiated power for bunches of arbitrary length by calculating the work done by the mutual forces with which different parts of the bunch act on each other. Such deceleration by the proper field determines the power I of energy loss to radiation, and in view of the stationary nature of the problem (motion on a circle) this quantity is equal to the radiated power, i.e., the flow of energy P through a distant fixed surface.

In the motion of bunches in a magnetic field the acceleration is directed normal to the trajectory, therefore the power is emitted mainly in the direction of the velocity, particularly for relativistic bunches. As a result the decrease in the radiated power, due to the interference of the fields of different parts of the bunch, depends first of all on the longitudinal dimension of the bunch. Moreover in most setups this dimension substantially exceeds the transverse one. For these reasons the problem may be simplified by confining the considerations to thread-like bunches, i.e., bunches whose transverse sectional area equals zero and for which all the elements of the charge move on one and the same circle (compare Refs. 1, 2). If the length of these bunches is comparable to or exceeds the wavelength of the radiation field then one may not use the familiar formulas for dipole, quadrupole, or magnetic dipole radiation[6] and it is necessary to perform the calculations in the manner indicated above.

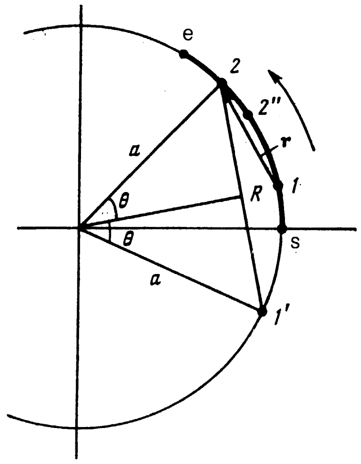

FIG. 1. For calculation of radiated power by relativistic bunches.

The geometry of the problem is shown in Fig. 1. The bunch rotates counter clockwise with speed u on the circle of radius a. The position of the bunch at the instant of time t is marked by the heavy line (the start of the bunch is the point "s" with angular coordinate $\varphi_s=0$, the end of the bunch is the point "e" with $\varphi_e=\Phi$ — the angular length of the bunch). The points 1 and 2 mark the first and second element of the charge between which the interaction is calculated at the time t, $\mathbf{r}$ is the radius-vector from the point 1 to the point 2. The point 1′ denotes the position of the first element of the charge at a previous instant of time t', chosen so that the retarded field arrives from this point at the point 2 at the time t. In this geometry the field of the first element chases after the second element, and therefore we shall refer to this picture as co-travelling motion. The point 2″ corresponds to the position of the second element of the charge at a previous time t'', such that the retarded field from the point 2″ reaches the point 1 at the time t. This picture will be referred to as meeting motion, since the field of the second element moves to meet the first element of the charge.

For long bunches the decelerating force $\mathbf{f}_e$ [see Eq. (4)] cannot be applied to the bunch as a whole, since the velocity $\mathbf{u}$ varies along the bunch while Eq. (4) was derived on the assumption of it being constant. However one may apply a convergent calculational procedure consisting of combining elements in pairs and for each pair calculating the work done by the mutual forces with which these elements of the charge act on each other. In this process there takes place mutual cancellation of all terms that become infinite as $r\to 0$, if use is made of Eq. (2), or as $R\to 0$, if use is made of Eq. (1). In order to take into account all possible pairing combinations of the elements of the charge it is necessary to integrate twice over the volume occupied by the charge and for the thread-like bunches under consideration we obtain the formula

$$I=-\int(\mathbf{jE})\,dV=-\frac{q^2}{2\Phi^2}\iint_0^{\Phi}[(\mathbf{u}_1\mathbf{E}_{21})+(\mathbf{u}_2\mathbf{E}_{12})]\,d\varphi_1\,d\varphi_2. \tag{5}$$

Here q is the charge of the bunch; φ_1 and φ_2 are the angular coordinates of the points 1 and 2; $\mathbf{u}_1$ and $\mathbf{u}_2$ are the velocities of the first and second elements of the charge at the time t; $\mathbf{E}_{21}$ and $\mathbf{E}_{12}$ are the fields at the points 1 and 2 at that same instant of time, produced by the second and first element of the charge respectively.

We calculate first I by substituting into Eq. (5) expression (2) for the field $\mathbf{E}(\mathbf{r},t)$ and taking into account that $\mathbf{E}_{12}=\mathbf{E}(\mathbf{r},\mathbf{u}_1)$ and $\mathbf{E}_{21}=\mathbf{E}(-\mathbf{r},\mathbf{u}_2)$, where the vector $\mathbf{r}$ is shown in Fig. 1. As was already mentioned, one may not take $\mathbf{u}_1=\mathbf{u}_2$ and therefore the decelerating force $\mathbf{f}_c$ cannot be used directly to calculate the slowing down of the bunch by its proper radiation field. The calculation shows, however, that nearly all the terms in the expressions for $(\mathbf{u}_1,\mathbf{E}_{21})$ and $(\mathbf{u}_2\mathbf{E}_{12})$ are odd functions of the difference $\varphi_2-\varphi_1$ and mutually cancel upon summation. There remain only two even terms, which coincide with the expression for $(\mathbf{u}_i\mathbf{f}_c^{(k)})$, where $i,k=1,2$, independently of whether it is calculated in point 1 or point 2. For circular motion of the charge we have $(\mathbf{u}\dot{\mathbf{u}})=0$ and but one term remains, proportional to $(\mathbf{u}_i\ddot{\mathbf{u}}_k)$, which gives after integration

$$I=\frac{2}{3}\frac{q^2\gamma^4u^4}{a^2c^3}\left[\frac{\sin(\Phi/2)}{\Phi/2}\right]^2=I_0\beta^4\left[\frac{\sin(\Phi/2)}{\Phi/2}\right]^2, \qquad (6)$$

where $\beta=u/c$ and $\gamma=(1-\beta^2)^{-1/2}$ is the relativistic factor.

Let us discuss this result. As was mentioned in Sec. 1, for $\beta\ll 1$ these formulas are valid at all points of the circle round which the bunch is rotating, i.e., the angular length Φ of the bunch may have any value. In particular, for $\Phi=2\pi$ it follows from Eq. (6) that $I=0$, which is quite natural since for this value of Φ we have a uniform ring of current which does not radiate electromagnetic waves. With decreasing Φ we are dealing with a long bunch and coherent spontaneous radiation takes place. The power of this radiation goes up with decreasing length of the bunch, and simultaneously the range of values of β for which formula (6) is applicable increases. In particular, for $\Phi\to 0$ the formula is applicable for any β and $I\to I_0\beta^4$, which coincides with the familiar formula for the radiation from a point charge.[6]

For small but finite values of Φ and $\beta\approx 1$ the form-factor in formula (6) leads to a certain decrease in the power I as compared to the point charge. However, on comparing Eqs. (3) and (6) in this case we see that Eq. (6) is only valid for sufficiently small Φ ($\Phi\ll 1/\gamma^3$) so that the form-factor is close to unity. The familiar asymptotic formula, first obtained by Schwinger (see, e.g., Ref. 1), is valid for $\Phi\gg 1/\gamma^3$. To calculate the radiated power for relativistic bunches of intermediate length, when $\Phi\sim 1/\gamma^3$, i.e., the length of the bunch is comparable to the wavelength at the maximum of the radiation spectrum (see Ref. 6), we make direct use of the initial formula (1) by calculating the retardation to higher precision than was done in deriving formula (2). In the case of the simplest circular trajectory this turns out to be possible.

To determine the retardation time we consider Fig. 1. For the co-travelling motion in the time $t-t'$ the light covers the distance R (the chord 1′–2), while the charge moves on the arc between the points 1′ and 1. In the meeting motion the light's path length $c(t-t'')$ is equal to the chord 2″–1 (this chord is not shown in Fig. 1, it corresponds to the angle $\tilde{\theta}$), while the charge moves on the arc 2″–2. Taking these geometric conditions into account one may write the transcendental equation

$$2\beta\sin\theta=2\theta-\psi,\quad 2\beta\sin\tilde{\theta}=\psi-2\tilde{\theta},\quad \psi=\varphi_2-\varphi_1 \qquad (7)$$

for the angles θ and $\tilde{\theta}$; given these angles and having calculated the length of the chord 1′–2 (see Fig. 1 for the co-travelling motion) and the chord 2″–1 which subtends the angle $2\tilde{\theta}$ for the meeting motion we can determine the retardation times $t-t'$ and $t-t''$.

We confine ourselves in the following to the ultrarelativistic case ($\gamma\gg 1$), and under that condition we can only consider sufficiently short bunches for which not only $\Phi\ll 1$ but also $\theta\ll 1$. This turns out to be possible because the coherent radiation begins to fall for $\Phi\sim 1/\gamma^3$, and both the indicated inequalities, as will be seen below, are consistent with the condition $\Phi\gg 1/\gamma^3$, i.e., they permit the solution of the problem for the full range of values of Φ of practical interest.

Exploiting the smallness of θ and $\tilde{\theta}$ we expand Eq. (7) in a power series accurate through cubic terms and solve the resultant cubic equations. We obtain

$$\theta_k=\theta(\Phi)=\frac{1}{\gamma}\{[\xi+(1+\xi^2)^{1/2}]^{1/3}+[\xi-(1+\xi^2)^{1/2}]^{1/3}\}=\frac{S}{\gamma},$$
$$\theta\approx\psi\gamma^2\left[1-\frac{1}{3}(\psi\gamma^3)^2\right],\quad \tilde{\theta}_k\approx\frac{\Phi}{4},\quad \tilde{\theta}\approx\frac{\psi}{4}, \qquad (8)$$

where $\xi=\frac{3}{2}\Phi\gamma^3$ and the second of formulas (8) is obtained from the first by the replacement $\Phi\to\psi$ and power series expansion in the small parameter $\psi\gamma^3$. It is not hard to see that the parameter ξ may take on values $\xi\gg 1$, since then $\theta_k\approx(3\Phi)^{1/3}$ and the initial assumptions are fulfilled provided $\Phi^{1/3}\ll 1$.

When that condition is satisfied we have $\theta_k\ll 1$, hence certainly $\tilde{\theta}_k\ll 1$, and therefore the trigonometric functions that enter the integrand in Eq. (5) — let us denote it by the letter Q — can also be expanded in a power series in θ and $\tilde{\theta}$. Changing at the same time the integration variables with the help of Eq. (7) we obtain

$$\frac{\partial\psi}{\partial\theta}\approx\frac{1}{\gamma^2}+\theta^2,\quad \frac{\partial\psi}{\partial\tilde{\theta}}\approx 4,$$
$$Q\approx\frac{4\gamma^2}{\gamma^2\theta^2(1+\gamma^2\theta^2)^2}-\frac{2}{\gamma^2\tilde{\theta}^2}. \qquad (9)$$

We break up the integral in Eq. (5) in two parts corresponding to the third formula in Eq. (9), and in each of the parts we successively go from the variables φ_1 and φ_2 to ψ and then to θ and $\tilde{\theta}$, respectively, making use of the first two formulas in Eq. (9). In this procedure the terms having θ- and $\tilde{\theta}$-dependent singularities at the lower limit of integration ($\theta\to 0$, $\tilde{\theta}\to 0$) mutually cancel and we obtain the final expression for the power loss due to radiation:

$$I=\frac{27I_0}{8\xi^2}\left\{S^2+2S-\ln\left(\frac{2\xi}{3S}\right)\right\}, \qquad (10)$$

where $I_0=2q^2c\gamma^4/3a^2$ is the power loss due to radiation for a point charge of the same size as the charge of the bunch and moving with the same velocity [more precisely, having the same energy — compare with Eq. (6) for $\beta\approx 1$] while ξ and S were defined in Eq. (8).

Let us compare Eq. (10) with known formulas. For $\Phi\to 0$ and $\xi\to 0$ one may find an approximate expression for S (accurate up to ξ^3) by making use of the second formula in Eq. (8). Substituting $S(\xi)$ into Eq. (10) and making use of the power series expansion in ξ we find $I=I_0$, i.e., Eq. (10)

correctly describes the radiation due to a point charge. For $\xi \gg 1$ the first formula in Eq. (8) yields $S \approx (2\xi)^{1/3}$. If $S \gg 1$ [but $S \ll \gamma$ since otherwise Eq. (10) is inapplicable] then the first term in Eq. (10) dominates all others and we obtain the asymptotic formula

$$\frac{I}{I_0} \approx \frac{2,7}{\xi^{4/3}} = \frac{1,6}{\Phi^{4/3}\gamma^4}, \tag{11}$$

which agrees with known results (see, e.g., Refs. 1 and 2). For intermediate values of ξ the expression (10) gives slightly bigger radiated power than was obtained in Ref. 2 by numerical calculation. For example, for $\xi = 15$ our value of I is larger by a factor of 1.5 than the value in Ref. 2. This difference is, apparently, related to the fact that in Ref. 2 was studied the radiation due to a thread-like bunch with parabolic charge density distribution along the length of the bunch while in the present work it is assumed that the charge density is constant. The fall in I starts for a length of the bunch for which $\xi \approx 1$. The physical significance of this result is obvious: this value of ξ corresponds to a wavelength for which the power of synchrotron radiation is maximal.[6]

Our formulas may be used to compare the power of coherent and incoherent spontaneous radiation by bunches. It is clear that with increasing ξ the quantity I does not tend to zero, as would follow from Eq. (11), but is bounded from below by the power of the incoherent radiation $\tilde{I} = I_0/N$, where N is the number of electrons in the bunch. Comparing this value with Eq. (11) we see that the coherent radiation dominates for $N > 0.6\Phi^{4/3}\gamma^4$. The high power of γ in this inequality leads to the result that for high-energy electrons (hundreds of MeV and above) the total energy losses are determined, in essence, by the incoherent radiation by the bunches, while at low electron energies the coherent energy losses, that dominate the long wavelength part of the spectrum, substantially exceed the incoherent losses. For example, in a cyclic electron accelerator — the microtron — the parameters of the bunches ($N \sim 10^8$, $\Phi \sim 5 \cdot 10^{-2}$, $\gamma \sim 20$) are such that the coherent radiation of the bunches exceeds in intensity the incoherent radiation by roughly five orders of magnitude (compare Ref. 2).

In some case our formulas may be used to calculate the energy loss to radiation in a plane undulator consisting of a set of magnets with uniform fields. This problem is non-stationary and for it the quantities I and P introduced above differ. However, the forced radiation is produced by reflecting mirrors with the result that the main part of the energy lost by the bunch remains within the working volume and acts again on the bunch. Therefore in calculating the initial conditions for the production of forced radiation it is precisely the magnitude of the losses I that matters, and not the energy flow P through a fixed surface.

Since the radiation by the bunch is formed in the angular length θ_k, determined by expression (8) and exceeding the length Φ of the bunch, Eq. (10) may be applied to the undulator provided $\theta_k \ll \Psi$, where Ψ is the angular length of the trajectory coinciding with the turning angle of the velocity vector in each of the magnets of the undulator. In that case the contribution from the transition segments (at the edges of the magnets) is not large, and the energy losses by the bunch due to coherent spontaneous radiation in each period of the undulator amount to $2Ia\Psi/c$, where the power I is given by formula (10) and a is the radius of curvature of the trajectory in the magnetic field of the undulator.

It is clear from physical considerations that the interference between fields from different parts of the bunch suppresses first of all the short-wavelength part of the spectrum, which serves to "prime" the forced undulator radiation. However, for a relativistic point charge in circular motion the synchrotron radiation spectrum has a sharp maximum in precisely this region (see, e.g., Ref. 6). After taking into consideration all the indicated factors in undulators with short electron bunches for comparatively low electron energy one should take into account the possible effect of spontaneous coherent radiation of the bunches (compare Ref. 8).

The author is grateful to A. F. Andreev, L. A. Vaĭnshteĭn and S. P. Kapitsa for discussion of the work and expressing critical remarks.

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Translated by Adam M. Bincer